

Problems With MAP Assessments and Consequences

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“By world standards, our current education system is mediocre... [We] could choose a different path, one with better-educated children, international economic leadership, and a faster growing economy. With this, we solve our fiscal and distributional problems not with battles over the balance of revenues and spending but by ensuring that the pie grows.” (Rick Hanushek, U.S. Senate Testimony, 3/8/2012)

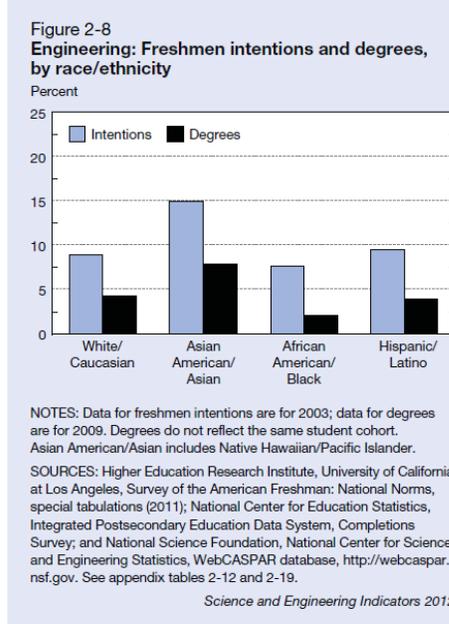
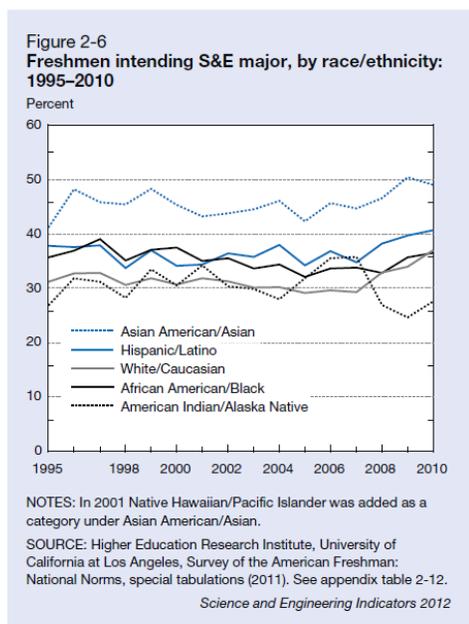
It is becoming increasingly clear that our current educational outcomes are far from what we need, and cost the nation dearly, particularly in STEM (Science, Technology, Engineering, Math) areas. ⁽¹⁾ To help ameliorate this the K-12 educational establishment targeted women and minorities in an attempt to broaden and strengthen our STEM outcomes, but it now appears that these focused programs have only made matters worse for women:

Women Engineering Graduates at 15-Year Low

“The number of male engineering graduates rose by 11% from 2004 to 2009, while the number of female engineering graduates actually fell by 5.2% over the same period, according to the National Center for Education Statistics.”

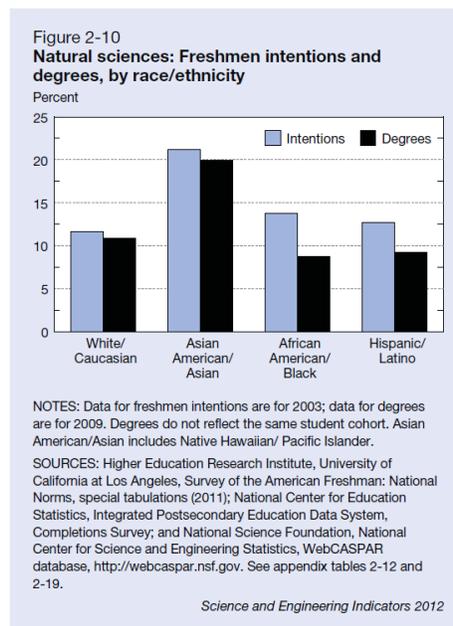
“In 2009, the percentage of undergraduate degrees from engineering schools that went to women hit 17.8%, a 15-year low, according to the American Society of Engineering Education.”

And we are getting similar, if not worse, results with the other targeted groups. There are also many further issues with the low degree attainment rates vs original intentions for all groups, but particularly minorities. Here is one that I consider particularly telling:



(1) For some projected consequences of this failure see [KR]

Compare this with the degree attainment rates vs intentions for social/behavioral sciences:



These data almost certainly represent a clear difference in K-12 preparation of students for the non-STEM and STEM areas. It appears evident that K-12 does a poor job of preparing students for STEM majors, or, at a minimum, does a poor job of advising students interested in STEM areas.

There is some indication that the creation and general acceptance of the new Core Standards is leading educators to claim they are now in a position to turn the situation around. But there is no evidence to support their belief, and considerable evidence that indicates improved outcomes are unlikely. For example recent work by T. Loveless, [L], gives overwhelming evidence that improved standards alone typically have no effect on outcomes.

It is worth putting forward conjectures as to what the underlying problem might be. One that is evident is that teachers can't teach what they don't know, and our teachers are poorly prepared in STEM areas. There is quite a bit of evidence for this – not the least of which is the fact that the typical elementary school teacher scored in the lowest third of her graduating high school class on the SAT's.⁽²⁾ This, together with the popular misconception that students can learn the math they need when they take more advanced classes that need it, provides a plausibility argument that education schools use to justify their assertions that they are not to blame for the problems above. In fact, they often go further, asserting that the problems are not that major. Sometimes they go even further, stating that the students are to blame, or their parents or their socio-economic classes.

More and more evidence, not the least of which is our increasing understanding of the structure of student outcomes in other countries, is making the last three claims less

⁽²⁾ See www.tampabay.com/blogs/gradebook/content/sat-scores-teacher-wannabes for details

and less likely. The second claim also fails because math, unlike many other areas, is rigidly sequential and cumulative. Success in grade 5 mathematics crucially depends on understanding virtually all the material in grade 4, and the same for the other grades. Moreover mastery in high school courses in engineering, chemistry, and physics, depends on mastery of, at a minimum, Algebra I and Geometry.

It becomes extremely difficult to catch up once a student falls behind at any level in mathematics. Indeed, before the student can learn further material he must correctly learn the material that caused his difficulties.

In this note we call into question the education schools themselves!

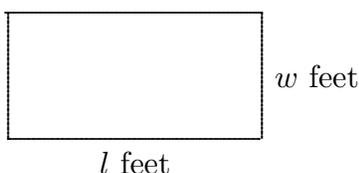
Recently, one of the two national consortia, (PARCC and SBAC), that have been awarded the responsibility for designing the assessments linked to the new Core Standards released a sample two hour assessment. The file can be found at:

<http://www.smarterbalanced.org/wordpress/wp-content/uploads/2012/02/DRAFT%20Math%20Item%20Specifications.pdf>.

The problems are very uneven with most of them requiring little more than fifth grade arithmetic and geometry, but the two that do test higher grade standards – problem 2 on page 31 and problem 6 on page 40 – show very poor mathematical judgment.

The first focuses on an entirely minor standard, one that is unique to Core Standards, but one that requires considerable sophistication on the part of both student and teacher to put it into proper context. The first part just tests the standard directly, but the second part is very hard to comprehend in any way that would make mathematical sense. Indeed,

Part B: The length, l , and the width, w , of the rectangle shown below have values that are rational numbers.



Construct an informal proof that shows the value of the area, in square feet, of the rectangle must be a rational number.

makes little sense. One constructs a *formal* proof by simply noting that the product of two fractions is a fraction, which follows from the definition of multiplication of fractions $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, and noting that the area, in square feet, is the product of the length in feet times the width in feet. I can't really imagine what an informal proof would be that was not a formal proof or was not mostly nonsense.

The second involves a reasoning argument in a geometry proof. It is far too subtle for almost all students, even those who have had a traditional geometry course. The reasoning

error in the two column proof is one that students simply will not see without focused test-prep. Far too few of the students will be able to say anything useful otherwise.

In the remainder of this article I give an analysis and a discussion of the problems presented by the MAP project in one of the two draft high school tests submitted for public examination and intended to align with and properly evaluate student mastery of the Core Standards.

It turns out that *the last seven of the ten major problems as well as at least one of the first 10 “routine” problems have serious issues* that usually make the problems nonsensical or the scoring rubrics incorrect. My conclusion is that among all the sample and actual tests I have examined – most where I was actually asked to examine the test items by state or national officials – including many of the current state assessments and a number of international exams, there was only one sample test that was worse and more mathematically illiterate than the MAP example.

It becomes clear as we work through the problems that the authors either do not understand the basic concepts that they are supposed to be testing, or they are motivated by non-mathematical considerations to present essentially nonsensical questions. We really do not know which is the case, but in spite of the Ph.D. in mathematics that one of the authors has, we would prefer to believe that these questions represent a failure of understanding, and not a deliberate attempt to misteach the subject.

The leaders of this project include some of the most prominent professors of mathematics education in this country and in the U.K. Thus we are forced to conjecture that the ultimate problem with our student outcomes lies with the education schools themselves. How can we expect better outcomes if many of the leading math education faculty at these institutions do not even have an adult understanding of fifth - tenth grade mathematics?

THE INTRODUCTORY MATERIAL IN THE CCR-C2 TEST

I had been looking at the MAP sample high school tests, (the 3-hour forms CCR-C1 and CCR-C2) at <http://map.mathshell.org/materials/tests.php>.

I became increasingly disturbed by what I was seeing so I decided to do a closer analysis and try to detail the kinds of issues I was having. I started with the first ten problems in the second of the MAP tests, CCR-C2. The first 10 problems in each of CCR-C1 and CCR-C2 are designed to be routine, and basically test “math facts.” However, look at CCR-C2, problem 8:

8. Use the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ to find the value of $\cos(x)$ if $\sin 2(x) + 2\cos(x) - 2 = 0$.

Two possibilities. First $\sin 2(x)$ might be a misprint and it should be $\sin^2(x)$, in which case $\cos(x) = 1$, since the equation then becomes $\sin^2(x) + 2\cos(x) - 2 = 0$ and replacing $\sin^2(x)$ by $1 - \cos^2(x)$ gives the equation $-(\cos^2(x) - 2\cos(x) + 1) = 0$ so $(\cos(x) - 1)^2 = 0$ and $\cos(x) = 1$. In fact, I would be pretty sure that this is the intent in problem 8 since the first 10 problems are meant to be routine. However, it is worth noting that core standards expect students to understand and prove the two angle addition and subtraction formulae, (F-TF.8 and F-TF.9), though these are (+) standards, hence focused on college intending students, and it is somewhat unclear what the PARCC and SBAC policy is on (+) standards.⁽³⁾

In any case, what kind of people would make a mistake like mistyping $\sin^2(x)$ as $\sin 2(x)$ on a major sample test for extensive public distribution, especially when the angle addition formulae are required by core standards? So I had to assume the expression is not a misprint. If so we get $\sin 2(x) = 2\cos(x)\sin(x)$ and $2\cos(x)\sin(x) + 2\cos(x) - 2 = 0$ or $\cos(x)(1 + \sin(x)) = 1$. Squaring both sides and replacing $\cos^2(x)$ by $1 - \sin^2(x)$ gives $(1 - \sin^2(x))(1 + \sin(x))^2 - 1 = 0$. Expanding and replacing $\sin(x)$ by y gives $y(y^3 + 2y^2 - 2) = 0$. So $y = 0$ and $y = .83928\dots$ are its real roots – the first since y is a factor, and the second by using Newton’s method or successive approximation starting with an *estimate* of the root from the graph. A simple check shows that 0 and $+\sqrt{1 - (.83928\dots)^2}$ are both solutions of the original problem, and things are equally fouled up, since the question asks for THE value of $\cos(x)$. (Yes, I know that there is an essentially 0% chance that a student taking this test would even be able to handle the algebra for the second interpretation. So by the process of elimination they HAD TO mean the first interpretation.)

Of course the real issues are with the final 10 problems. But certainly problem 8 is not a promising start.

The analysis that follows shows that the situation only gets worse in CCR-C2, part 2. In fact, fully seven of the ten problems there are so flawed that they will simply cause

⁽³⁾ In this regard, it is worth noting that a very high percentage of high school graduates list themselves as college intending. The Department of Labor in April, 2011 reported that 68.1% of 2010 high school graduates were enrolled in colleges and universities. But if one looks at college enrollment within 2 years after high school graduation the percentage is higher. C. Adelman reported an attendance rate of slightly over 80% for example, (private communication).

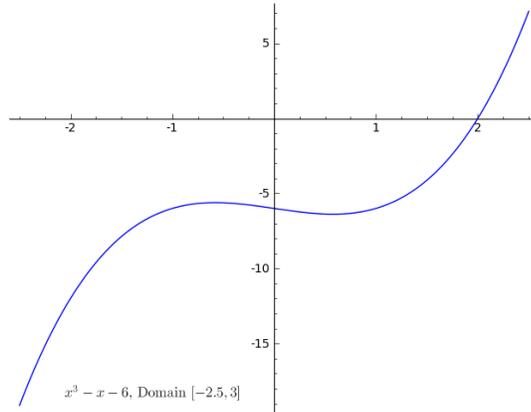
damage if they are treated as even representative of the mathematics that students need “in the 21st century.” Hence, 40% of the major questions are either not mathematics or are severely flawed mathematics. I would wonder if this is what we really want for our students.

To illustrate the fact that the same types of errors were equally prevalent in both exams, hence had to be associated with misconceptions on the part of the authors, I also include analyses of two of the main problems from the other released exam, CCR-C1. But it did not seem overly useful to do an equally exhaustive study of that document.

THE PROBLEMS IN THE MAIN PART OF THE EXAM

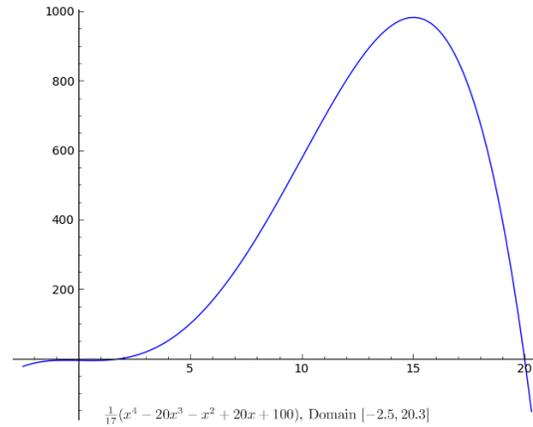
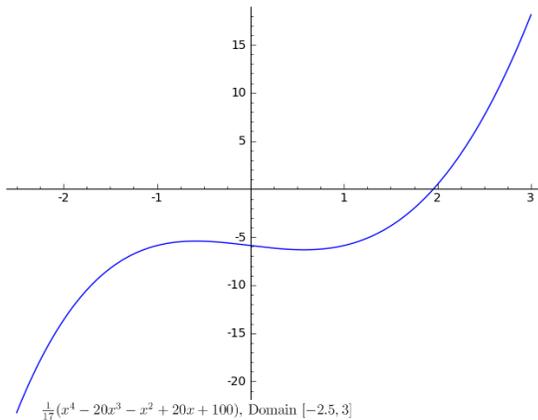
The fourth problem in the second part of CCR C2 is entitled **Cubic Graph**:

1. a. Show that $x = 2$ is a solution of the equation $x^3 - x - 6 = 0$.
- b. The diagram opposite (*here below*) shows the graph of $y = x^3 - x - 6$.



- i. Write down the coordinates of point A.
- ii. Use the graph to explain why there is only one solution to the equation. $x^3 - x - 6 = 0$.

Unfortunately, any argument only relying on the graph above *must be false*. This is widely misunderstood, so below we give an example consisting of another polynomial, $-\frac{1}{17}(x^4 - 20x^3 - x^2 + 20x + 100)$, which, in the same range as the graph above gives a very similar picture, but when we look in a larger range gives a completely different picture and actually has a second zero.



It is true that the left hand graph above is slightly different from the first graph on this page. But, as a matter of fact, it is not hard to construct a polynomial with graph arbitrarily close to the graph of $x^3 - x - 6$ in the range $[-2.5, 3]$, but which has at least one further root anywhere one desires outside the range $[-2.5, 3]$, say $x = 20$ for definiteness. There is one more issue well worth pointing out here. Many (most?) math educators love to say a function can be equally “represented” by tables, graphs, formulas, and verbal

descriptions. So, fundamentally they don't know that a graph usually can only give partial information. We've just seen one typical example of this, but it is also well worth knowing that actual physically constructed graphs are only approximations, due, if nothing else, to the thickness of the line. So it is incorrect to assert *the value of $f(x)$ at $x = x_0$ from the graph alone.* What actually happens is that students are aware of the popular – but mathematically ridiculous – assumption that if it appears the graph has an interesting point that seems to be near an integer, or a pair of integers, then it is, in fact, integral. This error will also appear multiple times in our analysis of the other problems in this sample exam. (In particular, see the second part of the final problem “Fearless Frames” where the authors themselves assert an answer of 128 when the computed answer to 8 places is 128.30005982 . . .) Likewise, it is incorrect to say that a *table* represents a function unless the function is only defined on the finite domain of points where the table gives its values. See the discussion of the problem **Sidewalk patterns** for a detailed discussion of this issue.

The difficulty above actually represents an area where the authors display consistent ignorance and lack of understanding. Look at the scoring rubric for this part of the “Cubic Graph” problem:

| Cubic Graph | Rubric | |
|---|--------|----------------|
| | Points | Section points |
| 1. Shows correct work such as: | | |
| a. $2^3 - 2 - 6 = 0$, so $x = 2$ is a solution to $x^3 - x - 6 = 0$ | 1 | |
| Gives correct answer such as: (2, 0) | 1 | |
| b. Gives correct explanation such as: | 1 | |
| Graph cuts $y = 0$ only once, so there is only one value of x for which $y = 0$. | | 3 |

From the examples above we see that the suggested argument here cannot actually work. In fact, while it may be possible to use considerations from school algebra to prove this assertion, I have no real idea of how to do it that way.⁽⁴⁾ By far, the most natural method involves calculus. The zeros of the derivative of the cubic, if distinct, are points where the cubic has a (local) maximum or minimum. This derivative is $3x^2 - 1$ which has zeros at exactly $\pm \frac{1}{\sqrt{3}}$. Since the cubic goes to $+\infty$ as $x \mapsto +\infty$ and $-\infty$ as $x \mapsto -\infty$ the point $-\frac{1}{\sqrt{3}}$ is a local maximum and $\frac{1}{\sqrt{3}}$ is a local minimum. Since the cubic is negative at both

⁽⁴⁾ Dick Askey points out that it is possible to use elementary methods to demonstrate that the function only crosses the x -axis once, but he admits that there is only the smallest chance that any high school student would discover it: “Note that here is how the first graph problem can be done without calculus. The function is $y = x^3 - x - 6$, . . . Shift the curve by 6 and you have a translated graph . . ., $y = x(x^2 - 1)$. For $x > 1$ both factors increase and for $0 < x < 1$ this function takes on values between 0 and -1 . It is odd so for $-1 < x < 0$ its values are between 0 and 1, so subtracting 6 gives a function which is negative between -1 and 1, and when $|x| > 1$ the function is monotone. Thus no calculus is needed to solve the problem.”

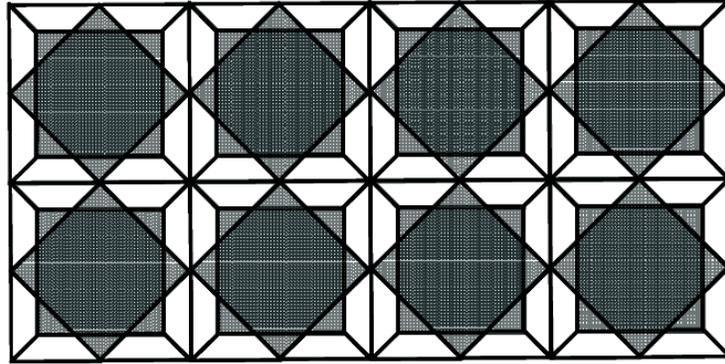
these points (which CAN be seen from the graph), and since the cubic monotone decreases for $x < -\frac{1}{\sqrt{3}}$ while it is monotone increasing for $x > \frac{1}{\sqrt{3}}$ the graph can only cut the x -axis at a single point and this point is greater than $\frac{1}{\sqrt{3}}$.

I have given the argument above in some detail simply to indicate the degree of sophistication, knowledge and basic mathematical skills necessary to give a proper proof of 1.b. This test is claimed to be appropriate for high school students, but a proper treatment of the question above could barely be handled by a high school graduate, and then only if the student had some familiarity with calculus, as well as a very sophisticated understanding of the actual mathematics involved.

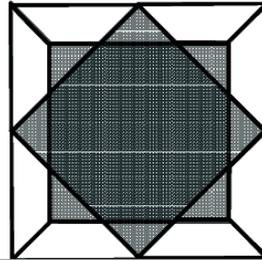
The fifth problem in part II, **Floor Pattern**, shows what amounts to exactly the same error as the cubic graph problem discussed above:

Floor Pattern

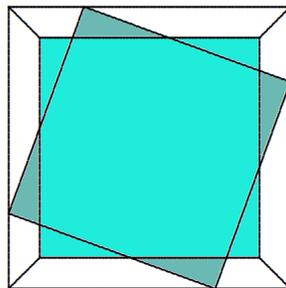
The diagram shows a floor pattern.



In the floor pattern, the shaded part is made by overlapping two equal squares.



The problem comes up in the phrase “In the floor pattern, the shaded part is made by overlapping two equal squares.” There are LOTS of ways to “overlap” the two squares if that is all that is specified. For example



The authors absolutely need to be more precise, for example, “rotate the first square about its center through an angle of 45° , then draw the square with sides parallel to those

of the first square, where the edges of the larger square contain the four vertices of the rotated square. Finally, draw the line segments from the vertices of the first square to the corresponding vertices of the larger square.”

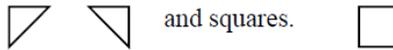
With this careful and precise description of the building blocks, the statements of the actual problem on the next page of CCR-C2 become easily verifiable by most students. But without this level of precision, all of the students would be *guessing* at what the authors meant, and this kind of thing results in exactly the kinds of errors we saw in the previous problem.

The errors of inadmissible imprecision that were illustrated in the **Floor Pattern** problem might be attributable to carelessness or proofreading issues. However, exactly the same errors occur in at least two of the problems in the other draft high school test that the authors provide, The fourth problem in part II of that test is called **Patchwork**. It has the previous types of issues. Indeed, it has even more kinds of errors. The same can be said for the sixth problem **Circles and Squares** in the other draft. So we have to conclude that these errors represent a far deeper misconception about mathematics. Here is the first part of **Patchwork**:

Patchwork

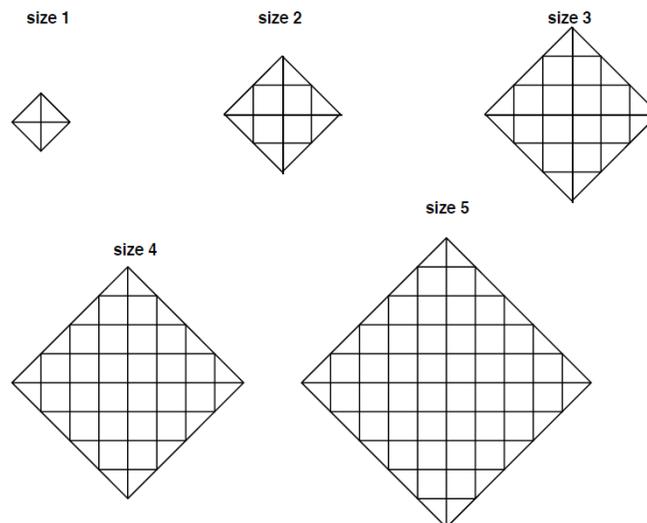
A sheet of square dot paper is provided for use with this item.

Kate makes patchwork cushions.
She uses right triangles



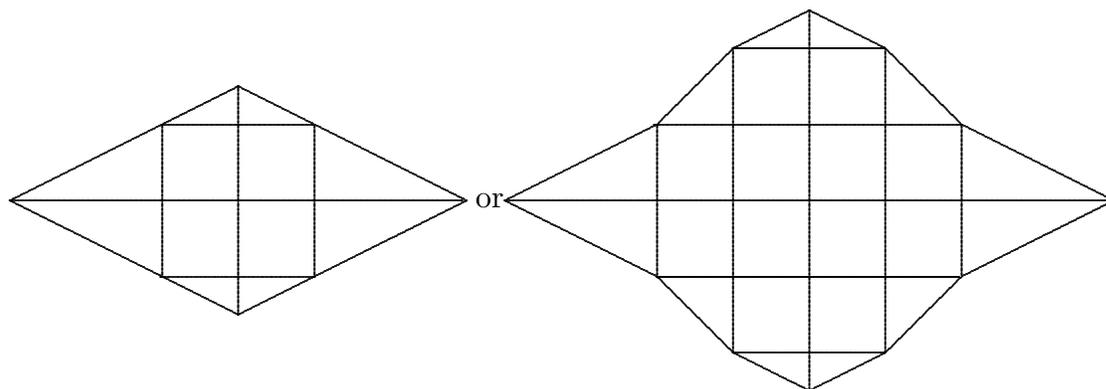
She uses triangles along the edges of each cushion. The rest is made from squares.

Note that the only thing specified is that we have right triangles and squares. Nothing is said about their size, but it appears that the authors assume the triangles are all the same and so are the squares. They also assume the triangles have base angles 45° . This is further “reinforced” by the diagrams



where it “appears” clear that the edges of the patches are straight lines. But they appear to be totally unaware that virtually the same pictures can occur with different sized triangles

and squares.



These were drawn to exaggerate the differences. If there are only very small deviations, it will be virtually impossible to detect that the edge lines are not straight in the right hand figure. So, to correctly pose the problem it is essential that further specifications occur.

One might well ask what further specifications are necessary. It turns out that one only needs to say same sized triangles and same sized squares, with the additional assumptions that the interiors of the triangles and squares do not overlap and that each vertex of any one of the squares is also a vertex of either three other squares, two other squares and a triangle, or two triangles. But something along these lines really needs to be present, if we are to avoid giving students the firm impression that mathematics is a guessing game.

Of course it is also true that the main questions here

2. Find a rule, or a formula, that will help Kate figure out the number of triangles that she needs for cushions of different sizes. Explain how you figured it out;
3. Use the number patterns in the table to find a rule, or a formula, that will help Kate figure out the number of squares she needs for cushions of different sizes. Explain why your rule works;

are correctly phrased to include any patterns the reader can come up with that start with the five cases, no matter how ill-defined those cases might be. For example, the rule can be that the five figures given are the totality of the sizes there are, or it can be that the sixth size consists of a single square, and the seventh of a block of four squares, and then these are all the sizes there are, and so on. The rule is only limited by the creativity of the student. All these are fine answers as long as the students state the rule they are using. Then using their rule and the numbers they have filled in in the included table, they can hopefully provide a correct answer for the additional figures that occur in their rule.

However, the rubric does not allow credit for any but the single rule the authors expect. Here are the rubrics:

2. Verbal rule: The number of triangles is four times the size of the cushion. or An algebraic rule: $t = 4n$ Explanation: Each cushion has four edges: each edge has the same number of triangles as the size. or From the table, as the size of the cushion increases by 1, the number of triangles increases by 4.

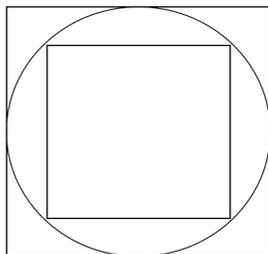
3. A stepwise verbal rule: The number of squares increases by 4, then 8, then 12, then 16 . . . : increasing multiples of 4. or The number of squares + the number of triangles for any size is equal to the number of squares for the next size. e.g.: $16 + 24 = 40$ or An algebraic rule: $s = 2n(n - 1)$ or equivalent algebraic rule. Explanations relating to the cushion design, such as the following. Stepwise rule: Each triangle of one size becomes a square in the next size. or Algebraic rule: Each cushion has four sections: if we put two sections together, we get two rectangles, size n by $(n - 1)$.

(In a real sense the author's required response is sort of nonsensical answer there will only be a finite number of sizes available in real situations. So the student responses suggested above would actually make more sense than the authors "correct answer.")

Here is the sixth problem in CCR C1 Part II, **Circles and Squares**

This diagram – which is scanned directly from the test – shows a circle with one square inside and one square outside.

- 1 . What is the ratio of the areas of the two squares? Show your work.



2. If a second circle is inscribed inside the smaller square, what is the ratio of the areas of the two circles? Explain your reasoning.

The correct answer to both (1) and (2) is “Who knows?” For example, look at the illustration. It shows a square *properly contained* entirely in the interior of a circle. In turn the circle might or might not be tangent to the external square at the four midpoints of its edges. We cannot tell due to the thickness of the lines.

In fact, if you look closely, you can see that the vertices of the smaller square do not appear to touch the circle! Since we do not know and cannot determine the side lengths of the second square, we cannot answer the question. The difficulty occurs because the original diagram is not fully described. If the authors had simply said the square is an *inscribed figure* in the circle while the circle is an *inscribed figure* in the larger square, the definition of “inscribed figure” would have made things precise. (See the Wikipedia article entitled *Inscribed figure*.)

The issue, as happens so many times in these “exams,” is that there is no rigorous description of the construction, and it follows that we don’t have any idea of the ratio of the smaller square to the bigger one. Likewise, the specification of the second circle is that it is inscribed inside the smaller square. This is, in fact, unambiguous, but since we don’t have any idea of the size of the small square, we have no idea of the ratio in part 2 either.

The remarks above give a mathematically correct discussion of the problem, but look at the rubric. No credit is given for the correct answer, and all the hidden assumptions that students have to guess at are required in order to receive credit.

Rubric

1. Gives correct answer: The ratio of the areas of the two squares is 1:2 Shows correct work such as:

Draws construction lines from the center of the circle to the vertices of the small square.

If the large square has side of length x , then, using the Pythagorean Theorem gives the length of the sides of the small square are $\sqrt{2}x/2$.

The area of the large square is x^2 .

The area of the small square is $x^2/2$

Accept alternative methods.

2. Gives correct answer: The ratio of the two areas is 1:2

If a second circle is inscribed in the smaller square, using the Pythagorean Theorem gives the radius of the small square is $\sqrt{2}x/4$

The area of the large circle is $\pi(x/2)^2 = \pi x^2/4$

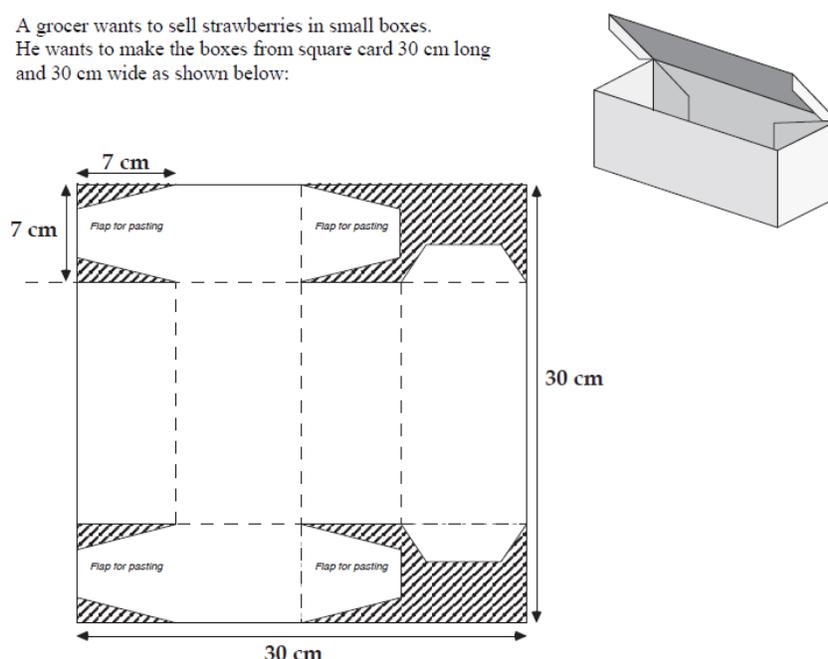
The area of the small circle is $\pi(\sqrt{2}x/4)^2 = \pi 2x^2/16 = \pi x^2/8$

Accept alternative methods.

So, again, we see the same collection of misunderstandings of the subject that we have seen in the previous questions.

The sixth problem in CCR C2 Part II is **Strawberry Boxes**

A grocer wants to sell strawberries in small boxes.
He wants to make the boxes from square card 30 cm long
and 30 cm wide as shown below:



The shaded areas are cut away and the rest is folded along the dashed lines.
The sides are folded up and stuck together using flaps.
The lid has two flaps that are not glued.

1. Calculate the volume of this box. Show your work.

This problem starts well, and the test authors give a sufficiently precise description of the construction that students should be able to figure out what is being asked:

However, the second part of the question:

2. Suppose the grocer starts with the same square of card, but changes the 7 cm to a different measurement. What is the largest volume he can make the box? Show your calculations.

is way above the level that one can legitimately expect from high school students. When 7 is replaced by x , the only things we can say are that $0 < x < 15$, and for x in this interval, the resulting volume of the box is $2x(15 - x)^2$. We are asked to find the maximum of this function over the interval. As was the case with the “Cubic Graph” problem, the normal way to do this is to take the derivative of the volume function and look for its 0’s. The derivative is $2(15 - x)^2 - 4x(15 - x) = 2(15 - x)(15 - 3x) = 6(15 - x)(5 - x)$, so it has zeros at 5 and 15, and, since as $x \mapsto \infty$ the function $2x(15 - x)^2$ also goes to ∞ while as $x \mapsto -\infty$ the functions goes to $-\infty$, it follows that 5 is a local maximum and 15 a local minimum. Since 15 is also the right hand end of the actual region, it follows that 5 is a maximum over the entire actual region.

Now, what did the test authors have in mind for this problem? Well, we actually have no way of knowing for sure since the rubric contains grading instructions for a problem

entitled **Fruit Boxes** that is clearly different from the problem **Strawberry Boxes**, but we can infer from the discussion in the rubric that the maximum point for Fruit Boxes is 6, and since the maximum point for Strawberry Boxes is 5, we see that the most likely expectation of the test authors is that the students will use “guess and check” with integer values, since they will have been trained to expect integer solutions – hardly realistic in the real world. But also, hardly correct!

The seventh problem in CCR C2 Part II, **Sidewalk Patterns**, exhibits yet another misconception about mathematics - and in a number of ways is the most seriously flawed of all the problems in this discouraging mess.

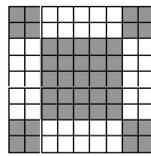
Sidewalk Patterns

In Prague some sidewalks are made of small square blocks of stone.

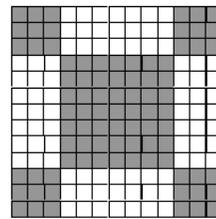
The blocks are in different shades to make patterns that are in various sizes.



Pattern #1

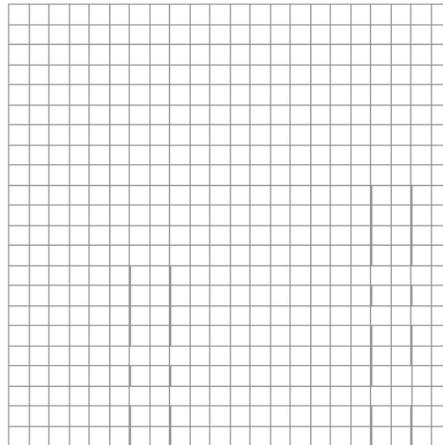


Pattern #2



Pattern #3

Draw the next pattern in this series. You may not need to use all the small squares.



Pattern #4

In the book [M] there is a discussion of these kinds of errors starting on page 98 (but it is probably worth the time to read the entirety of Chapter 3). Here is part of that discussion:

(3) This question is from a state eighth grade assessment exam.

Use the chart below to answer question 4.

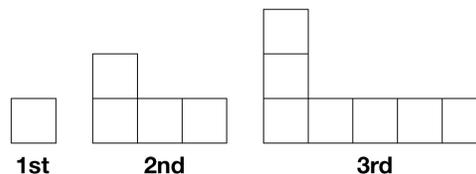
| | | | | | | |
|--------|----|----|----|----|-----|-----|
| Input | 3 | 4 | 5 | 6 | ... | n |
| Output | 10 | 13 | 16 | 19 | ... | ? |

- 4 If the input is n , what will the output be?
- A. $n + 3$
 - B. $n + 7$
 - C. $3(n + 2) + 1$
 - D. $3n + 1$

The difficulty here is that no rule is specified for generating the next term. The question is better than average in that the meaning of n is specified. In many questions of this type we are even required to speculate on whether n is the input value or whether it is the position of the element in the sequence. However, since no rule is given we are required to guess that the actual rule is contained in the list of possible answers. The question could be fixed by rephrasing it: *The chart above gives the first four input, output pairs for one of the rules below. If the input is n , which rule is it?*

(1) This is another question from the same eighth grade exam.

- 6 Each arrangement in this pattern is made up of tiles.



How many tiles will be in the 6th arrangement in the pattern?

This has exactly the same kinds of difficulties as the problem above. This time the variable n , the ordinal number above, has been vaguely indicated as the input, but no idea of the way in which the pattern is to be generated is given, and the problem is a *short response question*, which means that there is no help to be had from a list of possible answer choices.

H.-H. Wu has called this type of mistake part of the **Wishful Thinking Syndrome** in school mathematics education: give out partial information and students will automatically fill in the missing information to achieve a complete conceptual understanding on their own.

Of course, part of the analysis above shows that not only is there no “**unique**” extension, there is no “**best**” extension either.

It is worth noting that students who have been *trained* to respond to this kind of question “correctly” will get credit. Students who think more critically will get no credit.

This is more than amply illustrated in the grading rubric for **Sidewalk Patterns**. First there is “Draws correct pattern,” 1 point. Then, “Gives correct answers,” 2 points:

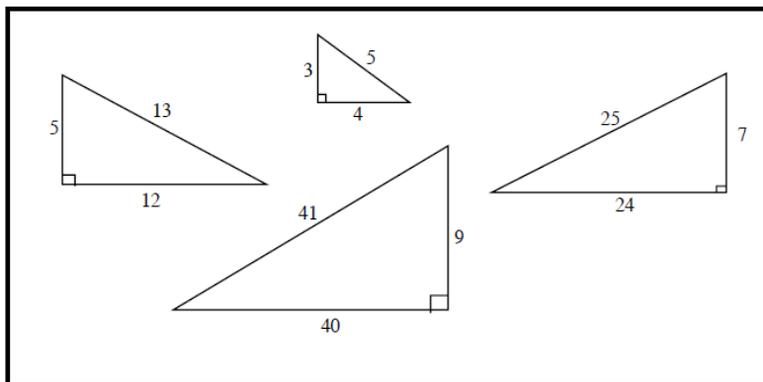
| Pattern number, n | 1 | 2 | 3 | 4 |
|------------------------|----|-----------|------------|------------|
| Number of white blocks | 12 | 40 | 84 | 144 |
| Number of gray blocks | 13 | 41 | 85 | 145 |
| Total number of blocks | 25 | 81 | 169 | 289 |

And so on. As Frank Quinn, [Q], points out in his opening sentence:

The reform approach to K-12 math is counterproductive as preparation for high-tech careers, even for high-achieving students. It should be considered a terminal track for students interested in such careers.

The eighth problem in CCR C2 Part II, **Pythagorean Triples**, also has severe difficulties, though these are of a different nature from what we've seen so far.

Pythagorean Triples



$(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$ and $(9, 40, 41)$ are called Pythagorean Triples because they satisfy the condition

$$c^2 = a^2 + b^2$$

1. Investigate the relationships between the lengths of the sides of triangles that belong to this set.

First, they are called Pythagorean or Euclidean triples because they satisfy the equation $c^2 = a^2 + b^2$ and all three are whole numbers. It is a known but deep result that all Pythagorean triples arise by choosing three whole numbers r , n and m with $n > m$, and taking the three resulting integers $c = r(n^2 + m^2)$, $a = r(n^2 - m^2)$, and $b = 2rnm$ if exactly one of n , m is odd, or $c = r\frac{(n^2+m^2)}{2}$, $a = r\frac{(n^2-m^2)}{2}$, and $b = rnm$ if both n , m are odd.

Now, it is pretty much absurd to assume that high school students would be able to reproduce this result from the paltry set of examples provided. However, one could note that in the examples the hypotenuse is always 1 more than one of the legs, and it is the case that $(n + 1)^2 - n^2 = 2n + 1$.

Thus, in this very special case, the student would only have to understand that if n is chosen so that $2n + 1$ is a square, then the result will be a Pythagorean triple. But even this is extremely sophisticated for high school students. So when we see, as part of the rubric, that for the third question, “Investigate rules for finding the perimeter and area of triangles that belong to this set when you know the length of the shortest side,” this is

clearly beyond reasonable expectation, especially when the rubric gives⁽⁵⁾

Searches for patterns.

Makes generalizations such as:

When $a = n$, $b = 1/2(n^2 - 1)$, $c = 1/2(n^2 + 1)$

Makes generalizations such as:

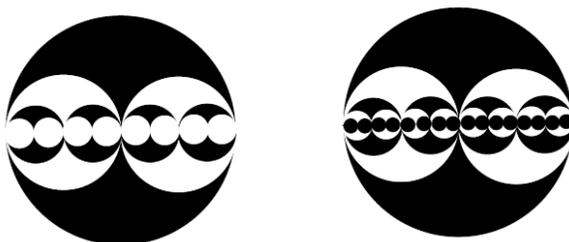
i The perimeter is $n^2 + n$

ii The area is $(1/4)n(n^2 - 1)$

⁽⁵⁾ Even here the authors confuse things as one sees by simply assuming n is even in the first case, which leads to b and c not being integers. But they have assumed the universe for this problem is Pythagorean triples!

The ninth problem in CCR C2 Part II is **Circle Pattern**. As the third question in this problem “Fill in the table to show what happens as the pattern continues” shows, we are once again dealing with extending a sort of pattern. But for this part there is no real issue because students are expected to only fill in the table for the figures where the details have already been given:

3. Fill in the table to show what happens as the pattern continues.



| Pattern | Black fraction | White fraction |
|-----------------------|----------------|----------------|
| One black circle | 1 | 0 |
| Two white circles | $\frac{1}{2}$ | $\frac{1}{2}$ |
| Four black circles | | |
| Eight white circles | | |
| Sixteen black circles | | |

However, the next part exactly repeats the difficulties that were discussed for the problem “Sidewalk Patterns:”

Write a description of what is happening to the black and white fractions as the pattern continues.

But this is slightly better since it seems to allow for the possibility that the test taker could select another method of continuing the construction aside from the “expected one” where successively 2^n black circles, each of radius one half the radius of the previous white circles, are placed with disjoint interiors, in the last set of white circles, two per circle, if n is even, while 2^n white circles are placed in the last set of black circles with similar constraints if n is odd.⁽⁶⁾

Of course, when we check the rubric, no credit is given for any of the results unless the students make the same hidden assumptions about the radii of the successive circles and their positions in the previous circles that the authors do.

⁽⁶⁾ Here I am ignoring the fact that these are hidden assumptions for all the reasons discussed earlier. The fact that the radii are successively multiplied by $\frac{1}{2}$ while the number of circles that are added at stage $n + 1$ is always twice the number at stage n , while the colors alternate from stage n to $n + 1$ is hidden and cannot be deduced from the pictures alone. Hence, as usual with this exam, none of the questions for this situation can be answered from just the data given.

Moreover, assuming that the question was sufficiently well posed that students understood that the continuation desired was this one, there is still a huge hurdle to overcome. Aside from the fact that the problem is incorrect as stated, if it were stated correctly it would still be too difficult for typical students to have any realistic chance of solving it. What students need to understand is that the formula for the area of the black regions at the n^{th} step is

$$A_{\text{Black}} = \pi r^2 \frac{1 - \left(\frac{-1}{2}\right)^{n+1}}{\frac{3}{2}}$$

while the area of the white regions at the n^{th} step is $\pi r^2 - A_{\text{Black}}$. Such formulae will not have been covered in high school unless the students have had the most advanced math course the high schools provide, and even if students have seen the “compound interest” formula for the partial sums of the geometric series $-\frac{1-x^{n+1}}{1-x}$ – they typically will not have seen nor been told that this also holds for *negative* x . Indeed, the standard A-SSE.4

Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*

is the only one in Common Core that relates to these issues, and it strongly emphasizes the case $0 \leq x, x \neq 1$.

The last problem in CCR C2 Part II, **Fearless Frames**, looks more innocent and correct than problems 4 - 9. It is:

Fearless Frames

Fearless Frames Inc makes metal frames for containers.

1 A client asks Fearless Frames to make a large container which is a rectangular prism with a square cross section.

The company has only 60 meters of suitable metal tubing in stock.

Find the dimensions of the container which holds the maximum volume the company can make using 60 meters of tubing.

Show how you figured it out.

But note that it asks for a “container” and only talks about “tubing.” So what is really being asked for are the dimensions of a *frame* for such a box under the constraints that the sum of the lengths of the 12 edges ≤ 60 and the volume is maximal for this constraint. But this is again a calculus problem, and hardly appropriate for virtually all high school students - at least in this country.

The volume is x^2y , and $8x+4y \leq 60$. Since we are maximizing volume, with respect to this constraint, it is clear that $8x+4y = 60$, so $2x+y = 15$ and $y = 15-2x$. It follows that we have $V = x^2(15-2x)$, a cubic polynomial, and we want to know the maximum value this polynomial takes in the range $0 \leq x \leq 7.5$. The derivative is $-6x^2 + 30x = 6x(5-x)$ with roots at 0 and 5. “Clearly,” (such things are never clear to tenth or even eleventh graders), for $x < 0$, the polynomial approaches $+\infty$ as $x \rightarrow -\infty$, and for $x > 5$, the polynomial approaches $-\infty$ as $x \rightarrow \infty$. Thus 0 is a local minimum and 5 is a local maximum.

But, as noted earlier in the analysis of **Cubic Graph** and **Strawberry Boxes**, the expected methods that students are supposed to use, as given in the scoring rubric are

- The graph of V against x shows that as x increases from 1 to 5 the volume increases, and then decreases for values of x from 5 to 7. V is max when $x = 5$.
- Alternatively May make a list showing the values $x = 4$ and volume 112 $x = 6$ and volume 108 When $x = 5$, $y = 5$ and $V = 125$
- States that for $P = 60$ meters, the maximum volume is 125 cubic meters.

So the same errors that were present in the discussion of the previous problems involving cubics are present here. Students are asked to find the maximum, but, as the rubric shows, they need to use the hidden assumption that whenever a question of this nature appears,

x is a whole number, so the answer can be found by checking whole number values, or looking at a graph. These are not lessons that anyone would like to see in the students we are training to become engineers or architects among other critical professions.

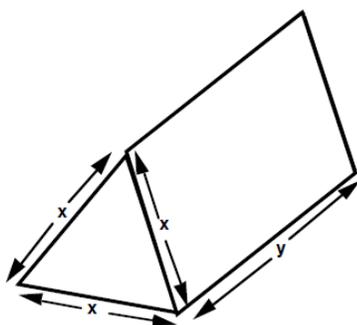
To further clarify this point, let us look at the second part of the **Fearless Frames** question:

Fearless Frames: (continued)

2. The client changes his mind!

He asks for a container that is a prism with a cross-section which is an equilateral triangle.

Investigate the maximum volume of the container that can be made using 60 meters of tubing for the frame.



and its scoring rubric:

- Shows that the height of the equilateral triangle is $\sqrt{3}x/2$.
- The volume of the prism (V) = $\sqrt{3}x^2y/4$,
- The perimeter of the prism (P) = $6x + 3y = 60$, so $y = 20 - 2x$.
- $V = \sqrt{3}x^2(20 - 2x)/4$
- V is maximum when $x = y = 6\frac{2}{3}$ (accept values 67) For perimeter 60 meters, the maximum volume is 128 cubic meters. Accept values 124 - 128

Note that all the same problems are present, but this time we finally get a situation where the maximum occurs at an irrational number. This would surely cause total confusion, but look at how the test authors try to get around it by again allowing students to just check whole number values of x . Unfortunately, even here the authors have problems. A calculator check shows they obtained the wrong value for the maximum. The exact value for the maximum volume is the irrational number $\frac{2000}{27}\sqrt{3} \sim 128.30005982$ which is not a whole number. Indeed, it is actually irrational since $\sqrt{3}$ is irrational and $\frac{2000}{27}$ is rational by definition. Moreover, Core Standards even has a standard to this effect, N-RN.3.

Finally, let us look at the third part of **Fearless Frames**. It is

3. What advice do you think Fearless Frames should offer to this customer? Show all your calculations.

This is phrased as asking for an opinion, not a fact, nor the result of a specific calculation. How can one grade an opinion as right or wrong? I can see grading on the basis of “dumb” or “not so dumb,” but not right or wrong. Yet the rubric is

Advise the customer that, using 60 meters of tubing, a container with a cross section which is an equilateral triangle holds a little more than one which is a square.

with one point given for this response, and no points for other responses.

Bibliography

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- [M] R. James Milgram, *The Mathematics Pre-Service Teachers Need to Know*, 2005, URL <ftp://math.stanford.edu/pub/papers/milgram/FIE-book.pdf>
- [Q] Quinn, F. The reform curriculum is counterproductive for high-tech careers.