Homework 4 Solutions

Math 171, Spring 2010
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26.5. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be absolutely convergent series. Prove that the series $\sum_{n=1}^{\infty} \sqrt{|a_n b_n|}$ converges.

Solution. Since $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge, by Theorem 23.1, $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges. Note that for all n, we have

$$\sqrt{|a_n b_n|} = \sqrt{|a_n||b_n|} \leq \sqrt{(|a_n| + |b_n|)(|a_n| + |b_n|)} = |a_n| + |b_n|.$$

So by Theorem 26.3 (Comparison Test), since $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges absolutely, we see that $\sum_{n=1}^{\infty} \sqrt{|a_n b_n|}$ converges absolutely, and hence converges as all terms are positive.

27.1. Find the radius of convergence R of each of the power series. Discuss the convergence of the power series at the points $|x - t| = R$.

Solution.
(a) $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$
Note that

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)-1}}{(2(n+1)-1)!}}{\frac{x^{2n-1}}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+1)(2n)} \right| = 0$$

so by the ratio test, the series converges absolutely for all x, and so $R = \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n}$
Note that

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)(x-1)^{(n+1)}}{2^{(n+1)}}}{\frac{n(x-1)^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{x-1}{2} \right| = \left| \frac{x-1}{2} \right|$$

so by the ratio test, the series converges absolutely if $|x - 1| < 2$ and diverges if $|x - 1| > 2$. Thus $R = 2$. If $|x - 1| = 2$ then the power series diverges, since $\sum_{n=1}^{\infty} n$ and $\sum_{n=1}^{\infty}(-1)^n n$ diverge.

(c)$\sum_{n=1}^{\infty} \frac{n!(x+2)^n}{2^n}$
Note that

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!(x+2)^{(n+1)}}{2^{(n+1)}}}{\frac{n!(x+2)^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+2)}{2} \right| = \begin{cases} 0 & \text{if } x = -2 \\ \infty & \text{otherwise} \end{cases}$$

so by the ratio test, the series diverges if $|x + 2| > 0$. Thus $R = 0$. If $|x + 2| = 0$ then the power series converges by the ratio test, or by the comment after Definition 27.1.

(d) $\sum_{n=1}^{\infty}(n^{1/n} - 1)x^n$
By Theorems 16.7 and 25.1 (Alternating Series Test), the series converges when $x = -1$.
By equality (16.3) in the proof of Theorem 16.7, we know $(1 + \frac{1}{n})^n \leq n$. Taking n-th roots of both sides and subtracting 1, we get $\frac{1}{n} \leq n^{1/n} - 1$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, hence so does $\sum_{n=1}^{\infty} n^{1/n} - 1$. Thus the series diverges when $x = 1$.

Hence by Theorem 27.2, we have $R = 1$, and we have already discussed the behavior of the power series when $|x| = 1$.

(e) $\sum_{n=2}^{\infty} \frac{(x-\pi)^n}{n(n-1)}$
Note that

$$\lim_{n\to\infty} \left| \frac{\frac{(x-\pi)^{(n+1)}}{(n+1)n}}{\frac{(x-\pi)^n}{n(n-1)}} \right| = \lim_{n\to\infty} \left| \frac{n-1}{n+1}(x-\pi) \right| = |x-\pi|$$

so by the ratio test, the series converges absolutely if $|x-\pi| < 1$ and diverges if $|x-\pi| > 1$. Thus $R = 1$. If $|x-\pi| = 1$ then the power series converges, since $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n-1)}$ converge.

27.2. Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-t)^n$ has radius of convergence R. Let p be an integer. Prove that the power series $\sum_{n=0}^{\infty} n^p a_n (x-t)^n$ has radius of convergence R.

Solution. We consider thhree cases.

First case: $0 < R < \infty$. By Theorem 27.2, we have $\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$. Note that $\lim_{n\to\infty} n^{p/n} = (\lim_{n\to\infty} n^{1/n})^p = 1^p = 1$ by Exercise 17.4 and Theorem 16.7. Note also that $\{|a_n|^{1/n}\}$ is a bounded sequence as $\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} < \infty$. Hence Theorem 20.8 applies, and we get

$$\limsup_{n\to\infty} (n^p|a_n|)^{1/n} = \limsup_{n\to\infty} n^{p/n}|a_n|^{1/n} = \lim_{n\to\infty} n^{p/n} \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$$

So $\sum_{n=0}^{\infty} n^p a_n (x-t)^n$ has radius of convergence R.

Second case: $R = \infty$. The method in the case above works, after replacing $\frac{1}{R}$ with zero.

Third case: $R = 0$. Hence by Definition 21.2 the sequence $\{|a_n|^{1/n}\}$ is not bounded above. Let K be a positive number. Since $\lim_{n\to\infty} n^{p/n} = 1$, there exists some N such that $n^{p/n} \geq \frac{1}{2}$ for all $n \geq N$. Since $\{|n^p a_n|^{1/n}\}$ is not bounded above, there exists some $M > N$ such that $|M^p a_M|^{1/M} > 2K$ (for otherwise $\max\{|1^p a_1|^{1/1}, \cdots, |N^p a_N|^{1/N}, 2K\}$ would be an upper bound for $\{|n^p a_n|^{1/n}\}$). Hence $|M^p a_M|^{1/M} = M^{p/M}|M^p a_M|^{1/M} > \frac{1}{2}2K = K$. This shows that $\{|n^p a_n|^{1/n}\}$ is not bounded above, so $\sum_{n=0}^{\infty} n^p a_n(x-t)^n$ has radius of convergence 0.

28.1. Prove that the following series converge conditionally.

Solution.

(a) $1 + \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{6}} + \cdots$

Let $\{a_n\}$ be the sequence $1, 1, -2, 1, 1, -2, \ldots$. Let $b_n = \frac{1}{\sqrt{n}}$. Note that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded and $\{b_n\}$ is a decreasing sequence with limit 0. We apply Corollary 28.3 (Dirichlet's Test) to see that

$$1 + \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{6}} + \cdots = \sum_{n=1}^{\infty} a_n b_n$$

converges.

(b) $\sum_{n=1}^{\infty} (-1)^n n^{(1-n)/n}$

Let $a_n = (-1)^n n^{-1}$ and let $b_n = n^{1/n}$. Note that $\sum_{n=1}^{\infty} a_n$ converges by Theorem 25.1 (Alternating Series Test) and that $\{b_n\}$ is a bounded monotone sequence for n sufficiently large by Theorem 16.7. We apply Corollary 28.6 (Abel's Test) to see that

$$\sum_{n=1}^{\infty} (-1)^n n^{(1-n)/n} = \sum_{n=1}^{\infty} (-1)^n n^{-1} n^{1/n} = \sum_{n=1}^{\infty} a_n b_n$$

converges.

28.3. Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a positive sequence $\{b_n\}$ such that $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} a_n b_n$ diverges.

Solution. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$ and $b_n = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{\sqrt{n}} & n \text{ even} \end{cases}$.

Then $\sum_{n=1}^{\infty} a_n$ converges by Theorem 25.1 (Alternating Series Test) and $\lim_{n\to\infty} b_n = 0$. Note $\sum_{n=1}^{\infty} a_n b_n = \sum_{m=1}^{\infty} \frac{1}{2m}$ diverges by Corollary 24.3.

28.6. Let $\{a_n\}$ be a periodic sequence, $a_n = a_{n+p}$ for all n and for fixed p, for which $\sum_{n=1}^{p} a_n = 0$. Let $\{b_n\}$ be a decreasing sequence with limit 0. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution. Consider the partial sums s_n of the series $\sum_{n=1}^{\infty} a_n$. Since $\{a_n\}$ is periodic and $\sum_{n=1}^{p} a_n = 0$, we see that $s_{n+p} = s_n + \sum_{i=1}^{p} a_{n+i} = s_n$ for all n. Hence $s_n \in \{s_1, ..., s_p\}$ for all n. This shows that the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is bounded. Hence Corollary 28.3 (Dirichlet's Test) applies, and we conclude $\sum_{n=1}^{\infty} a_n b_n$ converges.

29.3. Prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for every x and

$$(\sum_{n=0}^{\infty} \frac{x^n}{n!})(\sum_{n=0}^{\infty} \frac{y^n}{n!}) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$$

Solution. Note that

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$$

so by the ratio test, the series converges absolutely for every x.

By letting $a_n = \frac{x^n}{n!}$ and $b_n = \frac{y^n}{n!}$ in Theorem 29.9, we get that

$$\begin{aligned}
(\sum_{n=0}^{\infty} \frac{x^n}{n!})(\sum_{n=0}^{\infty} \frac{y^n}{n!}) &= (\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) \\
&= \sum_{n=0}^{\infty} c_n \\
&= \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i b_{n-i}) \\
&= \sum_{n=0}^{\infty} (\sum_{i=0}^{n} \frac{x^i}{i!} \frac{y^{n-i}}{(n-i)!}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}) \\
&= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}
\end{aligned}$$

where the last equality is by the binomial theorem.

29.5. Give an example of convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ whose Cauchy product diverges.

Solution.

Let $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{n+1}}$. This series converges by Theorem 25.1 (Alternating Series Test). The n-th term in the Cauchy product is

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n}(-1)^k(-1)^{n-k}\frac{1}{\sqrt{k+1}}\frac{1}{\sqrt{n+k+1}} = (-1)^n \sum_{k=0}^{n}\frac{1}{\sqrt{k+1}}\frac{1}{\sqrt{n-k+1}}$$

Since $\frac{1}{\sqrt{k+1}} \geq \frac{1}{\sqrt{n+1}}$ and likewise $\frac{1}{\sqrt{n-k+1}} \geq \frac{1}{\sqrt{n+1}}$, we have

$$|c_n| \geq \sum_{k=0}^{n}\frac{1}{n+1} = (n+1)\frac{1}{n+1} = 1$$

So $\lim_{n\to\infty} c_n \neq 0$, so the series $\sum_{n=0}^{\infty} c_n$ diverges by Theorem 22.3.