Review Problems

1. Consider the 1st-order ODE

$$y' = (y+1)(y-3)(y-a).$$

(a) How many equilibrium solutions are there? (Your answer should depend on a)
(b) Sketch solution curves (in the ty-plane) for different values of a.
(c) Draw the bifurcation diagram.

2. Consider the IVP

$$\begin{aligned} y' &= 3x^2(y^2+1) \\ y(0) &= 0 \end{aligned}$$

(a) Use the Improved Euler Method (with one step) to approximate $y(0.1)$.
(b) Find the exact value of $y(0.1)$.

3. Use Euler's method (with one step) to approximate $\ln(1.1)$.

4. Find the general solution of the 2nd-order ODE

$$2y'' - 12y' + 18y = 0.$$

5. Consider the 2nd-order IVP

$$\begin{aligned} y'' + 9y &= 0 \\ y(0) &= -2 \\ y'(0) &= 6. \end{aligned}$$

(a) Find the solution.
(b) Find the smallest value of $t > 0$ for which $y(t)$ is a local maximum.
(c) Graph the solution (in the ty-plane).

6. Consider the 2nd-order ODE

$$y'' - (2a-1)y' + a(a-1)y = 0.$$

(a) Determine the values of a, if any, for which all solutions tend to zero as $t \to \infty$.
(b) Determine the values of a, if any, for which all (non-zero) solutions become unbounded as $t \to \infty$.

7. Consider the 1st-order IVP

$$\begin{aligned}(\cos t)y' + (\sin t)y &= \sin t \cos t \\ y(\pi) &= 0.\end{aligned}$$

Find an interval on which the IVP will have exactly one solution.

8. Consider the 1st-order IVP

$$\begin{aligned}y' &= \sqrt{4 - t^2 - y^2} \\ y(1) &= 1.\end{aligned}$$

Prove that there is some interval $(1-h, 1+h)$ on which the IVP has a unique solution $y(t)$.

9. Consider the 2nd-order ODE

$$y'' - 5y' + 6y = 2e^t.$$

(a) Find the general solution using variation of parameters.
(b) Find the general solution using undetermined coefficients.

10. Consider the 2nd-order ODE

$$y'' + 2y' = 3 + 4\sin 2t.$$

Find the general solution using undetermined coefficients.

Answers

1. (a) If $a \neq -1$ and $a \neq 3$, then there are three equilibrium solutions. If $a = -1$ or if $a = 3$, then there are only two equilibrium solutions.

2. (a) $y(0.1) = \frac{2}{3000} = 1.5 \times 10^{-3}$.
(b) Separation of variables $\Longrightarrow$ $y(t) = \tan(x^3)$ $\Longrightarrow$ Exact value is $\tan(\frac{1}{1000})$.

3. Use $y = \ln x$, so $y'(x) = 1/x$ and $y(1) = 0$. Answer is $\ln(1.1) \approx 1/10$.

4. $y(t) = c_1 e^{3t} + c_2 t e^{3t}$.

5. (a) $y(t) = -2\cos(3t) + 2\sin(3t)$.
(b)-(c). Write $y(t)$ in phase-amplitude form to get $y(t) = 2\sqrt{2}\cos(3t - \frac{3\pi}{4})$. One can then graph this.

6. Characteristic equation $\lambda^2 - (2a-1)\lambda + a(a-1) = 0$ has roots $\lambda = a$, $\lambda = a - 1$, which are real and distinct, so

$$y(t) = c_1 e^{at} + c_2 e^{(a-1)t}.$$

Answer to (a): $a < 0$.
Answer to (b): $a > 1$.

7. Divide by $\cos t$ to get

$$y' + (\tan t)y = \sin t.$$

Answer is the interval $(\pi/2, 3\pi/2)$.

8. Let $f(t, y) = \sqrt{4 - t^2 - y^2}$. Calculate $\partial f / \partial y$. See that they're both continuous on the open disk $t^2 + y^2 < 4$ in the ty-plane. Notice that the initial value $(1, 1)$ lies in the interior of this disk.

9. In both (a) and (b), the general solution is the same:

$$y(t) = c_1 e^{2t} + c_2 e^{3t} + e^t.$$

10. The general solution is

$$y(t) = (c_1 + c_2 e^{-2t}) + (-\tfrac{1}{2}\sin 2t - \tfrac{1}{2}\cos 2t + \tfrac{3}{2}t).$$