TRIGONOMETRIC SUMS WITH MULTIPLICATIVE COEFFICIENTS:
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Abstract. Notes from Antal Balog’s talk given at the Clay-Fields Conference on Additive Combinatorics, Number Theory, and Harmonic Analysis given on April 6th, 2008. His talks were titled Trigonometric sums with multiplicative coefficients.

This material is new to some, but it is old stuff. It is one of his favorite subjects in the last decade but still not done. The first problem is to estimate from below the $L^1$ mean

\begin{equation}
\int_0^1 \left| \sum_{u \leq v} \mu(u)e(\alpha u) \right| \, d\alpha.
\end{equation}

We will start with new results and then go back but this is all for a reason.

Remark. Anytime we spoke someone had an idea and then it gave him an excuse to talk about it again.

The most recent result is from 2001 and is due to Balog and Ruzsa and it is based on the convolution. Set $f(\alpha) = \sum a_n e(n\alpha)$ and $g(\alpha) = \sum b_m e(m\alpha)$. Then

$$
\int_0^1 f(\alpha\beta)g(\beta)\,d\beta = \sum_n \int_0^1 a_nb_ne(n(\alpha-\beta))e(m\beta)\,d\beta = \sum_n a_nb_n \cdot 2\pi i \cdot e(n\alpha).
$$

So $M(\alpha) = \sum_{n \leq X} \mu(n)e(n\alpha)$ implies that $M \ast M(\alpha) = \sum_{n \leq X} \mu^2(n)e(n\alpha)$. Hence

$$
\int_0^1 \left| \int_0^1 M(\alpha-\beta)M(\beta)\,d\beta \right| \, d\alpha \leq \int_0^1 \int_0^1 |M(\alpha-\beta)M(\beta)| \, d\alpha \, d\beta = \left( \int_0^1 |M(\alpha)| \, d\alpha \right)^2.
$$

Hence enough to get lower bound for the $L^1$ norm of $\sum \mu^2(n)e(n\alpha)$, but Bruiden-Granville-Pandle-Vaughn-Wooley treated this and proved that

$$
X^{1/4} \ll \int_0^1 \left| \sum \mu^2(n)e(n\alpha) \right| \, d\alpha \ll X^{1/3-\epsilon}.
$$

So we can get $L^1$ mean $\gg X^{1/8}$ and $\ll X^{1/2}$. Can think of this as little bit in terms of Littlewood Conjecture.

Upper bound in many author paper implies asymptotics for number of representations of $n$ as a sum $a_1 + \cdots + a_k$ with $a_j$ an $r_j$-free number and $k \geq 2$. We remark that rarely does the circle method work for $k = 2$, but it succeeds here.

Still not known about what numbers are the sum of three cubes. These results also give the number of representations of $n$ as $a^3 + b^3 + c^3$ and $a, b, c$ are square-free. This is a beautiful paper.

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but it annoys him because they don’t mention our application. The lower bound is not the right lower bound. With Ruzsa we get

$$X^{1/3} \ll \int_0^1 \left| \sum_{n \leq X} \mu^2(n) e(n\alpha) \right| \, d\alpha \ll X^{1/3}.$$ 

This is based on $\mu^2(n) = \sum_{d|n} \mu(d)$ switching order of summation and getting $D(\alpha) = \sum_{n \leq X} e(n\alpha)$.

To improve upper bound just needed to do careful splitting. For lower bound use a slightly more delicate refinement. Step back to his favorite approach. This is due to Balog and Ruzsa from 1999. It uses approximation and convolution.

Find $F(\alpha) = \sum_{n \leq x} c_n e(n\alpha)$ with $c_n = 0$ for square-free $n$ and it approximates $D(\alpha)$ with

$$|F(\alpha) - D(\alpha)| \ll X^{3/4} \log^2(X).$$

What is the use? Compute

$$\int_0^1 F(\alpha - \beta)M(\beta) \, d\beta = M(\alpha)$$

and

$$\int_0^1 F(\alpha - \beta)M(\beta) \, d\beta = 0.$$

So $M(\alpha) = \int_0^1 (D(\alpha - \beta) - F(\alpha - \beta))M(\beta) \, d\beta$. Now $D - F$ is small so we have

$$M(\alpha) \ll X^{3/4} \log^2(X) \int_0^1 |M(\beta)| \, d\beta.$$

So if $L^1$ norm is small then it is uniformly small. But we have

$$\frac{6}{\pi^2} X \sim \int_0^1 |M(\alpha)|^2 \, d\alpha \leq \sup |M(\alpha)| \int_0^1 |M(\alpha)| \, d\alpha \leq X^{3/4} \log^2(X) \left( \int_0^1 |M(\alpha)| \right)^2.$$

Which implies that

$$\int_0^1 |M(\alpha)| \, d\alpha \gg X^{1/8}/\log(X).$$

Here there is maybe room for improvement if you can construct a better approximation, but you cannot approximate as good as $X^{1/2}$ we didn’t really use $\mu$ at all our $\mu$ was pretty arbitrary. Our best we can hope for is $X^{2/3}$ resulting in $X^{1/6}$. Both approaches use almost nothing about $\mu$. Did not even need multiplicative. Compare to Littlewood Conjecture... very little.. you just need a bunch of zeros in the coefficients and they have some sort of structure.

Step back to first approach to get the weakest result. But this is adaptable to multiplicative functions. This is due to Balog and Paredi in 1998. Use idea of Vaughan from 1988. Who was looking at $\int |\sum A(n)e(n\alpha)| \, d\alpha \ll \sqrt{X \log(X)}$ and have proved $\gg \sqrt{X}$. Not surprising since this function has peaks near rational $\alpha$ and has size roughly $X/\varphi(q)$. For lower bound

$$\sum_{q \leq Q} \sum_{a \sim q} \int_{-1/x+a/q}^{1/x+a/q} |\cdots| \, d\alpha \approx \sum_{q \leq Q} \sum_{a \sim q} \frac{2}{x} \frac{x}{\varphi(q)} \approx Q.$$

But you loose control for $Q$ a small power of $X$. So you use triangle inequality and sum inside integral.

Now $\mu$ is always smaller so what to do? If there are no Siegel zeros then $|M(\alpha)| \ll X/e^{\sqrt{\log X}}$. So suppose there is one and use it to find peaks. So if there is a Siegel zero then collecting all the peaks from this zero to get $\int |M(\alpha)| \, d\alpha \gg X^{1/2-\epsilon}$. IF not Siegel zeros define major and minor arcs with $R \leq \sqrt{X}$.

**Three Improvements.**

(1) Better lower bound than $X^{1/6}$. 
(2) General theorem for all multiplicative functions into $S^1$... or maybe Nilsequences.
(3) $L^3$ norm for $\sum \mu^2(n)e(n\alpha)$. This was suggested by Wooley.
I also wonder what can be said in the number field setting or for Beurling sequences.