

MATH 171: MIDTERM EXAMINATION

APRIL 22, 2010

This is a closed book, closed notes test. You are free to use results proved in class, but you must state clearly the result that you are using. Complete proofs are expected for all questions, and you are also expected to write in a clear style with complete, grammatical sentences. There will be partial credit, so please include approaches to the problem, and/or heuristic reasons for what the result might be or why it might be true. All the best!

1. (30 points) (a) State the definition of a countable set. Is the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ countable or uncountable? Explain your answer.

(b). State the definition of a Cauchy sequence. Let a_n be a Cauchy sequence. Is the sequence $b_n = a_n(1 + 1/\sqrt{n})$ also a Cauchy sequence? Give a proof or a counterexample.

(c). State the Bolzano-Weierstrass theorem. Let a_n be a bounded sequence of real numbers. Is it true that there is a subsequence a_{n_k} ($k \in \mathbb{N}$) which is convergent and such that for every k , n_k is a perfect square? Explain your answer.

2. (20 points) Let n be a natural number. A *binary expansion* of n is an expression of the form $n = \sum_{j=0}^k a_j 2^j$ with each $a_j = 0$ or 1 and $a_k = 1$.

(a). Prove that every natural number has a unique binary expansion.

(b). Given a natural number n and its binary expansion $n = \sum_{j=0}^k a_j 2^j$ as above, we define $b(n) = k + 1$. That is $b(n)$ denotes the number of binary digits of n . Determine, with proof, whether the following series converge or diverge:

$$(i). \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{b(n)},$$

$$(ii). \quad \sum_{n=1}^{\infty} \frac{1}{nb(n)},$$

$$(iii). \quad \sum_{n=1}^{\infty} \frac{1}{nb(n)^2}.$$

3. (20 points). Let a_n be a sequence of non-negative real numbers and suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n}$ also diverges.

4. (30 points) Let, as in problem 2, $b(n)$ denote the number of binary digits of the natural number n . Define $a(n) = (-1)^{b(n)}$ and

$$A(N) = \frac{1}{N} \sum_{n=1}^N a(n).$$

(a). For what values of N is $A(N) \leq A(N + 1)$, and for what values of N is $A(N) \geq A(N + 1)$?

(b). Determine $\limsup_{N \rightarrow \infty} A(N)$ and $\liminf_{N \rightarrow \infty} A(N)$.