MATH 171: MIDTERM EXAMINATION, SOLUTIONS.

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1. (30 points) (a) State the definition of a countable set. Is the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ countable or uncountable? Explain your answer.

Solution. A set is called countable if it is finite, or if there is a bijection between it and the set of natural numbers. Recall that the set of rational numbers is countable. Hence, for any given a, the set $\{a + b\sqrt{2} : b \in \mathbb{Q}\}$ is also countable. The set $\mathbb{Q}(\sqrt{2})$ is the union over all rational numbers a of the countable set $\{a + b\sqrt{2} : b \in \mathbb{Q}\}$. Since the countable union of countable sets is countable we conclude that $\mathbb{Q}(\sqrt{2})$ is countable.

(b). State the definition of a Cauchy sequence. Let a_n be a Cauchy sequence. Is the sequence $b_n = a_n(1 + 1/\sqrt{n})$ also a Cauchy sequence? Give a proof or a counterexample.

Solution. A sequence a_n Is called Cauchy if for every $\epsilon > 0$ there exists a natural number N such that $|a_m - a_n| < \epsilon$ for all $m,\ n > N$. We established in class that a sequence is Cauchy if and only if it is convergent. If a_n is a convergent sequence, then since $1+1/\sqrt{n}$ is also a convergent sequence and the product of two convergent sequences is convergent, we conclude that b_n is also a convergent sequence.

(c). State the Bolzano-Weierstrass theorem. Let a_n be a bounded sequence of real numbers. Is it true that there is a subsequence a_{n_k} $(k \in \mathbb{N})$ which is convergent and such that for every k, n_k is a perfect square? Explain your answer.

Solution. The Bolzano-Weierstrass theorem asserts that every bounded sequence has a convergent subsequence. Given the bounded sequence a_n consider the subsequence a_{n^2}. This too is a bounded sequence of real numbers, and hence by Bolzano-Weierstrass there is a convergent subsequence of a_{n^2}; that is there is a convergent sequence a_{n_k} where each n_k is a perfect square.

2. (20 points) Let n be a natural number. A *binary expansion* of n is an expression of the form $n = \sum_{j=0}^{k} a_j 2^j$ with each $a_j = 0$ or 1 and $a_k = 1$.

(a). Prove that every natural number has a unique binary expansion.

Solution. We prove the existence of a binary expansion by complete induction on n. The case $n = 1$ is settled by writing $1 = 1 \cdot 2^0$. Suppose the result has been established for all natural numbers up to n, and we now want to establish it for n. Let 2^k be the largest power of two not exceeding n. Thus $2^k \le n < 2^{k+1}$. We then write $n = 2^k + \ell$. If $\ell = 0$ then $n = 2^k$ is a binary expansion of n. If $\ell \neq 0$ then $\ell < 2^{k+1} - 2^k = 2^k$ so that $\ell < n$ and by induction hypothesis ℓ has a binary expansion $\ell = \sum_{j=0}^{r} a_j 2^j$ with $a_j = 0$ or 1, $a_r = 1$ and $r < k$. Now $n = 2^k + \sum_{j=0}^{r} a_j 2^j$ is a binary expansion for n.

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The uniqueness of a binary expansion may also be established by induction. Note that if $n = \sum_{j=0}^{k} a_j 2^j$ with $a_k = 1$ and $a_j = 0$ or 1, we have $2^k \le n \le 2^k + 2^{k-1} + \ldots + 1 = 2^{k+1} - 1$. Thus k is uniquely determined as the largest power of 2 not exceeding n. Then using, by induction hypothesis, the uniqueness of the binary expansion for $n - 2^k$ we conclude that the binary expansion is unique.

(b). Given a natural number n and its binary expansion $n = \sum_{j=0}^{k} a_j 2^j$ as above, we define $b(n) = k+1$. That is $b(n)$ denotes the number of binary digits of n. Determine, with proof, whether the following series converge or diverge:

$$(i). \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{b(n)},$$

$$(ii). \qquad \sum_{n=1}^{\infty} \frac{1}{nb(n)},$$

$$(iii). \qquad \sum_{n=1}^{\infty} \frac{1}{nb(n)^2}.$$

Solution. (i). Since $1/b(n)$ is a monotone decreasing sequence, the alternating series test gives that $\sum_{n=1}^{\infty} (-1)^n/b(n)$ is convergent.

(ii). We use the 2^n test. Note that $b(2^n) = (n+1)$, and therefore

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^n b(2^n)} = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

is divergent, as is $\sum_{n=1}^{\infty} 1/(nb(n))$.

(iii). Again we use the 2^n test. Here we obtain that

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^n b(2^n)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

which converges and therefore so does the series $\sum_{n=1}^{\infty} 1/(nb(n)^2)$.

3. (20 points). Let a_n be a sequence of non-negative real numbers and suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n}$ also diverges.
Solution. Suppose instead that the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges. Then there exists a natural number N with $\sqrt{a_n} < 1$ for all $n > N$. It follows that $a_n < \sqrt{a_n} < 1$ for all $n > N$, and therefore $\sum_{n=N+1}^{\infty} a_n$ converges. Hence adding the finitely many terms $a_1, \ldots, a_N$ to this sum we obtain that $\sum_{n=1}^{\infty} a_n$ converges which is a contradiction.

4. (30 points) Let, as in problem 2, $b(n)$ denote the number of binary digits of the natural number n. Define $a(n) = (-1)^{b(n)}$ and

$$A(N) = \frac{1}{N} \sum_{n=1}^{N} a(n).$$

(a). For what values of N is $A(N) \le A(N+1)$, and for what values of N is $A(N) \ge A(N+1)$?

Solution. Many of you found this difficult, and in general in solving problems it is useful to work out the first few terms to get an idea of what the answer should be. Here are the first few values of $a(n)$ and $A(n)$:

$$a(1) = -1,\ a(2) = 1,\ a(3) = 1,\ a(4) = -1,\ a(5) = -1,\ a(6) = -1,\ a(7) = -1,$$
$$a(8) = 1,\ a(9) = 1,\ a(10) = 1,\ a(11) = 1,\ a(12) = 1,\ a(13) = 1,\ a(14) = 1, \ldots.$$

$$A(1) = -1,\ A(2) = 0,\ A(3) = \frac{1}{3},\ A(4) = 0,\ A(5) = -\frac{1}{5},\ A(6) = -\frac{1}{3},\ A(7) = -\frac{3}{7},$$
$$A(8) = -\frac{1}{4},\ A(9) = -\frac{1}{9},\ A(10) = 0,\ A(11) = \frac{1}{11},\ A(12) = \frac{1}{6},\ A(13) = \frac{3}{13},\ A(14) = \frac{2}{7} \ldots.$$

You can see from this that $A(N)$ increases for N from 1 to 3, and decreases for N from 3 to 7, and seems to increase from 7 to 14 (and keeps increasing until 15) and so on. Also the values of $A(n)$ seem to get as high as $1/3$, and as low as a little below $-1/3$. Now for the proofs.

The condition $A(N) \leq A(N+1)$ is equivalent to

$$(N+1) \sum_{n=1}^{N} a(n) \leq N \sum_{n=1}^{N+1} a(n),$$

and this simplifies to the condition

$$\sum_{n=1}^{N} a(n) \leq N a(N+1).$$

If $a(N+1) = 1$ then the condition above holds, and if $a(N+1) = -1$ then we have the opposite implication that $A(N) \geq A(N+1)$. Thus we find that $A(N) \leq A(N+1)$ if $N+1$ has an even number of binary digits, and $A(N) \geq A(N+1)$ if $N+1$ has an odd number of binary digits. In other words, if $k \geq 1$ is an integer then $A(N) \leq A(N+1)$ for $2^{2k-1} \leq (N+1) \leq 2^{2k} - 1$, and $A(N) \geq A(N+1)$ for $2^{2k} \leq N+1 \leq 2^{2k+1} - 1$.

(b). Determine $\limsup_{N\to\infty} A(N)$ and $\liminf_{N\to\infty} A(N)$.

Solution. For an integer $k \geq 1$ we know from our work for the first part that $A(2^{2k-1} - 1) \leq A(2^{2k}) \leq \ldots \leq A(2^{2k} - 1)$, and that $A(2^{2k} - 1) \geq A(2^{2k}) \geq \ldots \geq A(2^{2k+1} - 1)$. Now note that

$$\begin{aligned} A(2^{2k} - 1) &= \frac{1}{2^{2k} - 1} \Big(-1 + (4-2) - (8-4) + \ldots + (2^{2k} - 2^{2k-1}) \Big) \\ &= \frac{1}{2^{2k} - 1} \Big(2^{2k-1} - 2^{2k-2} + \ldots + 2 - 1 \Big) = \frac{1}{3}, \end{aligned}$$

and

$$\begin{aligned} A(2^{2k+1} - 1) &= \frac{1}{2^{2k+1} - 1} \Big(-1 + (4-2) + \ldots - (2^{2k+1} - 2^{2k}) \Big) \\ &= \frac{1}{2^{2k+1} - 1} \Big(-2^{2k} + 2^{2k-1} + \ldots + 2 - 1 \Big) = \frac{1}{2^{2k+1} - 1} \Big(-\frac{2^{2k+1} + 1}{3} \Big). \end{aligned}$$

Thus $A(2^{2k} - 1) = 1/3$ and $A(2^{2k+1} - 1)$ tends to $-1/3$ as $k \to \infty$, and other values of $A(n)$ fluctuate between these values (write this up carefully for your benefit). Therefore $\limsup_{N\to\infty} A(N) = 1/3$ and $\liminf_{N\to\infty} A(N) = -1/3$.