Homework 3 Solutions

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17.4. Let $\{a_n\}$ be a sequence with positive terms such that $\lim_{n\to\infty} a_n = L > 0$. Let x be a real number. Prove that $\lim_{n\to\infty} a_n^x = L^x$.

Solution. Let $\epsilon > 0$. By Theorem 17.4, note that $L < (L^x + \epsilon)^{1/x}$ and $L > (L^x - \epsilon)^{1/x}$. Since $\lim_{n\to\infty} a_n = L$, there exists some N such that $n \geq N$ implies $a_n < (L^x+\epsilon)^{1/x}$ and $a_n > (L^x-\epsilon)^{1/x}$. Hence by Theorem 17.4, for $n \geq N$ we have $a_n^x < L^x+\epsilon$ and $a_n^x > L^x-\epsilon$. This shows $\lim_{n\to\infty} a_n^x = L^x$.

19.3. Let $0 \leq \alpha < 1$, and let f be a function from $\mathbb{R} \to \mathbb{R}$ which satisfies

$$|f(x) - f(y)| \leq \alpha|x - y|$$

for all $x, y, \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$, and let $a_{n+1} = f(a_n)$ for $n = 1, 2, \ldots.$ Prove that $\{a_n\}$ is a Cauchy sequence.

Solution. First we prove by induction on n that $|a_{n+1} - a_n| \leq \alpha^{n-1}|a_2 - a_1|$ for all $n \in \mathbb{N}$. The base case $n = 1$ is obvious. Assuming the formula is true when $n = k$, we show it is true for $n = k + 1$:

$$|a_{k+2} - a_{k+1}| = |f(a_{k+1}) - f(a_k)| \leq \alpha|a_{k+1} - a_k| \leq \alpha\alpha^{k-1}|a_2 - a_1| = \alpha^k|a_2 - a_1|$$

Hence, by induction, this formula is true for all n.
Note that if $|a_2 - a_1| = 0$, then $a_n = a_1$ for all n, and so the sequence is clearly Cauchy. Hence we consider the case when $|a_2 - a_1| \neq 0$. Now, given any $\epsilon > 0$, pick N such that $\alpha^{N-1} < \frac{\epsilon(1-\alpha)}{|a_2-a_1|}$, which we can do because $0 \leq \alpha < 1 \Longrightarrow \lim_{n\to\infty} \alpha^n = 0$. Then, for any $m, n \geq N$, with $m \geq n$, we have

$$\begin{aligned}
|a_m - a_n| &\leq \sum_{i=n}^{m-1} |a_{i+1} - a_i| \quad \text{by the triangle inequality} \\
&\leq \sum_{i=n}^{m-1} \alpha^{i-1}|a_2 - a_1| \quad \text{by our formula above} \\
&= |a_2 - a_1| \sum_{i=n}^{m-1} \alpha^{i-1} \\
&\leq |a_2 - a_1| \sum_{i=N}^{\infty} \alpha^{i-1} \\
&= |a_2 - a_1| \frac{\alpha^{N-1}}{1-\alpha} \quad \text{by the formula for the sum of an infinite geometric series} \\
&< \epsilon \quad \text{by our choice of } N.
\end{aligned}$$

Hence $\{a_n\}$ is a Cauchy sequence.

20.6. Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$, where a_n is ...

Solution.
(a) $\frac{1}{n}$

Since we know $\lim_{n\to\infty} \frac{1}{n} = 0$, we know $\limsup_{n\to\infty} \frac{1}{n} = 0 = \liminf_{n\to\infty} \frac{1}{n}$ by Theorem 20.4.

(b) $(1+\frac{1}{n})^n$
Since we know $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, we know $\limsup_{n\to\infty}(1+\frac{1}{n})^n = e = \liminf_{n\to\infty}(1+\frac{1}{n})^n$ by Theorem 20.4.

(c) $(-1)^n(1-\frac{1}{n})$
Since $-1 \le (1+\frac{1}{n})^n \le 1$ for all n, we have $\liminf_{n\to\infty}(1+\frac{1}{n})^n \ge -1$ and $\limsup_{n\to\infty}(1+\frac{1}{n})^n \le 1$. Since the subsequence $\{a_{2n-1}\}$ has limit -1 and the subsequence $\{a_{2n}\}$ has limit 1, we have $\liminf_{n\to\infty}(1+\frac{1}{n})^n \le -1$ and $\limsup_{n\to\infty}(1+\frac{1}{n})^n \ge 1$. Hence $\liminf_{n\to\infty}(1+\frac{1}{n})^n = -1$ and $\limsup_{n\to\infty}(1+\frac{1}{n})^n = 1$.

20.7. Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ and $\mathcal{L}_a$, where $a_1, a_2, \dots$ is an enumeration of the rational numbers in the closed interval $[0,1]$.

Solution. We show that $\mathcal{L}_a = [0,1]$. It is easy to show $\mathcal{L}_a \subset [0,1]$.
To show $\mathcal{L}_a \supset [0,1]$, let $s \in [0,1]$. First we consider the case $s > 0$. By Theorem, 17.1, there exists an increasing rational sequence $\{r_n\}$ with limit s. As $s > 0$, for n sufficiently large we have $r_n \ge 0$, so we may assume that $r_n \ge 0$ for all n, hence $r_n \in [0,1]$ for all n. By induction on n, we define a sequence $\{b_n\}$ which is a subsequence of both $\{a_n\}$ and $\{r_n\}$. For the base case, set $b_1 = r_1 = a_k$ for some integer k. For the inductive step, suppose we have defined $b_1, \dots, b_n$ and $b_n = r_l = a_k$. Since $a_1, a_2, \dots$ is an enumeration of the rational numbers, and since the set $\{r_{l+1}, r_{l+1}, \dots\}$ is infinite but $\{a_1, \dots a_k\}$ is finite, there exists some $k' > k$ such that $a_{k'} = r_{l'}$ for some $l' > l$. Set $b_{n+1} = r_{l'} = a_{k'}$. Note that $\{b_n\}$ is a subsequence of both $\{a_n\}$ and $\{r_n\}$. Since $\{b_n\}$ is a subsequence of $\{r_n\}$, we have $\lim_{n\to\infty} b_n = \lim_{n\to\infty} r_n = s$. Since $\{b_n\}$ is a subsequence of $\{a_n\}$, this shows $s \in \mathcal{L}_a$. The case $s = 0$ is analogous. Hence $\mathcal{L}_a \supset [0,1]$, so $\mathcal{L}_a = [0,1]$.
Since $\mathcal{L}_a = [0,1]$, we have $\limsup_{n\to\infty} a_n = \operatorname{lub} \mathcal{L}_a = 1$ and $\liminf_{n\to\infty} a_n = \operatorname{glb} \mathcal{L}_a = 0$.

20.9. Let $\{a_n\}$ be a bounded sequence such that every convergent subsequence of $\{a_n\}$ has a limit L. Prove that $\lim_{n\to\infty} a_n = L$.

Solution.
Method 1: Note that $\mathcal{L}_a = \{L\}$. Hence $\limsup_{n\to\infty} a_n = \operatorname{lub}(\mathcal{L}_a) = L = \operatorname{glb}(\mathcal{L}_a) = \liminf_{n\to\infty} a_n$. So by Theorem 20.4, $\lim_{n\to\infty} a_n = L$.

Method 2: Suppose for a contradiction that $\{a_n\}$ does not have limit L. Hence there exists an $\epsilon > 0$ such that for any integer N, there exists some $n > N$ with $|a_n - L| > \epsilon$. This allows us to define $n_1 < n_2 < n_3 \dots$ such that $|a_{n_i} - L| > \epsilon$ for all i. Since $\{a_{n_i}\}$ is a bounded sequence, by Bolzano-Weierstrass it has a convergent subsequence, which clearly does not converge to L. This is a contradiction, and so it must be that $\lim_{n\to\infty} a_n = L$.

20.13. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\{a_n\}$ is convergent and $\{b_n\}$ is bounded. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

Solution. Since $\{a_n\}$ is convergent, it is bounded. So Theorem 20.6 gives us

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Let $\lim_{n\to\infty} a_n = L$. Note if $l \in \mathcal{L}_b$, then $l = \lim_{k\to\infty} b_{n_k}$ for some n_k. Hence

$$\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k} = L + l$$

so $L + l \in \mathcal{L}_{a+b}$. This shows that

$$\limsup_{n\to\infty}(a_n + b_n) = \operatorname{lub}\mathcal{L}_{a+b} \geq L + \operatorname{lub}\mathcal{L}_b = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

We've shown

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

The proof that

$$\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

is analogous.

20.20. Let $\{a_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} = L$.

Solution. Let $\epsilon > 0$. Then there exists some N such that $n \geq N$ implies $a_n \leq L + \epsilon$. Note we have

$$\begin{aligned}(a_1 \cdots a_{N+m})^{1/(N+m)} &= (a_1 \cdots a_N)^{1/(N+m)}(a_{N+1} \cdots a_{N+m})^{1/(N+m)} \\ &\leq (a_1 \cdots a_N)^{1/(N+m)}(L+\epsilon)^{m/(N+m)} \\ &= (a_1 \cdots a_N(L+\epsilon)^{-N})^{1/(N+m)}(L+\epsilon)\end{aligned}$$

Hence we have

$$\begin{aligned}\limsup_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} &= \limsup_{m\to\infty}(a_1 \cdots a_{N+m})^{1/(N+m)} \\ &\leq \limsup_{m\to\infty}(a_1 \cdots a_N(L+\epsilon)^{-N})^{1/(N+m)}(L+\epsilon) \quad \text{by Theorem 20.5} \\ &= L + \epsilon \quad \text{by Theorem 16.4}\end{aligned}$$

Hence $\limsup_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} \leq L + \epsilon$ for all $\epsilon > 0$, so $\limsup_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} \leq L$. Analogously, one can show that $\liminf_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} \geq L$. Hence by Theorem 20.2 and 20.4, we have $\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} = L$.

21.2. Let $A_n = \operatorname{lub}\{a_n, a_{n+1}, \dots\}$ and $B_n = \operatorname{glb}\{a_n, a_{n+1}, \dots\}$ for $n = 1, 2, \dots$. Compute A_n, B_n, $\lim_{n\to\infty} A_n$, and $\lim_{n\to\infty} B_n$, where $a_n =$

Solution.

(a) $(-1)^n$

Clearly $A_n = 1$ and $B_n = -1$ so $\lim_{n\to\infty} A_n = 1$ and $\lim_{n\to\infty} B_n = -1$.

(b) $\frac{1}{n}$

Clearly $A_n = \frac{1}{n}$ and $B_n = 0$ so $\lim_{n\to\infty} A_n = 0$ and $\lim_{n\to\infty} B_n = 0$.

(c) $(1 + \frac{1}{n})^n$

By Theorem 16.6 the sequence $\{(1 + \frac{1}{n})^n\}$ is increasing and convergent with limit e. So $A_n = e$ and $B_n = (1 + \frac{1}{n})^n$ so $\lim_{n\to\infty} A_n = e$ and $\lim_{n\to\infty} B_n = e$.

(d) $\frac{(-1)^n}{n}$

We have $A_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ \frac{1}{n+1} & n \text{ odd} \end{cases}$ and $B_n = \begin{cases} \frac{-1}{n+1} & n \text{ even} \\ \frac{-1}{n} & n \text{ odd} \end{cases}$. So $\lim_{n\to\infty} A_n = 0$ and $\lim_{n\to\infty} B_n = 0$.

(e) $(-1)^n(1 - \frac{1}{n})$

We have $A_n = 1$ and $B_n = -1$, so $\lim_{n\to\infty} A_n = 1$ and $\lim_{n\to\infty} B_n = -1$.

22.4. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its sum.

Solution. We use partial fractions to rewrite the terms. We try

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Longrightarrow 1 = (n+1)A + nB = A + n(A+B) \Longrightarrow A = 1 \text{ and } B = -1.$$

Indeed, we have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So the n-th partial sum is

$$\begin{aligned} s_n &= a_1 + a_2 + ... + a_n \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

This is an example of a *telescoping* series. Since

$$\lim_{n=1}^{\infty} s_n = \lim_{n=1}^{\infty} \left(1 - \frac{1}{n+1}\right) = 1,$$

we have that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

23.5. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

Solution. Let $a_n = 1$ and $b_n = -1$ for all n. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are geometric series with $r = 1$, and hence diverge by Theorem 22.4. However, $a_n + b_n = 0$ for all n so the n-th partial sum of $\sum_{n=1}^{\infty}(a_n + b_n)$ is zero for all n, giving $\sum_{n=1}^{\infty}(a_n + b_n) = 0$ converges.

24.9. Prove that if $\{a_n\}$ is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges if $0 \le s \le 1$.

Solution. Let $\epsilon > 0$. Let $s_n = \sum_{k=1}^{n} a_k$. Since $\{s_n\}$ converges, $\{s_n\}$ is Cauchy so there exists some N such that $n, m \ge N$ implies $|s_n - s_m| < \epsilon$. In particular, for $n \ge N$ we have $(n - N)a_n \le a_{N+1} + \cdots + a_n = |s_n - s_N| < \epsilon$. Hence $\lim_{n\to\infty}(n - N)a_n = 0$. By Theorem 22.3, we have $\lim_{n\to\infty} Na_n = N \lim_{n\to\infty} a_n = 0$ too. So by Theorem 12.2, we have

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} (n - N)a_n + \lim_{n\to\infty} Na_n = 0$$

so we're done.

Now, note that for $0 \le s \le 1$, we have $\lim_{n\to\infty} n\frac{1}{n^s} = \lim_{n\to\infty} n^{1-s} \ne 0$. By the above, this means that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ cannot converge, that is, it diverges.

25.2. Let $\{a_n\}$ satisfy the hypotheses of the alternating series test. Let $\{s_n\}$ denote the sequence of partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Prove that the sequence $\{s_{2n-1}\}$ is decreasing and bounded below by 0.

Solution. Note $s_{2(n+1)-1} = s_{2n-1} - a_{2n} + a_{2n+1} \le s_{2n-1}$ since $a_{2n+1} \le a_{2n}$. Hence the sequence $\{s_{2n-1}\}$ is decreasing.

Note $s_{2n-1} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-3} - a_{2n-2}) + a_{2n-1}$, where each term in parenthesis is bounded below by 0, and a_{2n-1} is also bounded below by 0. Hence $\{s_{2n-1}\}$ is bounded below by 0.

25.4. Give an example of a sequence $\{a_n\}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$, but the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ diverges.

Solution. Let $a_n = \begin{cases} \frac{2}{n+1} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$. Then $\lim_{n\to\infty} a_n = 0$ but the $(2n-1)$-th partial sum of $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is equal to the n-th partial sum of $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by Corollary 24.3, $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ also diverges.

26.4. Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Solution. By Theorem 22.3, we have $\lim_{n\to\infty} |a_n| = 0$. So by Theorem 13.2, $\{|a_n|\}$ is bounded. Hence $\{a_n\}$ is bounded. We apply part (i) of Theorem 26.4 (replacing $\{b_n\}$ with $\{a_n\}$) to see that $\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$ converges absolutely. Since all terms are non-negative, $\sum_{n=1}^{\infty} a_n^2$ converges.

26.8. Let $\{a_n\}$ be a sequence of positive numbers. Prove that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ then $\lim_{n\to\infty} a_n^{1/n} = L$. Deduce $\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = e$.

Solution. Define the sequence $\{b_n\}$ by $b_1 = a_1$ and $b_n = \frac{a_n}{a_{n-1}}$ for $n \geq 2$. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, we have $\lim_{n\to\infty} b_n = L$. Note that $a_n = b_1 b_2 \cdots b_n$. Applying exercise 20.20 to the sequence $\{b_n\}$, we get

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} (b_1 b_2 \cdots b_n)^{1/n} = L$$

Now, let $a_n = \frac{n^n}{n!}$. Note that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{n+1}}{n^n} = \lim_{n\to\infty} (\frac{n+1}{n})^n = \lim_{n\to\infty} (1+\frac{1}{n})^n = e$$

By the conclusion above, we have

$$\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \lim_{n\to\infty} a_n^{1/n} = e$$