Homework 2 Solutions

Math 171, Spring 2010
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9.6. Prove that if A and B are countable sets, then $A \times B$ is countable.

Solution. For a fixed $a \in A$, let $B_a = \{(a, b) \in A \times B \mid b \in B\}$. Since B is countable, each B_a is countable. Note that $\cup_{a \in A} B_a$ is the countable union of countable sets, and hence is countable by Theorem 9.5. Since $A \times B = \cup_{a \in A} B_a$, we have that $A \times B$ is countable.

9.11. Prove that the collection of finite subsets of $\mathbb{P}$ is countable. Deduce that the collection of inifinite subsets of $\mathbb{P}$ is uncountable.

Solution. For any $n \in \mathbb{P}$, let A_n be the set of all subsets of $\mathbb{P}$ that are bounded by n, that is, that have no element larger than n. Note that A_n is a finite set of size 2^n, and hence is countable. Hence $\cup_{n=1}^{\infty} A_n$ is countable by Theorem 9.5. Note also that $\cup_{n=1}^{\infty} A_n$ is the set of finite subsets of $\mathbb{P}$, as any finite subset is bounded by some n. Hence the collection of finite subsets of $\mathbb{P}$ is countable.

To see that the collection of inifinite subsets of $\mathbb{P}$ is uncountable, first note that

$$P(\mathbb{P}) = (\text{collection of finite subsets of } \mathbb{P}) \cup (\text{collection of infinite subsets of } \mathbb{P})$$

So, suppose for a contradiction that the collection of infinite subsets of $\mathbb{P}$ were countable. Then, by Corollary 9.6 and the first half of this problem, this would imply that $P(\mathbb{P})$ were countable. This contradicts Corollary 8.5. Hence it must be that the collection of inifinite subsets of $\mathbb{P}$ is uncountable.

10.12. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} |a_n| = |L|$.

Solution. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there exists some N such that $n \geq N$ implies $|a_n - L| < \epsilon$. Hence for $n \geq N$, we have $||a_n| - |L|| \leq |a_n - L| < \epsilon$. By the definition, this shows that $\lim_{n\to\infty} |a_n| = |L|$.

11.8. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose that

$$\lim_{n\to\infty} a_{2n} = L = \lim_{n\to\infty} a_{2n-1}$$

Prove that $\lim_{n\to\infty} a_n = L$.

Solution. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_{2n} = L$, there exists a positive integer N such that if $n \geq N$, then $|a_{2n} - L| < \epsilon$. Similarly, since $\lim_{n\to\infty} a_{2n-1} = L$, there exists a positive integer N' such that if $n \geq N'$, then $|a_{2n-1} - L| < \epsilon$. Note for all $n \geq \max\{2N, 2N' - 1\}$, we have $|a_n - L| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have shown that $\lim_{n\to\infty} a_n = L$.

11.12 Prove that the set of subsequences of $\{1/n\}_{n=1}^{\infty}$ is uncountable.

Solution. Suppose for a contradiction that the set of subsequences of $\{1/n\}_{n=1}^{\infty}$ were countable. Then we could enumerate all of the subsequences as $\{a_n^{(1)}\}, \{a_n^{(2)}\}, \{a_n^{(3)}\}, \ldots$. We will now define a subsequence $\{b_n\}$ which is not in the list. If $a_1^{(1)} = \frac{1}{k}$, then define $b_1 = \frac{1}{k+1}$. Note $b_1 \neq a_1^{(1)}$ so $\{b_n\} \neq \{a^{(1)}\}$. Inductively, if $a_n^{(n)} = \frac{1}{k}$ and $b_{n-1} = \frac{1}{k'}$, then define $b_n = \frac{1}{\max\{k,k'\}+1}$. Note that $b_n < b_{n-1}$ and so $\{b_n\}$ is indeed a subsequence of $\{1/n\}$. Note also that $b_n \neq a_n^{(n)}$ so $\{b_n\} \neq \{a^{(n)}\}$

for all n. So $\{b_n\}$ is a subsequence of $\{1/n\}$ that is not in our list, a contradiction. Hence the set of subsequences of $\{1/n\}_{n=1}^{\infty}$ is uncountable.

This is an example of a diagonalization argument, as is Example 8.3.

12.4 Find $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$.

Solution. Note

$$\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, this means that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

12.7 If $\lim_{n\to\infty}(a_{n+1}/a_n) = 0$, find $\lim_{n\to\infty} a_n$.

Solution. Since $\lim_{n\to\infty}(a_{n+1}/a_n) = 0$, then there exists some N such that for all $n \geq N$, we have $|a_{n+1}/a_n| < \frac{1}{2}$. Hence for m positive, $|a_{N+m}| < |\frac{a_N}{2^m}|$. Note that $\lim_{m\to\infty} \frac{1}{2^m} = 0$, either by Theorem 16.3 or the example on the bottom of page 47. Hence $\lim_{m\to\infty} |\frac{a_N}{2^m}| = 0$, and so $\lim_{n\to\infty} a_n = 0$.

13.1 Prove that if $\{a_n\}$ and $\{b_n\}$ are bounded sequences and c is a real number, then $\{ca_n\}$, $\{a_n + b_n\}$, and $\{a_nb_n\}$ are bounded sequences.

Solution. Since $\{a_n\}$ and $\{b_n\}$ are bounded sequences, there exist numbers M and M' such that $|a_n| \leq M$ and $|b_n| \leq M'$ for all n.

Given $c \in \mathbb{R}$, we have $|ca_n| = |c||a_n| \leq |c|M$ for all n, so $|c|M$ is a bound for $\{ca_n\}$ and so $\{ca_n\}$ is bounded.

Note $|a_n + b_n| \leq |a_n| + |b_n| \leq M + M'$ for all n, so $M + M'$ is a bound for $\{a_n + b_n\}$ and so $\{a_n + b_n\}$ is bounded.

Note $|a_nb_n| \leq |a_n||b_n| \leq MM'$ for all n, so MM' is a bound for $\{a_nb_n\}$ and so $\{a_nb_n\}$ is bounded.

13.3 Give an example of sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ is bounded and $\{b_n\}$ is convergent, but $\{a_n + b_n\}$ and $\{a_nb_n\}$ are divergent.

Solution. Let

$$a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Let $b_n = 1$ for all n. Then $\{a_n\}$ is bounded by 1 and $\{b_n\}$ is converges to 1. However,

$$a_n + b_n = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

so $\{a_n + b_n\}$ is divergent and

$$a_nb_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

so $\{a_nb_n\}$ is divergent.

15.4 Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = \infty = \lim_{n\to\infty} b_n$. Prove that $\lim_{n\to\infty} a_nb_n = \infty$.

Solution. Let M be a real number. Since $\lim_{n\to\infty} a_n = \infty$, there exists a positive integer N such that if $n \geq N$ then $a_n > 1$. Since $\lim_{n\to\infty} b_n = \infty$, there exists a positive integer N' such that if $n \geq N'$ then $b_n > M$. Hence, if $n \geq \max\{N, N'\}$, then $a_nb_n > 1 \cdot M = M$. This shows that $\lim_{n\to\infty} a_nb_n = \infty$.

16.5 Find the following limits.

Solution. (a) $\lim_{n\to\infty}(1+\frac{1}{n^2})^{n^2}$

Note the sequence $(1+\frac{1}{n^2})^{n^2}$ is a subsequence of $(1+\frac{1}{n})^n$, which has limit e. Hence $\lim_{n\to\infty}(1+\frac{1}{n^2})^{n^2} = e$ by Theorem 11.2.

(b) $\lim_{n\to\infty}(1+\frac{1}{n+1})^n$

Note $(1+\frac{1}{n+1})^n = (1+\frac{1}{n+1})^{n+1}/(1+\frac{1}{n+1})$. By Theorem 12.9,

$$\lim_{n\to\infty}(1+\frac{1}{n+1})^n = \lim_{n\to\infty}(1+\frac{1}{n+1})^{n+1}/\lim_{n\to\infty}(1+\frac{1}{n+1}) = e/1 = e$$

(c) $\lim_{n\to\infty}(1+\frac{1}{n})^{n+1}$

Note $(1+\frac{1}{n})^{n+1} = (1+\frac{1}{n})^n(1+\frac{1}{n})$. By Theorem 12.6,

$$\lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = \lim_{n\to\infty}(1+\frac{1}{n})^n \lim_{n\to\infty}(1+\frac{1}{n}) = e\cdot 1 = e$$

(d) $\lim_{n\to\infty}(1+\frac{1}{n^2})^n$

By Theorem 16.6, we know that $1 \le (1+\frac{1}{n^2})^{n^2} \le e$ for all n positive. Taking n-th roots, we get $1 \le (1+\frac{1}{n^2})^n \le e^{1/n}$. Note $\lim_{n\to\infty} e^{1/n} = 1$ by Theorem 16.4. Hence by Theorem 14.3, the Squeeze Theorem, we have $\lim_{n\to\infty}(1+\frac{1}{n^2})^n = 1$.

(e) $\lim_{n\to\infty}(1+\frac{1}{n})^{n^2}$

Since $\lim_{n\to\infty}(1+\frac{1}{n})^n = e > 2$, there exists some N such that $(1+\frac{1}{n})^n > 2$ for all $n \ge N$. Hence $(1+\frac{1}{n})^{n^2} > 2^n$ for all $n \ge N$. Since $\lim_{n\to\infty} 2^n = \infty$, this means that $\lim_{n\to\infty}(1+\frac{1}{n})^{n^2} = \infty$.

(f) $\lim_{n\to\infty}(1+\frac{1}{2n})^n$

Note $(1+\frac{1}{2n})^n = ((1+\frac{1}{2n})^{2n})^{1/2}$. Hence

$$\lim_{n\to\infty}(1+\frac{1}{2n})^n = \lim_{n\to\infty}((1+\frac{1}{2n})^{2n})^{1/2} = (\lim_{n\to\infty}(1+\frac{1}{2n})^{2n})^{1/2} = e^{1/2} = \sqrt{e}$$

where the second equality is from Exercise 12.6, or from the fact that the square root is a continuous function.

16.13 Let $\{a_n\}$ be any sequence of real numbers such that $\lim_{n\to\infty} na_n = 0$. Prove that $\lim_{n\to\infty}(1+\frac{1}{n}+a_n)^n = e$.

Solution. First, note that for $a > 0$, $(1+\frac{a}{n})^n = (1+\frac{1}{n/a})^n = ((1+\frac{1}{n/a})^{\frac{n}{a}})^a$. Also note that $\lim_{n\to\infty}(1+\frac{1}{n/a})^{\frac{n}{a}} = e$ since $x \to (1+\frac{1}{x})^x$ is an increasing function on $(0,\infty)$. Hence

$$\lim_{n\to\infty}(1+\frac{a}{n})^n = \lim_{n\to\infty}((1+\frac{1}{n/a})^{\frac{n}{a}})^a = e^a$$

by Exercise 17.4, which is an extension of Corollary 12.7 to real numbers.

So, given $\epsilon > 0$, pick $1 > \delta > 0$ sufficiently small such that $e^{1+\delta} < e+\epsilon$ and $e^{1-\delta} > e-\epsilon$. Since $\lim_{n\to\infty} na_n = 0$, since $\lim_{n\to\infty}(1+\frac{1+\delta}{n})^n = e^{1+\delta}$, and since $\lim_{n\to\infty}(1+\frac{1-\delta}{n})^n = e^{1-\delta}$, there exists an N such that $n \ge N$ implies $|na_n| < \delta$, implies $(1+\frac{1+\delta}{n})^n < e^{1+\delta}+\epsilon$, and implies $(1+\frac{1-\delta}{n})^n > e^{1-\delta}-\epsilon$. Then $n \ge N$ implies

$$(1+\frac{1}{n}+a_n)^n = (1+\frac{1+na_n}{n})^n < (1+\frac{1+\delta}{n})^n < e^{1+\delta}+\epsilon < e+2\epsilon$$

and

$$(1+\frac{1}{n}+a_n)^n = (1+\frac{1+na_n}{n})^n > (1+\frac{1-\delta}{n})^n > e^{1-\delta} - \epsilon > e - 2\epsilon$$

Hence $\lim_{n\to\infty}(1+\frac{1}{n}+a_n)^n = e$.

16.15 Let $a_1 > 1$. Let $a_{n+1} = 2 - 1/a_n$ for $n = 1, 2, ...$ Prove that $\{a_n\}$ is a bounded monotone sequence and find the limit.

Solution. Note if $a_n > 1$, then $1/a_n < 1$ so $a_{n+1} = 2 - 1/a_n > 2 - 1 = 1$. Hence by induction on n, we have $a_n > 1$ for all n. So $\{a_n\}$ is bounded below by 1.

Note $a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$ implies $a_n^2 \geq 2a_n - 1$ implies $a_n \geq 2 - 1/a_n = a_{n+1}$ as a_n is positive for all n. Hence $\{a_n\}$ is a decreasing sequence. So $\{a_n\}$ is bounded above by a_1, and so $\{a_n\}$ is a bounded sequence.

By Theorem 16.2, $\{a_n\}$ has a limit; call it L. We have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(2 - 1/a_n) = 2 - 1/L$$

by the results of Chapter 12. We solve for L to see $L = 1$.

18.1. Show (by example) that "bounded" cannot be omitted from the hypotheses of the Bolzano-Weierstrass theorem.

Solution. Consider the increasing sequence (u_n) given by $u_n = n$. A subsequence of (u_n) is of the form $(u_{f(n)})$ where $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. No subsequence of (u_n) is bounded, since given any M, we have

$$u_{f(M+1)} \geq u_{M+1} = M + 1 > M.$$

So any subsequence of (u_n) is monotone but not bounded, and so by Theorem 16.2, any subsequence of (u_n) is not convergent.

18.5. Let $\{a_n\}$ be a sequence such that for some $\epsilon > 0$, $|a_n - a_m| \geq \epsilon$ for all $n \neq m$. Prove that $\{a_n\}$ has no convergent subsequence.

Solution. Because any subsequence of $\{a_n\}$ has the same property (above) as $\{a_n\}$, it suffices to prove that $\{a_n\}$ does not converge (for then any subsequence of $\{a_n\}$ will not converge by the same proof).

Suppose for a contradiction that $\{a_n\}$ has a limit L. Then by the definition of a limit, there exists some integer N such that for all $n \geq N$, $|a_n - L| < \frac{\epsilon}{2}$. In particular, we have $|a_N - L| < \frac{\epsilon}{2}$. However, this implies that

$$|a_{N+1} - L| \geq |a_{N+1} - a_N| - |a_N - L| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},$$

where we used the triangle inequality. This is a contradiction, and so $\{a_n\}$ cannot converge.

Note: another easy way to see that $\{a_n\}$ does not converge would be to show that the sequence $\{a_n\}$ is not Cauchy and then use Theorem 19.3.