1) Let \( f(x) \) be a smooth \( 2\pi \)-periodic function. Recall that the \( N \)th partial Fourier sum is the function

\[
S_N(f; x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ikx}.
\]

As explained briefly in class, this can be represented as a convolution,

\[
S_N(f; x) = D_N f(x),
\]

for the explicit \( 2\pi \)-periodic function \( D_N(x) = \frac{1}{2\pi} \sin((N + \frac{1}{2})x)/\sin(x/2) \).

(This is discussed in §7.4.2 in the book.) Explain carefully why this is true.

Next, define

\[
F_N(f; x) = \frac{S_0(f; x) + \ldots + S_N(f; x)}{N + 1}.
\]

This too can be represented in the form \( C_N f(x) \). Show how to derive the explicit expression for \( C_N \), and in particular, show that it is a nonnegative function (this turns out to be very important in applications).

Hint: Write out the integral defining \( \hat{f}(k) \) for each \( k \) and insert it into the equation representing \( f \) in terms of its Fourier series. For \( D_N \), you just need to sum a finite geometric series. For the second you’ll have to work a bit harder, but the idea ultimately is the same, to find an explicit sum of a finite series.

2) Modify the Shannon-Whittaker interpolation formula to show how to reconstruct the \( L \)-bandlimited function \( f(x) \) from the samples \( f(x_0 + n\pi/L), n \in \mathbb{Z} \).

3) Suppose that \( f \) is \( L \)-periodic (and smooth) and \( N \)-bandlimited (i.e. its Fourier coefficients \( \hat{f}(k) \) vanish when \( |k| \geq N \)). Let \( x_1, \ldots, x_{2N-1} \) be an arbitrary set of distinct points in the interval \([0, L] \). Prove whether it is possible to reconstruct the function \( f(x) \) from the samples \( f(x_j), j = 1, \ldots, 2N - 1 \). If this is possible, discuss in what sense this representation is better when the points are equally spaced? (For this, you should review the version of the Nyquist theorem for periodic functions, Theorem 8.3.1.)