Homework 5 Solutions

Math 171, Spring 2010
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29.15. Let $a_1 = 1$ and $a_n = -1/2^{n-1}$ for $n \geq 2$. Let $\{b_{m,n}\}$ be the double sequence whose matrix is

$$\begin{pmatrix} a_1 & a_1 & a_1 & a_1 & \dots \\ 0 & a_2 & a_2 & a_2 & \dots \\ 0 & 0 & a_3 & a_3 & \dots \\ 0 & 0 & 0 & a_4 & \dots \end{pmatrix}$$

Show that the sum by columns of the series $\sum_{m,n} b_{m,n}$ exists, but that the sum by rows does not exist.

Solution. Since $\sum_{m=1}^{\infty} b_{m,n} = \sum_{m=1}^{n} a_m = 1 - \frac{1}{2} - \cdots - \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}}$ converges, and since $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} b_{m,n}) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1-\frac{1}{2}} = 2$ converges, the sum by columns of the series $\sum_{m,n} b_{m,n}$ exists. Since $\sum_{n=1}^{\infty} b_{1,n} = \sum_{n=1}^{\infty} 1$ diverges, the sum by rows does not exist.

35.3. Let d be a metric on a set M. Prove that $|d(x,z) - d(y,z)| \leq d(x,y)$ for all $x, y, z \in M$.

Solution. By the triangle inequality, we have $d(x,z) \leq d(x,y) + d(y,z)$, hence $d(x,z) - d(y,z) \leq d(x,y)$. Also by the triangle inequality, we have $d(y,z) \leq d(y,x) + d(x,z)$, hence $-d(x,y) = -d(y,x) \leq d(x,z) - d(y,z)$. Together, these inequalities show $|d(x,z) - d(y,z)| \leq d(x,y)$ for all $x, y, z \in M$.

35.7. Let (M, d) be a metric space. Prove that

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)} \qquad d''(x,y) = \min\{d(x,y), 1\}$$

define metrics on M. Prove that d' and d'' are bounded by 1.

Solution. Since d maps into $[0, \infty)$ and satisfies (i) and (ii) of Definition 35.1, it is clear that d' and d'' map into $[0, \infty)$ and satisfy (i) and (ii).

Note the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{x}{1+x}$ is increasing as $f'(x) = \frac{1}{(1+x)^2} > 0$. Therefore since $d(x,z) \leq d(x,y) + d(y,z)$, we have

$$\begin{aligned} d'(x,z) &= \frac{d(x,z)}{1 + d(x,z)} \\ &\leq \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} \\ &= \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)} \\ &\leq \frac{d(x,y)}{1 + d(x,y)} + \frac{d(y,z)}{1 + d(y,z)} \\ &= d'(x,y) + d'(y,z) \end{aligned}$$

This gives the triangle inequality for d'.

Note

$$d''(x,z) = \min\{d(x,z), 1\} \leq \min\{d(x,y) + d(y,z), 1\} \leq \min\{d(x,y), 1\} + \min\{d(y,z), 1\} = d''(x,y) + d''(y,z)$$

This gives the triangle inequality for d'.

Hence d' and d'' define metrics on M. It is clear that d' and d'' are bounded by 1.

35.8. Let (M_1, d_1) and (M_2, d_2) be metric spaces. Prove that $(M_1 \times M_2, d)$ is a metric space, where d is defined by the formula

$$d[(x_1, x_1), (y_1, y_2)] = d_1(x_1, y_1) + d_2(x_2, y_2)$$

The space $(M_1 \times M_2, d)$ is called the product metric space.

Solution. Since d_1 and d_2 map into $[0, \infty)$ and satisfy (i) and (ii) of Definition 35.1, it is clear that d maps into $[0, \infty)$ and satisfies (i) and (ii). To see the triangle inequality, note

$$\begin{aligned}
d[(x_1, x_2), (z_1, z_2)] &= d_1(x_1, z_1) + d_2(x_2, z_2) \\
&\leq d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) \\
&= d_1(x_1, y_1) + d_2(x_2, y_2) + d_1(y_1, z_1) + d_2(y_2, z_2) \\
&= d[(x_1, x_2), (y_1, y_2)] + d[(y_1, y_2), (z_1, z_2)]
\end{aligned}$$

Hence $(M_1 \times M_2, d)$ is a metric space.

35.9. Let H^∞ denote the set of all real sequences $\{a_n\}$ such that $|a_n| \leq 1$ for every positive integer n. H^∞ is called the *Hilbert cube.*

(a) Let $\{a_n\}$, $\{b_n\} \in H^\infty$. Prove that the series $\sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$ converges.

Solution. Since $\frac{|a_n - b_n|}{2^n} \leq \frac{2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{2}{2^n}$ converges, we know that $\sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$ converges by Theorem 26.3 (Comparison Test).

(b) Prove that $d(\{a_n\}, \{b_n\}) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$ defines a metric on H^∞.

Solution. It is clear that d maps into $[0, \infty)$ and satisfies (i) and (ii) of Definition 35.1. To see the triangle inequality, note

$$\begin{aligned}
d(\{a_n\}, \{c_n\}) &= \sum_{n=1}^{\infty} \frac{|a_n - c_n|}{2^n} \\
&\leq \sum_{n=1}^{\infty} \frac{|a_n - b_n| + |b_n - c_n|}{2^n} \\
&= \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n} + \sum_{n=1}^{\infty} \frac{|b_n - c_n|}{2^n} \\
&= d(\{a_n\}, \{b_n\}) + d(\{b_n\}, \{c_n\})
\end{aligned}$$

Hence d defines a metric on H^∞.

36.4. Prove that if $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n/n$ converges absolutely.

Solution. Since $\sum_{n=1}^{\infty} a_n^2$ converges, $\{a_n\} \in l^2$. Since $\sum_{n=1}^{\infty} (1/n)^2$ converges, $\{1/n\} \in l^2$. Hence by Theorem 36.3, $\sum_{n=1}^{\infty} a_n/n$ converges absolutely.

36.8. Let $\{a_n\} \in l^1$ and $\{b_n\} \in l^\infty$. Prove that $\{a_n b_n\} \in l^1$.

Solution. This is just a restatement of Theorem 26.4(i). Since $\{a_n\} \in l^1$, we know $\sum_{n=1}^{\infty} a_n$ converges absolutely. Since $\{b_n\} \in l^\infty$, we know $\{b_n\}$ is bounded. By Theorem 26.4(i), $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely, and hence $\{a_n b_n\} \in l^1$.

36.11. Let $\{a_n\}$ be a sequence such that $\{a_nb_n\} \in l^1$ for every sequence $\{b_n\} \in l^1$. Prove that $\{a_n\} \in l^\infty$. Show (by example) that the above statement is false if l^∞ is replaced by c_0.

Solution. Suppose for a contradiction that $\{a_n\} \notin l^\infty$, that is, $\{a_n\}$ is not bounded. So for all positive integers k and N there exists some $n \geq N$ such that $|a_n| > k$. Let N_1 be such that $a_{N_1} > 1$. Let $N_2 > N_1$ be such that $a_{N_2} > 2$. Continuing as such, let $N_k > N_{k-1}$ be such that $a_{N_k} > k$. Define the sequence $\{b_n\}$ by

$$b_n = \begin{cases} \frac{1}{k^2} & \text{if } n = N_k \\ 0 & \text{otherwise} \end{cases}$$

Note that $\sum_{n=1}^{\infty} |b_n| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\{b_n\} \in l^1$. However, note that

$$|a_nb_n| = \begin{cases} \frac{|a_n|}{k^2} \geq \frac{1}{k} & \text{if } n = N_k \\ 0 & \text{otherwise} \end{cases}$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ diverges, this shows that $\sum_{n=1}^{\infty} |a_nb_n|$ diverges by Theorem 26.3 (Comparison Test), and so $\{a_nb_n\} \notin l^1$. This is a contradiction, and so it must be the case that $\{a_n\} \in l^\infty$.

To see that the statement is false if l^∞ is replaced by c_0, let $a_n = 1$ for all n. Then $\{a_n\}$ is a sequence such that $\{a_nb_n\} \in l^1$ for every sequence $\{b_n\} \in l^1$, but $\{a_n\} \notin c_0$ as the terms do not converge to 0.

37.4. Show (by examples) that Theorem 37.2 does not generalize to l^2, c_0, or l^∞.

Solution. Let $\{\delta^{(k)}\}_{k=1}^{\infty}$ be defined by $\delta_n^{(k)} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$. Let $a_n = 0$ for every positive integer n and let $a = \{a_n\}$. It is clear that $\lim_{k\to\infty} \delta_n^{(k)} = a_n$ for every positive integer n.

Note $\{\delta^{(k)}\}_{k=1}^{\infty}$ is a sequence of points in l^2, but $\{\delta^{(k)}\}$ does not converge to a in l^2 since $d_{l^2}(\delta^{(k)}, a) = \sqrt{1^2} = 1$ for every positive integer k.

Note $\{\delta^{(k)}\}_{k=1}^{\infty}$ is a sequence of points in l^∞, but $\{\delta^{(k)}\}$ does not converge to a in l^∞ since $d_{l^\infty}(\delta^{(k)}, a) = 1$ for every positive integer k.

Note $\{\delta^{(k)}\}_{k=1}^{\infty}$ is a sequence of points in c_0, but $\{\delta^{(k)}\}$ does not converge to a in c_0 since $d_{c_0}(\delta^{(k)}, a) = 1$ for every positive integer k.

Hence Theorem 37.2 does not generalize to l^2, c_0, or l^∞.

37.9. (a) Prove that the equation $d'(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ defines a metric on $\mathbb{R}^n$.

Solution. It is clear that d maps into $[0, \infty)$ and satisfies (i) and (ii) of Definition 35.1. To see the triangle inequality, note that

$$d'(x,z) = \sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i| = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d'(x,y) + d'(y,z)$$

Hence d' defines a metric on $\mathbb{R}^n$.

(b) Let $\{a^{(k)}\}$ be a sequence of points in $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$. Prove that $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d)$ if and only if $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d')$, where d is the usual metric for $\mathbb{R}^n$.

Solution. By Theorem 37.2, it suffices to show that $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d')$ if and only if $\lim_{k\to\infty} a_j^{(k)} = a_j$ for $j = 1, \ldots, n$. We mimic the proof of Theorem 37.2.

First suppose that $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d')$. Let $\epsilon > 0$. There exists a positive integer N such that if $k \geq N$, then $d'(a^{(k)}, a) < \epsilon$. Let j be a positive integer such that $1 \leq j \leq n$. If $k \geq N$, then

$$|a_j^{(k)} - a_j| \le \sum_{i=1}^{n} |a_i^{(k)} - a_i| = d'(a^{(k)}, a) < \epsilon$$

and thus $\lim_{k\to\infty} a_j^{(k)} = a_j$ for $j = 1, \ldots, n$.

Now suppose that $\lim_{k\to\infty} a_j^{(k)} = a_j$ for $j = 1, \ldots, n$. Let $\epsilon > 0$. For each j there exists a positive integer N_j such that if $k \ge N_j$, then $|a_j^{(k)} - a_j| < \epsilon/n$. If $k \ge \max\{N_1, \ldots, N_n\}$, then

$$d'(a^{(k)}, a) = \sum_{i=1}^{n} |a_i^{(k)} - a_i| < \sum_{i=1}^{n} \epsilon/n = \epsilon$$

and thus $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d')$.

37.10. Let (M, d) be a metric space, and let d' and d'' be defined as in Exercise 35.7. Let $\{a_n\}$ be a sequence in M and let $a \in M$. Prove that the following statements are equivalent.
(a) $\{a_n\}$ converges to a in (M, d).
(b) $\{a_n\}$ converges to a in (M, d').
(c) $\{a_n\}$ converges to a in (M, d'').

Solution. Let "iff" mean "if and only if".

To see (a) is equivalent to (b), note that $\{a_n\}$ converges to a in (M, d) iff $\lim_{n\to\infty} d(a_n, a) = 0$, iff $\lim_{n\to\infty} d'(a_n, a) = \lim_{n\to\infty} \frac{d(a_n,a)}{1+d(a_n,a)} = 0$, iff $\{a_n\}$ converges to a in (M, d').

To see (a) is equivalent to (c), note that $\{a_n\}$ converges to a in (M, d) iff $\lim_{n\to\infty} d(a_n, a) = 0$, iff $\lim_{n\to\infty} d''(a_n, a) = \lim_{n\to\infty} \min\{d(a_n, a), 1\} = 0$, iff $\{a_n\}$ converges to a in (M, d'').