Research Statement

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0. Introduction

My research is in differential geometry and geometric analysis, specifically the interrelated areas of special holonomy and calibrated geometry.

Special holonomy forms an important part of differential geometers' ongoing quest for Einstein metrics (especially Ricci-flat ones), while calibrated geometry occupies a special place in the theory of minimal submanifolds (especially those of higher codimension).

$$\text{Special Holonomy} \subset \text{Einstein Metrics}$$
$$\text{Calibrated Geometry} \subset \text{Minimal Submanifolds}$$

Both areas are intimately related to physics, specifically to string theory and M-Theory.

In recent work [29], I prove the local existence of a new infinite-dimensional family of Einstein metrics in dimension 6. These metrics are *nearly-Kähler*, meaning that they are the cross-sections of (Ricci-flat) cones with G_2 holonomy. In §1, this existence theorem is described, along with further results from [29] on the geometry of these new metrics.

These (local) results represent an important step in the larger search for new (global) compact, simply-connected, nearly-Kähler 6-manifolds. Indeed, constructing such manifolds has proven unusually challenging: only six examples are currently known [13].

My current work, outlined in §2, seeks the construction of new compact examples. In §2.1, §2.2, and §2.3, I describe how the results of [29] suggest several possible approaches towards such a construction. In §2.4, I discuss how a positive solution of this problem would immediately give rise to several interesting further questions.

Finally, I propose in §3 several future investigations in calibrated geometry. Some of these projects are natural extensions of recent results to less symmetric settings, while others are perhaps more conjectural and ambitious.

1. Previous Work: Local Geometry of Nearly-Kähler 6-Manifolds

1.1 Background: Nearly-Kähler 6-Manifolds.

Nearly-Kähler 6-manifolds are a class of Riemannian 6-manifolds (M^6, g) whose geometry is modeled on the round 6-sphere. Like the round $\mathbb{S}^6$, they carry a triple (J, Ω, Υ) consisting of a compatible almost-complex structure J, a non-degenerate 2-form Ω, and a complex volume form Υ, and these are asked to satisfy the defining differential equations

$$d\Omega = 3\,\mathrm{Im}(\Upsilon)$$
$$d\,\mathrm{Re}(\Upsilon) = 2\,\Omega \wedge \Omega.$$

Here, the almost-complex structure J is **not** integrable, and the 2-form Ω is **not** closed.

Yet, in spite of these two shortcomings, nearly-Kähler 6-manifolds enjoy several remarkable properties that have led to increased attention as of late, especially in connection with special holonomy. Indeed, they are exactly the links of (7-dimensional) cones whose reduced holonomy is contained in G_2. In particular, nearly-Kähler 6-manifolds are Einstein of positive scalar curvature.

1.2 Previous Work

A central problem in the study of nearly-Kähler 6-manifolds is the present lack of compact, simply-connected examples. Indeed, only six such examples are presently known. Four of these are the homogeneous spaces [38]

$$\mathbb{S}^6 = \frac{\mathrm{G}_2}{\mathrm{SU}(3)}, \qquad \mathbb{S}^3 \times \mathbb{S}^3 = \frac{\mathrm{SU}(2)^3}{\Delta\mathrm{SU}(2)}, \qquad \mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1) \times \mathrm{Sp}(1)}, \qquad \mathrm{Flag}(\mathbb{C}^3) = \frac{\mathrm{SU}(3)}{T^2},$$

and it has been shown [8] that these are the only possible homogeneous examples.

Following work of Podestà and Spiro [32] [33], Conti and Salamon [9], and Fernández, Ivanov, Muñoz and Ugarte [11], recently Foscolo and Haskins [13] succeeded in constructing *inhomogeneous* nearly-Kähler metrics on $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ which are cohomogeneity-one. From this point of view, the next natural question is the following:

Important Problem: Construct compact, simply-connected nearly-Kähler 6-manifolds of cohomogeneity-two.

As a first step in this direction, I study in [29] the local geometry of nearly-Kähler 6-manifolds M of **cohomogeneity-two**, meaning that M admits a faithful action of a connected Lie group G whose generic orbits have codimension two. This G-action on M is presumed to preserve (J, Ω, Υ), and we suppose G is closed in the isometry group of (M, g).

I show in [29] that if M is compact, then the generic orbits in M are **coisotropic**, meaning that the 4-form $\Omega \wedge \Omega$ vanishes on these orbits. Thus, in the quest for compact examples, attention should be restricted to those M satisfying this necessary condition.

This orbit condition makes possible an analysis by the method of moving frames. By adapting frames to the geometry of the generic orbits and examining the corresponding structure equations, I show that there is not much freedom in the choice of the Lie group G:

Proposition 1 [29]**:** The acting Lie group G is 4-dimensional and non-abelian. Moreover, if M is complete, then G is a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$.

The next question to address that of local existence. That is, on sufficiently small open sets of $\mathbb{R}^6$ we ask whether cohomogeneity-two nearly-Kähler metrics can exist at all. If so, what is the initial data required to construct these metrics as solutions to (a sequence of) Cauchy problems?

In [29], I approach this problem by an application of Cartan's Third Theorem [7]. This result of Cartan generalizes Lie's Third Theorem on the "integration" of Lie algebras to local Lie groups. Its primary hypothesis is that "mixed partials commute," meaning the satisfaction of a set of integrability conditions (analogous to the Jacobi identity for Lie algebras).

In the case of cohomogeneity-two nearly-Kähler metrics, these integrability conditions form a system of roughly 70 quadratic equations on 55 functions. Careful study of this system led me to partition the class of metrics under consideration into three types, called Types I, II, and III.

I show in [29] that metrics of each Type exist locally and in abundance: each Type is an infinite-dimensional family. Somewhat more precisely:

Theorem 2 [29]**:** On sufficiently small open sets in $\mathbb{R}^6$, cohomogeneity-two nearly-Kähler metrics of Type I and Type II exist locally.

Moreover, generic metrics of Types I and II are uniquely determined by initial data consisting of 2 arbitrary functions of 1 variable.

The criterion of Proposition 1 shows that metrics of Types I and II are incomplete. In fact:

Proposition 3 [29]**:** If M is of Type I, then G is a discrete quotient of $\mathrm{H}_3 \times \mathbb{R}$, where H_3 is the real Heisenberg group. If M is of Type II, then G is solvable.

In particular, metrics of Types I and II are incomplete.

Thus, if compact examples exist, then they belong to Type III. In this setting, the Lie group G may be a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$, but other Lie groups are also possible. In the case of most interest, the analogous result holds:

Theorem 4 [29]**:** Suppose G is a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$. On sufficiently small open sets in $\mathbb{R}^6$, cohomogeneity-two nearly-Kähler structures (of Type III) exist locally.

Moreover, generic metrics in this class are uniquely determined by initial data consisting of 2 arbitrary functions of 1 variable.

2. Current Work: Global Geometry of Nearly-Kähler 6-Manifolds

The main goal of my current work is to solve the "Important Problem" of §1.2 – that is, to construct compact, simply-connected nearly-Kähler 6-manifolds of cohomogeneity-two – by building upon the results of [29]. I am currently pursuing several avenues to this end, which I describe in §2.1, §2.2, and §2.3 below. In §2.4, analogues and extensions of the "Important Problem" are discussed.

2.1 Holomorphic Interpretations: Elliptic PDE Systems.

The dependence on 2 functions of 1 variable in Theorem 4 above – the same initial data required to construct holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ locally – suggests the intriguing possibility that cohomogeneity-two nearly-Kähler 6-manifolds may be recovered from holomorphic data.

Indeed, one can ask whether cohomogeneity-two nearly-Kähler metrics might be reconstructed from solutions to an elliptic PDE system on a Riemann surface. In turn, might solutions to this PDE system be interpreted as pseudo-holomorphic curves in some almost-complex manifold?

The answers to these two questions are expected to be "yes," and demonstrating this properly is work in progress. In the Type I setting, I verified this in [29]:

Theorem 5 [29]**:** Across their principal loci, cohomogeneity-two nearly-Kähler structures of Type I are solutions to a certain (explicit) quasilinear elliptic PDE system on a Riemann surface.

Similar PDE systems (and interpretation via pseudo-holomorphic curves) for Type II and Type III metrics are forthcoming.

Thus, one potential method for constructing new compact nearly-Kähler manifolds would be to derive *a priori* estimates for the relevant elliptic PDE system (for Type III metrics) to prove the existence of solutions defined on the entire principal locus. One would then show that at least one of these solutions extends smoothly across the singular G-orbits in M.

2.2 Holomorphic Interpretations: Weierstrass Representation Formulas.

Another holomorphic interpretation of cohomogeneity-two nearly-Kähler metrics would be a **Weierstrass formula**. Classically, the original Weierstrass representation

$$\varphi = \mathrm{Re} \int \left(\frac{1}{2} f(1-g^2), \frac{i}{2} f(1+g^2), fg \right) dz$$

constructs immersions φ of minimal surfaces in $\mathbb{R}^3$ from an arbitrary holomorphic function f and meromorphic g with fg^2 holomorphic. Analogous representation formulas have also been discovered, for example, for minimal surfaces in $\mathbb{S}^4$ in [5], for pseudo-holomorphic curves in $\mathbb{S}^6$ in [6], and for ruled pseudo-Lagrangians in $\mathbb{S}^6$ in [26]. The natural question is:

Problem 6: Does any class of cohomogeneity-two nearly-Kähler metrics admit a Weierstrass representation formula?

This is also work in progress. If Problem 6 admits a positive solution, then it may be possible to construct compact nearly-Kähler manifolds using complex analytic methods.

2.3 Other Approaches.

In [29], I show that the class of Type III nearly-Kähler metrics with G a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$ includes a one-parameter family of metrics which are cohomogeneity-one under a larger group (aside from an endpoint which represents the homogeneous $\mathbb{S}^3 \times \mathbb{S}^3$). This one-parameter family is described by relatively simple structure equations, so there is hope to understand its geometry in some detail.

Analyzing perturbations of this family of metrics may prove fruitful in understanding the PDE system for Type III nearly-Kähler metrics. This direction is under investigation.

In a different vein, recent calculations (unpublished) suggest the possibility that Type III nearly-Kähler metrics may arise as solutions to a single second-order elliptic PDE. Such a differential equation might be considerably simpler to analyze than the systems described above. Discovery of this PDE is also a work in progress.

2.4 Analogues and Extensions.

The project of §2.1-§2.3, if successful, would immediately raise several interesting follow-up problems. Namely, if M is a compact nearly-Kähler 6-manifold, one would like to:

- Construct pseudo-holomorphic curves in M.
- Construct pseudo-Lagrangians in M.
- Construct pseudo-Hermitian-Yang-Mills connections on bundles over M.
- Study the deformation theory of M.

Separately, one might also seek the construction of compact cohomogeneity-3 nearly-Kähler 6-manifolds. In [29], it is remarked that for such a manifold, the 3-form $\mathrm{Im}(\Upsilon)$ must vanish on the principal orbits, which suggests studying those whose principal orbits are pseudo-special Lagrangian. General considerations (also described in [29]) show that the acting Lie group G must satisfy $3 \leq \dim G \leq 6$.

3. Future Work: Calibrated Geometry

3.1 Background: Calibrated Geometry.

A **calibration** is a closed differential k-form $\varphi \in \Omega^k(M)$ on a Riemannian n-manifold (M^n, g) that satisfies $\varphi|_V \leq \mathrm{vol}|_V$ for every oriented k-plane $V \subset T_xM$, where vol is the volume form of (M, g). A k-dimensional submanifold $\Sigma^k \subset M^n$ is φ-**calibrated** if $\varphi|_{T_xM} = \mathrm{vol}|_{T_xM}$ for every $x \in \Sigma$.

Calibrated submanifolds enjoy the rare and remarkable property of being area minimizing in their homology class [16]. In particular, they are stable minimal submanifolds.

The study of special holonomy metrics and calibrated submanifolds go hand-in-hand. Indeed, Riemannian manifolds with special holonomy come equipped with natural parallel calibrations: simply parallel translate a (suitably scaled) Hol-invariant k-form at a point. The table below gives a few examples of this relationship.

Holonomy of M	$\dim M$	Calibrated Submanifolds of M
$\mathrm{U}(n)$	$2n$	Holomorphic submanifolds
$\mathrm{SU}(n)$	$2n$	Special Lagrangian n-folds
G_2	7	Associative 3-folds, Coassociative 4-folds
Spin(7)	8	Cayley 4-folds

3.2 Gauss Map of Calibrated Submanifolds.

In classical minimal surface theory, the celebrated **Bernstein Theorem** states that a minimal hypersurface in $\mathbb{R}^3$ which is an entire graph must be a hyperplane. Work of several mathematicians extended this result to hypersurfaces in $\mathbb{R}^n$ for $n \leq 8$ [1], [34], [36], and showed its failure in dimensions $n \geq 9$ [4].

If one asks the same question for minimal submanifolds of higher codimension, the result again fails [24], even in the calibrated setting [16]: there exist entire graphical (Lipschitz) coassociative cones in $\mathbb{R}^7$.

However, certain weak Bernstein Theorems are nevertheless true. That is: If the tangent space to a minimal submanifold is constrained to lie in a sufficiently small region of the Grassmannian, then the submanifold must be an affine plane. Such theorems are shown, for example, in [12], [20], [21], [40], and [37].

In the setting of coassociative 4-folds in $\mathbb{R}^7$, it would be interesting to strengthen the currently known results. More broadly, it would be good to have a precise understanding of how restrictions on the Gauss map of a calibrated submanifold constrain its shape. In these directions, I plan to investigate the following problems:

Problem 7: Prove a sharper weak Bernstein Theorem for coassociative 4-folds in $\mathbb{R}^7$.

Problem 8: Classify those coassociative 4-folds in $\mathbb{R}^7$ whose Gauss map has low rank. Is a similar classification possible for maps other than the usual Gauss map?

3.3 Calibrated Cones in Curved Ambient Spaces.

In calibrated geometry, there is interest in having an ample collection of calibrated cones. Cones provide examples of the simplest type of singularity, and may be used to construct new (smooth) nearby examples (e.g., by singular perturbation techniques).

The construction of such cones reduces to the construction of their cross-sections, which I call **pseudo-calibrated links**. These links are frequently minimal submanifolds (of codimension ≥ 2), and so are of interest in their own right. In this direction, it would be good to know the answer to the following:

Problem 9: Characterize those SU(3)-structures and G_2-structures for which all pseudo-calibrated submanifolds (of a given type) therein are minimal.

The table below describes the calibrated cones $C(\Sigma)$ (and their pseudo-calibrated links Σ) which arise in a conical manifold $C(M)$ having holonomy SU(3), G_2, or Spin(7).

Σ	M	$C(\Sigma)$	$C(M)$
Special Legendrian	Sasaki-Einstein	Special Lagrangian	Calabi-Yau
Pseudo-Holomorphic Curve	Nearly-Kähler	Associative	G_2
Pseudo-Lagrangian	Nearly-Kähler	Coassociative	G_2
Pseudo-Associative	Nearly-G_2	Cayley	Spin(7)

When the ambient space is the flat $\mathbb{R}^n$ (resp., the round $\mathbb{S}^{n-1}$), there is by now an extensive literature on calibrated cones (resp., pseudo-calibrated links). To name but a few of examples: Special Lagrangian cones are studied in [17], [18], [19], [22], associative cones in [6], [28], coassociative cones in [30], [26], [14], and Cayley cones in [27], [15].

By contrast, the study of calibrated cones in curved ambient (conical) spaces seems to have only just begun. Pseudo-holomorphic curves in CP^3 are studied by Xu in [39], and those in $\mathbb{S}^3 \times \mathbb{S}^3$ by Bolton, Dioos, and Vrancken in [3]. Very recently, pseudo-Lagrangians in $\mathbb{S}^3 \times \mathbb{S}^3$ were examined by Dioos, Vrancken, and Wang in [10]. Pseudo-associatives in the squashed 7-sphere $\mathbb{S}^7_{\text{sq}}$ were studied by Kawai in the recent work [23].

By analogy, I hope to address the following construction problems:

Problem 10: Construct large families of pseudo-Lagrangians in the nearly-Kähler $\mathrm{SU}(3)/T^2$ and CP^3.

Problem 11: Construct pseudo-holomorphic curves and pseudo-Lagrangians in the cohomogeneity one nearly-Kähler 6-manifolds of Foscolo-Haskins [13].

Problem 12: Construct large families of pseudo-associatives in the nearly-G_2 Aloff-Wallach spaces $N_{k,\ell}$ and in the Berger manifold $\mathrm{SO}(5)/\mathrm{SO}(3)$. Relate pseudo-associatives in $N_{k,\ell}$ to gauge-theoretic objects constructed in [2]. Construct additional examples of pseudo-associatives in the squashed $\mathbb{S}^7_{\mathrm{sq}}$, beyond those constructed in [23].

3.4 Curvature and Topology of Calibrated Submanifolds.

The theory of minimal surfaces in $\mathbb{R}^3$ is an old and highly well-developed area of differential geometry; a great deal is known about the curvature and topology of these submanifolds. For example, if one considers complete, embedded minimal surfaces in $\mathbb{R}^3$ of finite total curvature, many topological types are now known (see the survey [31]), as are various topological obstructions (e.g., [35], [25]).

Ideally, one would like to understand the calibrated submanifolds of $\mathbb{R}^n$ to a similar extent. The following two (very broad) problems are in this direction:

Problem 13: Classify calibrated submanifolds (say, special Lagrangian) of $\mathbb{R}^n$ subject to various curvature conditions.

Problem 14: Produce complete, embedded calibrated submanifolds in $\mathbb{R}^n$ of new topological types. In the other direction, are there topological obstructions to complete, embedded calibrated submanifolds of $\mathbb{R}^n$? What if asymptotic conditions are added?

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