ORTHOGONAL POLYNOMIALS
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Abstract. These are my notes from a talk by George Andrews on March 8, 2010 in the workshop on Mock Modular Forms and $q$-series at the American Institute of Mathematics. All mistakes are my own.

Orthogonal polynomials lead to Fourier series via the orthogonality relations. Expansion in terms of Legendre polynomials. Orthogonal polynomials have a recurrence relation

$$a_n p_{n+1}(x) + (x + b_n)p_n(x) + c_n p_{n-1}(x) = 0$$

A nice calculation is the following for $0 < q < 1$ and weights $(1 - q)q^i$ we carry out a Riemann-Steiljes integral as a $q$-integral. Namely,

$$\int_0^1 f(x) d_q x = \sum_{i=0}^{\infty} f(q^i)(1 - q)q^i$$

This is the $q$-dissection. As an example we have

$$\int_0^1 x^n dx = \lim_{q \to 1} \int_0^1 x^n d_q x$$

$$= \lim_{q \to 1} \sum_{i=0}^{\infty} q^{in}(1 - q)q^i$$

$$= \lim_{q \to 1} \frac{(1 - q)}{(1 - q^{n+1})} = \lim_{q \to 1} \frac{1}{1 + q + q^2 + \cdots + q^n}$$

Bailey pairs give the following relations. If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(aq; q)_{n-r}(aq; q)_{n+r}}$$

then

$$\sum_{n=0}^{\infty} a^n q^n \beta_n = \frac{1}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} a^n q^n \alpha_n.$$ 

Rogers second proof of Rogers-Ramanujan follows if $\beta_n = \frac{1}{(1-q)\cdots(1-q^n)}$ to complete the proof we need to apply Jacobi’s triple product formula.

There is a strong form of Bailey’s Lemma which hatches a new Bailey proof leading to an infinite family of identities.

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For example, if we let $\beta_n = \frac{1}{(1+q)\cdots(1+q^n)}$ the fifth order mock theta function, then what is $\alpha_n$? A compute algebra package will tell you that

$$
\begin{align*}
\alpha_0 &= 1 \\
\alpha_1 &= q^2 - 3q \\
\alpha_8 &= q^{100} - 2q^{99} + 2q^{96} - q^{92} - 2q^{88} + \cdots
\end{align*}
$$

Notice that most of the coefficients are 0! Furthermore, those that aren’t usually equal $\pm 1, \pm 2$.

From this one finds

$$
\prod (1 - q^n) = \sum_{n=0}^\infty \sum_{|j| \leq n} (-1)^n q^{n(5n+1)/2 - j^2} (1 - q^{4n+1}).
$$

These identities follow from iterative Bailey pairs and excitement of compute algebra packages.

We return to orthogonal polynomials and the identity

$$
(aq; q)_\infty \sum_{n \geq 0} \frac{q^{n^2} a^n (-yq; q)_n}{(q^2; q^2)_n} = \sum_{n \geq 0} \frac{(-1)^n q^n a^{2n} (a^2; q^2)_n (1 - aq^{2n})}{(q^2; q^2)_n} \times p_n(y; -\frac{a}{q}, -1; q)
$$

where $p_n$ are the little $q$-Jacobi polynomials. These polynomials have orthogonality properties. The case $y = 1$ and $a = 1$ yield the Rogers-Ramanujan identity and $y = -1$ and $a = 1$ yields the fifth order mock theta function.

Expanding as a function of $y$ gives

$$
\sum_{n \geq 0} c_n p_n(y; -\frac{a}{q}, -1; q)
$$

W. Hahn and Andrews-Askkey-Roy for orthogonality. This satisfies a 3-term recursion which can be found on page 167 of Gaspar-Rahman. With Askey looked at $q$-series expansions for little $q$-Jacobi polynomials. Watson proved $3\phi_7 \sim_4 \phi_3$ identity based on Whipple’s usual for hypergeometric series.

Returning to Bailey pairs and see inversion has

$$
\alpha \sim \sum \beta_j(*) (q^{-n}; q)_n (q^n; q)_n
$$

and the $\beta_j$ has no dependence on $n$. (This looks like a false theta function.) Slater’s compendium (38) and (39) look like mock theta functions but with the signs messed up

$$
\frac{1 - q + q^2 - q^5 + q^7 - q^{12} + \cdots}{1 - q - q^2 + q^5 + q^7 - q^{12} + \cdots} = 1 + 2q^2 + 2q^3 + 4q^4 + 4q^5 + \cdots
$$

The top has the messed up signs and the bottom is the Euler’s pentagonal number series. This comes from an application of BIG $q$-Jacobi polynomials.

There is a connection to spiral self-avoiding walks and statistical mechanics.

Askey-Wilson polynomials in their seminal paper give a large family for orthogonal polynomials.

Zagier made a remark at the end of the discussion that the definition of a false theta function should be

$$
\sum_{n \in \mathbb{Z}} \epsilon(n) q^{cn^2} = \sum_{n \in \mathbb{Z}} \epsilon_{\text{odd}}(n) q^{cn^2} + \sum_{n \in \mathbb{Z}} \epsilon_{\text{even}}(n) q^{cn^2}
$$

where $\epsilon$ is periodic. Better term would be half-theta function as you can write

$$
\sum_{n \geq A} \chi(n) q^{cn^2}.
$$
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