ANNALS OF PURE AND APPLIED LOGIC

ELSEVIER Annals of Pure and Applied Logic 79 (1996) 37–52

Systems of explicit mathematics with non-constructive μ-operator. Part II

Solomon Feferman[a,*], Gerhard Jäger[b,1]

[a] *Departments of Mathematics and Philosophy, Stanford University, Stanford, CA 94305, USA*
[b] *Institut für Informatik und angewandte Mathematik, Universität Bern, Neubrückstrasse 10, CH-3012 Bern, Switzerland*

Received 12 January 1995
Communicated by D. van Dalen

Abstract

This paper is mainly concerned with proof-theoretic analysis of some second-order systems of explicit mathematics with a non-constructive minimum operator. By introducing axioms for variable types we extend our first-order theory **BON** to the elementary explicit type theory **EET** and add several forms of induction as well as axioms for μ. The principal results then state: $\mathbf{EET}(\mu)$ plus set induction (type induction, formula induction) is proof-theoretically equivalent to Peano arithmetic **PA** (the second-order system $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_0}$, the second-order system $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_{\varepsilon_0}}$).

1. Introduction

For the general background to this work, which continues the metamathematical study of systems of explicit mathematics introduced in [3], see the introduction of Part I [9], which treated applicative theories with (and without) the non-constructive (i.e. unbounded) minimum operator μ. In this part we determine the effect of adding axioms for (variable) types (a.k.a. classes), together with several schemes for induction on the type N of natural numbers.

A base theory **EET**, called elementary explicit type theory, is introduced for this purpose. The schemes of induction on N considered are $(\mathbf{S}\text{-}\mathbf{I_N})$ (set induction), $(\mathbf{T}\text{-}\mathbf{I_N})$ (type induction) and $(\mathbf{F}\text{-}\mathbf{I_N})$ (formula induction). We determine the proof-theoretic strength of **EET** and $\mathbf{EET}(\mu)$ with each of these three schemes as follows:

$$\begin{aligned}
&\mathbf{EEE} + (\mathbf{S}\text{-}\mathbf{I_N}) \equiv \mathbf{PRA}, && \mathbf{EET}(\mu) + (\mathbf{S}\text{-}\mathbf{I_N}) \equiv \mathbf{PA},\\
&\mathbf{EET} + (\mathbf{T}\text{-}\mathbf{I_N}) \equiv \mathbf{PA}, && \mathbf{EET}(\mu) + (\mathbf{T}\text{-}\mathbf{I_N}) \equiv (\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_0},\\
&\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N}) \equiv (\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA}), && \mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N}) \equiv (\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_{\varepsilon_0}}.
\end{aligned}$$

* Corresponding author. E-mail: sf@csli.stanford.edu.
[1] E-mail: jaeger@iam.unibe.ch.

SSDI 0168-0072(95)00028-3

Thus the strongest of the systems considered is still weaker than full predicative analysis $(\mathbf{\Pi}^0_\infty\textbf{-CA})_{<\Gamma_0}$.

The system **EET** provides a conceptual framework for the direct representation of notions from analysis. It was shown in [4] how extensive portions of (Bishop-style) constructive analysis can be formalized directly in a version $\mathbf{EM}_0\!\upharpoonright$ of **EET** + $(\mathbf{T}\text{-}\mathbf{I_N})$. Correspondingly, it was indicated in [6] (see also [5]) how extensive portions of non-constructive classical and modern analysis can be formalized directly in a version **W** of $\mathbf{EET}(\mu)$ + $(\mathbf{S}\text{-}\mathbf{I_N})$. Thus in both cases, systems of strength **PA** suffice for these portions of mathematical practice.

2. The syntax of applicative theories with types

The language $\mathbb{L}_p$ of our applicative theories with types is a second-order extension of the first-order language L_p of the basic theory of partial operations and numbers which is described at full length in [9]. We add to L_p type variables and the binary relation symbol $\in$ for membership and a further binary relation symbol $\mathfrak{R}$. The formula $\mathfrak{R}(a, B)$ is used to express that the individual *a represents* the type B or is a *name* of B. Further new individual constants $\mathbf{c}_e$ $(e < \omega)$ for the uniform representation of elementary types are available as well.

Variables. The language $\mathbb{L}_p$ comprises individual variables $a, b, c, x, y, z, f, g, h, \ldots$ and type variables $A, B, C, X, Y, Z, \ldots$ (both possibly with subscripts). Types are supposed to range over collections of individuals.

Constants. In addition there are the individual constants $\mathbf{k}, \mathbf{s}$ (combinators), $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$ (pairing and unpairing), 0 (zero), $\mathbf{s}_N$ (numerical successor), $\mathbf{p}_N$ (numerical predecessor), $\mathbf{d}_N$ (definition by numerical cases), $\mathbf{r}_N$ (recursion), μ (unbounded minimum operator) and $\mathbf{c}_e$ $(e < \omega)$, the meaning of which will be explained later.

The *individual terms* $(r, s, t, r_0, s_0, t_0, \ldots)$ of $\mathbb{L}_p$ are generated as follows:

(1) All individual variables and all individual constants are individual terms.

(2) If s and t are individual terms, then so also is $(s \cdot t)$.

Thus the principal operation for the formation of individual terms is term application $(s \cdot t)$ which we often just write as (st) or st. In this simplified form we adopt the convention of association to the left so that $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 \cdot s_2) \ldots \cdot s_n)$. In the following we write (t_1, t_2) for $\mathbf{p} t_1 t_2$ and $(t_1, t_2, \ldots, t_n)$ for $(t_1, (t_2, \ldots, t_n))$. Further we put $t' := \mathbf{s}_N t$ and $1 := 0'$.

Relation symbols. The relation symbols of $\mathbb{L}_p$ are the unary $\downarrow$ and N as well as the binary $=$, $\in$ and $\mathfrak{R}$. The relation symbols $\downarrow$, N and $=$ apply to individuals whereas $\in$ and $\mathfrak{R}$ represent relations between individuals and types.

The *formulas* $(\varphi, \chi, \psi, \varphi_0, \chi_0, \psi_0, \ldots)$ of $\mathbb{L}_p$ are generated as follows:

(1) $t\downarrow$, $N(t)$, $(s = t)$, $(s \in A)$, $(A = B)$ and $\mathfrak{R}(s, A)$ are (atomic) formulas.

(2) If φ and ψ are formulas, then so also are $\neg\varphi$ and $(\varphi \vee \psi)$.

(3) If φ is a formula, then so also are $(\exists x)\varphi$ and $(\exists X)\varphi$.

The remaining logical operations are defined as usual. An $\mathbb{L}_p$ formula φ is called *stratified* if the relation symbol $\Re$ does not occur in φ; an $\mathbb{L}_p$ formula is called an *elementary formula* if it is stratified and does not contain bound type variables.

In the following we will make use of the logic of partial terms. Then $t\downarrow$ is read "t is defined" or "t has a value". The *partial equality relation* $\simeq$ is introduced by

$$(s \simeq t) := ((s\downarrow \vee t\downarrow) \rightarrow (s = t))$$

and $(s \neq t)$ is written for $(s\downarrow \wedge t\downarrow \wedge \neg(s = t))$. As additional abbreviations in connection with the relation symbol N for the natural numbers we will use

$$t \in N := N(t), \qquad A \subset N := (\forall x)(x \in A \rightarrow x \in N),$$

$$(\exists x \in N)\varphi := (\exists x)(x \in N \wedge \varphi), \quad (\forall x \in N)\varphi := (\forall x)(x \in N \rightarrow \varphi).$$

Further, the fact that an individual term t plays the role of a total function from N^m to N may be expressed as follows:

$$(t : N \rightarrow N) := (\forall x \in N)(tx \in N),$$

$$(t : N^{m+1} \rightarrow N) := (\forall x \in N)(tx : N^m \rightarrow N).$$

The logic of the applicative theories with types, which will be considered below, is the (classical) *logic of partial terms* due to Beeson [1], which is equivalent to the E^+-logic with equality and strictness of Troelstra and van Dalen [16].

3. Elementary explicit type theory EET

Now we turn to the elementary explicit type theory **EET**. It is a second-order theory whose first-order part (axioms (1)–(11) below) corresponds to the theory **BON** of Feferman and Jäger [9] and provides a series of axioms concerning the individuals and their applicative behaviour. We repeat them for completeness:

I. *Partial combinatory algebra*

(1) $\mathbf{k}xy = x$,

(2) $\mathbf{s}xy\downarrow \wedge \mathbf{s}xyz \simeq xz(yz)$.

II. *Pairing and projection*

(3) $(x, y)\downarrow \wedge \mathbf{p}_0(x, y) = x \wedge \mathbf{p}_1(x, y) = y$,

(4) $(x, y) \neq 0$.

III. *Natural numbers*

(5) $0 \in N \wedge (\forall x \in N)(x' \in N)$,

(6) $(\forall x \in N)(x' \neq 0 \wedge \mathbf{p}_N(x') = x)$,

(7) $(\forall x \in N)(x \neq 0 \rightarrow \mathbf{p}_N x \in N \wedge (\mathbf{p}_N x)' = x)$.

IV. *Definition by cases on N*

(8) $a \in N \wedge b \in N \wedge a = b \rightarrow \mathbf{d}_N xyab = x$,

(9) $a \in N \wedge b \in N \wedge a \neq b \rightarrow \mathbf{d}_N xyab = y$.

V. *Primitive recursion on N*

(10) $(f:N \to N) \wedge (g:N^3 \to N) \to (\mathbf{r}_N fg:N^2 \to N)$,

(11) $(f:N \to N) \wedge (g:N^3 \to N) \wedge x \in N \wedge y \in N \wedge h = \mathbf{r}_N fg \to hx0 = fx \wedge hx(y') = gxy(hxy)$.

Hence the individuals form a partial combinatory algebra equipped with pairing and unpairing, with natural numbers and their usual successor and predecessor functions and with definition by numerical cases; $\mathbf{r}_N$ acts as a recursion operator which guarantees closure under primitive recursion.

Remark 1. It is mentioned in [9] that the possibility of lambda abstraction and the recursion theorem follow from the axioms (1) and (2). Proofs of both results are given in [3, 1].

Now we turn to the second-order part of **EET**, i.e. we add axioms for types and names. As mentioned above, $\mathfrak{R}(s, A)$ informally means that s represents the type A or is a name of A. The internal representation of types by their names will be intensional whereas types themselves should be considered extensionally.

VI. *Extensionality*

(Ext) $\quad (\forall x)(x \in A \leftrightarrow x \in B) \to A = B.$

In the following axioms about explicit representation and elementary comprehension the relation symbol $\mathfrak{R}$ and the constants $\mathbf{c}_e$ play a major role. The explicit representation axioms state that every type has a name. The constants $\mathbf{c}_e$ serve to assign names to all type terms in such a way that this assignment is uniform in the parameters which occur in the type terms.

VII. *Explicit representation*

(E.1) $\quad (\exists x)\mathfrak{R}(x, A)$,

(E.2) $\quad \mathfrak{R}(a, B) \wedge \mathfrak{R}(a, C) \to B = C.$

It will be convenient for the naming process in connection with elementary comprehension, which is described below, to make use of the following conventions:

(1) We assume that there is some arbitrary but fixed standard assignment of Gödel numbers to the formulas of $\mathbb{L}_p$.

(2) We assume further that $v0, v1, \ldots$ and $V0, V1, \ldots$ are arbitrary but fixed enumerations of the individual and type variables. If φ is an $\mathbb{L}_p$ formula with no other individual variables than $v0, \ldots, vm$ and no other type variables than $V0, \ldots, Vn$ and if $\vec{x} = x_0, \ldots, x_m$ and $\vec{Y} = Y_0, \ldots, Y_n$, then we write $\varphi[\vec{x}, \vec{Y}]$ for the $\mathbb{L}_p$ formula which results from φ by a simultaneous replacement of vi by x_i and Vj by Y_j ($0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$).

(3) Finally, if $\vec{x} = x_0, \ldots, x_n$ and $\vec{X} = X_0, \ldots, X_n$, then $\mathfrak{R}(\vec{x}, \vec{X})$ stands for $\bigwedge_{i=0}^{n} \mathfrak{R}(x_i, X_i)$.

The following axioms depend on the Gödel numbering and the enumeration of the variables of $\mathbb{L}_p$. However, for obvious reasons this is not a serious restriction.

VIII. *Elementary comprehension.* Let $\varphi[x,\vec{y},\vec{Z}]$ be an elementary $\mathbb{L}_p$ formula and e its Gödel number; then we have

$$\text{(ECA.1)} \qquad (\exists X)(\forall x)(x \in X \leftrightarrow \varphi[x,\vec{a},\vec{B}])$$

$$\text{(ECA.2)} \qquad \Re(\vec{b},\vec{B}) \wedge (\forall x)(x \in A \leftrightarrow \varphi[x,\vec{a},\vec{B}]) \to \Re(\mathbf{c}_e(\vec{a},\vec{b}), A).$$

Hence the constants $\mathbf{c}_e$ serve to provide names for the types which are generated by the formulas of the corresponding Gödel numbers. This assignment of names to types is uniform in the individual and type variables which occur in the defining formula.

Remark 2. It is also possible to give a simple and very natural finite axiomatization of elementary comprehension. The basic idea is that one singles out a few operations, which are instances of (ECA.1) and (ECA.2), so that the full scheme can be obtained by combination of these operations. See the Appendix for more details.

The elementary explicit type theory **EET** is defined to be the $\mathbb{L}_p$ theory which consists of the individual axioms (1)–(11), extensionality, the explicit representation axioms (E.1) and (E.2) and the axioms about elementary comprehension (ECA.1) and (ECA.2). An essentially equivalent formulation of **EET** was first introduced in [11]; a similar theory is presented in [8]. The first-order theory **BON** of [9] is the subtheory of **EET** which consists of the axioms (1)–(11), restricted to the language L_p.

4. Forms of induction and minimum operator

In this section we repeat the exact formulations of set and formula induction on the natural numbers presented in [9] and introduce the new form of induction – type induction – which is specific for second-order theories. In addition, the axioms for the non-constructive unbounded minimum operator μ will be stated again.

First we recall from [9] that the *subsets of* N, in contrast to the subtypes of N, are identified with the characteristic functions which are total on N. Accordingly $P(N)$ is defined to be the type of all subsets of N, in the sense of characteristic functions, and we introduce the shorthand notation

$$a \in P(N) := (\forall x \in N)(ax = 0 \vee ax = 1).$$

The three principles of complete induction on the natural numbers which will be considered later are the following.

Set induction on N ($\mathbf{S\text{-}I_N}$):

$$a \in P(N) \wedge a0 = 0 \wedge (\forall x \in N)(ax = 0 \to ax' = 0) \;\; \to \;\; (\forall x \in N)(ax = 0),$$

Type induction on N $(\mathbf{T}\text{-}\mathbf{I_N})$:

$$0 \in A \wedge (\forall x \in N)(x \in A \rightarrow x' \in A) \;\rightarrow\; (\forall x \in N)(x \in A)$$

Formula induction on N $(\mathbf{F}\text{-}\mathbf{I_N})$:

$$\varphi(0) \wedge (\forall x \in N)(\varphi(x) \rightarrow \varphi(x')) \;\rightarrow\; (\forall x \in N)\varphi(x)$$

for all formulas φ of $\mathbb{L}_p$. Obviously $(\mathbf{S}\text{-}\mathbf{I_N})$ can be regarded as a special case of $(\mathbf{T}\text{-}\mathbf{I_N})$ and $(\mathbf{T}\text{-}\mathbf{I_N})$ as a special case of $(\mathbf{F}\text{-}\mathbf{I_N})$. Adding these induction principles to the theory **EET** yields the following new theories:

$$\mathbf{EET} + (\mathbf{S}\text{-}\mathbf{I_N}), \qquad \mathbf{EET} + (\mathbf{T}\text{-}\mathbf{I_N}), \qquad \mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N}).$$

In the following we will show that the theory $\mathbf{EET} + (\mathbf{S}\text{-}\mathbf{I_N})$ is a conservative extension of $\mathbf{BON} + (\mathbf{S}\text{-}\mathbf{I_N})$ and the theory $\mathbf{EET} + (\mathbf{T}\text{-}\mathbf{I_N})$ a conservative extension of $\mathbf{BON} + (\mathbf{F}\text{-}\mathbf{I_N})$. In view of the results of Feferman and Jäger [9] this means that $\mathbf{EET} + (\mathbf{S}\text{-}\mathbf{I_N})$ is proof-theoretically equivalent to primitive recursive arithmetic **PRA** and $\mathbf{EET} + (\mathbf{T}\text{-}\mathbf{I_N})$ to Peano arithmetic **PA**. We will also see that $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N})$ is of the same proof-theoretic strength as the theory $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})$ of second-order arithmetic with arithmetic comprehension.

Now we continue to follow the lines of [9] and add the non-constructive unbounded minimum operator μ. This functional assigns a natural number μf to each total function from N to N so that $f(\mu f) = 0$ if there exists an $x \in N$ with the property $fx = 0$. The exact axiomatization is as follows:

Axioms of the unbounded minimum operator:

$(\mu.1)$ $\quad (f : N \rightarrow N) \;\rightarrow\; \mu f \in N,$

$(\mu.2)$ $\quad (f : N \rightarrow N) \wedge ((\exists x \in N)(fx = 0) \;\rightarrow\; f(\mu f) = 0.$

We shall write $\mathbf{EET}(\mu)$ for $\mathbf{EET} + (\mu.1, \mu.2)$. In this paper we provide a proof-theoretic characterization of this system extended by $(\mathbf{S}\text{-}\mathbf{I_N})$, $(\mathbf{T}\text{-}\mathbf{I_N})$ and $(\mathbf{F}\text{-}\mathbf{I_N})$.

Remark 3. In [12] a stronger form of the unbounded minimum operator μ is considered in which the axiom $(\mu.1)$ is replaced by the equivalence

$$(f : N \rightarrow N) \;\leftrightarrow\; \mu f \in N.$$

However, it turns out that this modification does not affect the proof-theoretic strength of the theories considered in this article.

5. The proof-theoretic strength of EET and EET(μ) with set and with type induction

Determination of the proof-theoretic strength of **EET** and $\mathbf{EET}(\mu)$ with set and type induction on N is most simply established by a model-theoretic argument, showing that these second order theories are conservative over suitable first-order extensions of **BON**. A careful analysis of the following arguments makes clear that they can be formalized in suitable theories thus providing proof-theoretic reductions.

Lemma 4. *Let $\mathcal{M}$ be a model of* **BON**. *Then there exists a model $\mathcal{M}^*$ of* **EET** *which has the following properties*:

1. *$\mathcal{M}^*$ is an extension of $\mathcal{M}$ in the sense that we have for all sentences φ of L_p*

$$\mathcal{M}^* \models \varphi \;\Leftrightarrow\; \mathcal{M} \models \varphi.$$

2. *If $\mathcal{M}$ is a model of* $(\mathbf{F}\text{-}\mathbf{I}_{\mathrm{N}})$ *with respect to L_p, then $\mathcal{M}^*$ is a model of* $(\mathbf{T}\text{-}\mathbf{I}_{\mathrm{N}})$ *with respect to $\mathbb{L}_p$.*

Proof. Let $\mathcal{M}$ be a model of **BON** and assume that M is the universe of $\mathcal{M}$. As a first step we assign to each constant $\mathbf{c}_e$ an element $\hat{\mathbf{c}}_e$ of M so that (i) there is no conflict with the interpretation of the constants of L_p in $\mathcal{M}$ and (ii) $\hat{\mathbf{c}}_m v$ and $\hat{\mathbf{c}}_n w$ are different in $\mathcal{M}$ for all $m \neq n$ and all $v, w \in M$. Then, if T is a subset of the power set of M and R a subset of $M \times T$, we write $(\mathcal{M}, T, R)$ for the $\mathbb{L}_p$ structure in which the types range over T, the relation symbol $\in$ is interpreted as the restriction of the usual element relation to $M \times T$ and the symbols $\mathfrak{R}$ and c_e are interpreted as R and $\hat{\mathbf{c}}_e$, respectively.

The next step is to define, by induction on $k < \omega$, subsets R_k of M together with subsets $ty(w)$ of M for all $w \in R_k$; furthermore we write T_k for $\{ty(w) : w \in R_k\} : k = 0$. For every formula $\varphi[x, \vec{y}\,]$ of L_p with Gödel number e and for all $\vec{v} \in M$ we have $\hat{\mathbf{c}}_e(\vec{v}\,) \in R_0$ and set

$$ty(\hat{\mathbf{c}}_e(\vec{v}\,)) := \{m \in M : \mathcal{M} \models \varphi[m, \vec{v}\,]\}.$$

$k > 0 : R_k$ contains R_{k-1}. In addition, for every elementary formula $\varphi[x, \vec{y}, \vec{Z}\,]$ of $\mathbb{L}_p$ with Gödel number e and for all $\vec{v} \in M$ and $\vec{w} \in R_{k-1}$ we have $\hat{\mathbf{c}}_e(\vec{v}, \vec{w}\,) \in R_k$ and set

$$ty(\hat{\mathbf{c}}_e(\vec{v}, \vec{w})) := \{m \in M :\ (\mathcal{M}, T_{k-1}, \emptyset) \models \varphi[m, \vec{v}, ty(\vec{w}\,)]\}.$$

Here we use the shorthand notation that $ty(\vec{w}\,)$ stands for $ty(w_1), \ldots, ty(w_n)$ if $\vec{w}$ is the sequence $w_1, \ldots, w_n$.

Based on these definitions we now introduce collections T and R in order to interpret the types and the representation relation:

$$T := \bigcup_{k<\omega} T_k \quad \text{and} \quad R := \bigcup_{k<\omega} \{(w, ty(w)) :\ w \in R_k\}.$$

It is then easily seen that $\mathcal{M}^* := (\mathcal{M}, T, R)$ provides a model of **EET** which possesses all the claimed properties. □

Observe that the axioms of the unbounded minimum operator are first order. Because of its first part, the previous lemma therefore also applies to **BON**(μ) and **EET**(μ). It immediately implies the following theorem which characterizes some extensions of the second-order theories **EET** and **EET**(μ) in terms of extensions of the first-order theories **BON** and **BON**(μ), respectively.

Theorem 5 (Conservative extensions). *We have*:
1. **EET** + (**S-I**$_{\mathrm{N}}$) *is a conservative extension of* **BON** + (**S-I**$_{\mathrm{N}}$);
2. **EET** + (**T-I**$_{\mathrm{N}}$) *is a conservative extension of* **BON** + (**F-I**$_{\mathrm{N}}$);
3. **EET**(μ) + (**S-I**$_{\mathrm{N}}$) *is a conservative extension of* **BON**(μ) + (**S-I**$_{\mathrm{N}}$);
4. **EET**(μ) + (**T-I**$_{\mathrm{N}}$) *is a conservative extension of* **BON**(μ) + (**F-I**$_{\mathrm{N}}$).

In view of the results of Feferman and Jäger [9] we now obtain a proof-theoretic characterization of some explicit type theories based on **EET**. In the following corollary $(\mathbf{\Pi}^0_\infty\textbf{-CA})_{<\varepsilon_0}$ is the well-known system of second-order arithmetic for iterated arithmetic comprehension through each ordinal less than ε_0 which is discussed in detail for example in [2, 10].

Corollary 6. *We have the following proof-theoretic equivalences*:

$$\mathbf{EET} + (\mathbf{S\text{-}I_N}) \equiv \mathbf{PRA}, \qquad \mathbf{EET}(\mu) + (\mathbf{S\text{-}I_N}) \equiv \mathbf{PA},$$

$$\mathbf{EET} + (\mathbf{T\text{-}I_N}) \equiv \mathbf{PA}, \qquad \mathbf{EET}(\mu) + (\mathbf{T\text{-}I_N}) \equiv (\mathbf{\Pi}^0_\infty\textbf{-CA})_{<\varepsilon_0}.$$

Lemma 4 cannot be used, however, to analyse the theories **EET** + (**F-I**$_{\mathrm{N}}$) and **EET**(μ) + (**F-I**$_{\mathrm{N}}$). We cannot expect that the structures $\mathcal{M}^*$ defined in Lemma 4 satisfy formula induction, even when $\mathcal{M}$ does.

6. The proof-theoretic strength of EET and EET(μ) with formula induction

The proof-theoretic strength of **EET** + (**F-I**$_{\mathrm{N}}$) has already been determined, among other things, in [14] so that the corresponding result is stated here for completeness only. The main focus in this section is on the proof-theoretic analysis of **EET**(μ) + (**F-I**$_{\mathrm{N}}$).

6.1. Lower bounds

The lower bounds of the theories **EET** + (**F-I**$_{\mathrm{N}}$) and **EET**(μ) + (**F-I**$_{\mathrm{N}}$) are easily established by embedding suitable systems of second-order arithmetic in them. We do not want to give a very detailed description of these systems now and confine ourselves to mentioning the basic principles. All details are already presented in [9].

The language $\mathcal{L}_2$ of second-order arithmetic contains number variables $(x, y, z, \dots)$, set variables $(X, Y, Z, \dots)$, the constant 0 and symbols for all primitive recursive functions and relations. An $\mathcal{L}_2$ formula is called *arithmetic* if it does not contain bound set variables, whereas free set variables are permitted.

The first such theory, which will be important for us, is the system $(\mathbf{\Pi}^0_\infty\textbf{-CA})$ which contains the usual machinery of primitive recursive functions and relations, the scheme of complete induction on the natural numbers for all $\mathscr{L}_2$ formulas plus the scheme of *arithmetic comprehension* for all arithmetic $\mathscr{L}_2$ formulas:

$$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x)).$$

For defining the second theory, we refer to (initial segments of) the standard well-ordering $\prec$ of order type Γ_0, which is described, for example, in [15]. There one also finds the definition of the ordinal number $\varepsilon_{\varepsilon_0}$, which is crucial for us here. Finally, if n is a natural number, then we write $\prec_n$ for the restriction of $\prec$ to numbers the $m \prec n$, and *transfinite induction* $TI(\alpha, \varphi)$ up to α with respect to φ is the formula

$$(\forall x \prec n)((\forall y \prec x)\varphi(y) \rightarrow \varphi(x)) \rightarrow (\forall x \prec n)\varphi(x),$$

provided that the order type of the well-ordering $\prec_n$ is the ordinal α; for each $\alpha < \Gamma_0$ there is exactly one such n.

$(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_{\varepsilon_0}}$ is the extension of $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})$ obtained by adding axioms which state that the arithmetic comprehension axioms can be iterated through all ordinals less than $\varepsilon_{\varepsilon_0}$, plus transfinite induction $TI(\alpha, \varphi)$ for all $\alpha < \varepsilon_{\varepsilon_0}$ and all $\mathscr{L}_2$ formulas φ. More details on such second-order theories can be found in [2, 9, 10].

Now we consider two forms of interpreting the language $\mathscr{L}_2$ of second-order arithmetic into $\mathbb{L}_p$: In both cases the number variables of $\mathscr{L}_2$ are interpreted as ranging over N and the primitive recursive functions and relations are represented by individual terms of $\mathbb{L}_p$, which is possible by making use of the recursion operator $\mathbf{r}_N$. However, in the first case the set variables of $\mathscr{L}_2$ are interpreted as ranging over subtypes of N, whereas in the second case the set variables of $\mathscr{L}_2$ are interpreted as ranging over $P(N)$.

Hence every $\mathscr{L}_2$ formula φ is translated according to the first interpretation into an $\mathbb{L}_p$ formula $\varphi^{\mathscr{N}}$ so that

$$((\exists x)\varphi(x))^{\mathscr{N}} := (\exists x \in N)\varphi^{\mathscr{N}}(x),$$

$$((\exists Y)\varphi(Y))^{\mathscr{N}} := (\exists Y)(Y \subset N \wedge \varphi^{\mathscr{N}}(Y)).$$

This is in contrast to the second interpretation which translates the same $\mathscr{L}_2$ formula φ into an $\mathbb{L}_p$ formula (even an L_p formula) φ^N which deals with the number and set variables according to

$$((\exists x)\varphi(x))^N := (\exists x \in N)\varphi^N(x),$$

$$((\exists Y)\varphi(Y))^N := (\exists y)(y \in P(N) \wedge \varphi^N(y)).$$

Observe that these two interpretations do not differ for arithmetic formulas of $\mathscr{L}_2$ without free set variables.

It is now easy to see that the theory $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})$ is contained in $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I}_\mathbf{N})$ via the first of these interpretations. Elementary comprehension in $\mathbf{EET}$ takes care of the scheme of arithmetic comprehension in $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})$, and all other steps are straightforward.

Proposition 7. *We have for all $\mathscr{L}_2$ sentences φ:*

$$(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA}) \vdash \varphi \;\Rightarrow\; \mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N}) \vdash \varphi^{\,\mathsf{I}}.$$

Now we consider $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$. It is a well-known result in proof theory that $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})$ proves $(\forall X)TI(\alpha, X)$ for all $\alpha < \varepsilon_{\varepsilon_0}$; cf. e.g. [15]. Hence we may conclude that $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N})$ proves $TI(\alpha, \varphi)$ for all elementary $\mathbb{L}_p$ formulas φ and all $\alpha < \varepsilon_{\varepsilon_0}$.

Therefore the stage is set for embedding $(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_{\varepsilon_0}}$ into $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$. One only has to follow the proof of Theorems 9 and 10 of [9] in order to make sure that (the translation of) the arithmetic comprehension axioms can be iterated through all ordinals less than $\varepsilon_{\varepsilon_0}$. The proof is exactly as in [9], with the only difference being that in $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$ we have transfinite induction for elementary formulas for all ordinals less than $\varepsilon_{\varepsilon_0}$, whereas in the theory $\mathbf{BON}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$ of [9] it was available only for all $\alpha < \varepsilon_0$.

Proposition 8. *We have for all $\mathscr{L}_2$ sentences φ:*

$$(\mathbf{\Pi}^0_\infty\text{-}\mathbf{CA})_{<\varepsilon_{\varepsilon_0}} \vdash \varphi \;\Rightarrow\; \mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N}) \vdash \varphi^N.$$

6.2. *The theory* $\widehat{\mathbf{E\Omega}}$

In this section we introduce the theory $\widehat{\mathbf{E\Omega}}$ of second-order arithmetic plus ordinals, which will be used later to determine the upper bound for $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$. The theory $\widehat{\mathbf{E\Omega}}$ corresponds to the second-order version of the system $\mathbf{PA}^w_\Omega$ of [9] with the possibility to form subsets of the natural numbers by means of elementary comprehension. $\widehat{\mathbf{E\Omega}}$ and related systems are fully described and proof-theoretically analysed in [13].

Let P be a new n-ary relation symbol, i.e. a relation symbol which does not belong to the language $\mathscr{L}_2$. Then $\mathscr{L}_2(P)$ is the extension of $\mathscr{L}_2$ by P. An $\mathscr{L}_2(P)$ formula is called *P-positive* if each occurrence of P in this formula is positive. We call P-positive formulas without free or bound set variables, which contain at most $\vec{x}$ free, *inductive operator forms*, and let $A(P, \vec{x})$ range over such forms.

$\mathscr{L}_\Omega$ results from $\mathscr{L}_2$ by adding a new sort of *ordinal variables* $(\alpha, \beta, \gamma, \alpha_0, \beta_0, \gamma_0, \ldots)$, a new binary relation symbol $<$ for the less relation on the ordinals[1] and an $(n+1)$-ary relation symbol P_A for each inductive operator form $A(P, \vec{x})$ for which P is n-ary.

The *number terms* $(s, t, s_0, t_0, \ldots)$ of $\mathscr{L}_\Omega$ are the number terms of $\mathscr{L}_2$, and the *formulas* $(\varphi, \psi, \chi, \varphi_0, \psi_0, \chi_0, \ldots)$ of $\mathscr{L}_\Omega$ are inductively generated as follows:

1. All atomic formulas of $\mathscr{L}_2$ are atomic formulas of $\mathscr{L}_\Omega$; further atomic formulas of $\mathscr{L}_\Omega$ are $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_A(\alpha, \vec{s})$; we write $P^\alpha_A(\vec{s})$ for $P_A(\alpha, \vec{s})$.

[1] It will always be clear from the context whether $<$ denotes the less relation on the nonnegative integers or on the ordinals.

2. If φ and ψ are formulas of $\mathscr{L}_\Omega$, then $\neg\varphi$ and $\varphi \vee \psi$ are formulas of $\mathscr{L}_\Omega$.

3. If φ is a formula of $\mathscr{L}_\Omega$, then $(\exists x)\varphi$, $(\forall x)\varphi$, $(\exists X)\varphi$, $(\forall X)\varphi$, $(\exists\alpha)\varphi$ and $(\forall\alpha)\varphi$ are formulas of $\mathscr{L}_\Omega$.

If $\varphi(P)$ is an $\mathscr{L}_2(P)$ formula and $\psi(\vec{x})$ an $\mathscr{L}_\Omega$ formula (where P is n-ary and $\vec{x}$ is the sequence $x_1, \ldots, x_n$), then $\varphi(\psi)$ denotes the result of substituting $\psi(\vec{s})$ for every occurrence of $P(\vec{s})$ in $\varphi(P)$. For every $\mathscr{L}_\Omega$ formula φ we write φ^α to denote the $\mathscr{L}_\Omega$ formula which is obtained by replacing all unbounded ordinal quantifiers $(Q\beta)$ in φ by $(Q\beta < \alpha)$. Additional abbreviations are

$$P_A^{<\alpha}(\vec{s}) := (\exists\beta < \alpha)P_A^\beta(\vec{s}) \quad \text{and} \quad P_A(\vec{s}) := (\exists\alpha)P_A^\alpha(\vec{s}).$$

An $\mathscr{L}_\Omega$ formula without free or bound set variables is called a Δ_0^Ω formula if all its ordinal quantifiers are bounded; it is called a Σ^Ω formula if all positive universal ordinal quantifiers and all negative existential ordinal quantifiers are bounded; correspondingly, it is called a Π^Ω formula if all negative universal ordinal quantifiers and all positive existential ordinal quantifiers are bounded. The *elementary* $\mathscr{L}_\Omega$ formulas are the $\mathscr{L}_\Omega$ formulas without bound set variables; free set variables, however, are permitted in elementary formulas. See [13] for the precise definitions.

The theory $\widehat{\mathbf{E\Omega}}$ is formulated in the language $\mathscr{L}_\Omega$ and contains the usual axioms of predicate logic with equality plus the following non-logical axioms.

Number-theoretic axioms. These comprise the axioms of Peano arithmetic **PA** with the exception of complete induction on the natural numbers.

Inductive operator axioms. For all inductive operator forms $A(P, \vec{x})$:

$$P_A^\alpha(\vec{s}) \leftrightarrow A(P_A^{<\alpha}, \vec{s}).$$

Σ^Ω *reflection axioms.* For every Σ^Ω formula φ:

$$(\Sigma^\Omega\text{-Ref}) \qquad \varphi \rightarrow (\exists\alpha)\varphi^\alpha.$$

Linearity of the relation $<$ *on the ordinals*

$$(\text{LO}) \qquad \alpha \nless \alpha \wedge (\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha).$$

Elementary comprehension. For every elementary formula $\varphi(x)$ of $\mathscr{L}_\Omega$:

$$(\text{ECA}) \qquad (\exists X)(\forall y)(y \in X \leftrightarrow \varphi(y)).$$

Full induction on the natural numbers. For all $\mathscr{L}_\Omega$ formulas $\varphi(x)$:

$$(\mathscr{L}_\Omega\text{-}\mathbf{I}_\mathbf{N}) \qquad \varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x')) \;\rightarrow\; (\forall x)\varphi(x).$$

Δ_0^Ω *induction on the ordinals.* For all Δ_0^Ω formulas $\varphi(\alpha)$:

$$(\Delta_0^\Omega\text{-}\mathbf{I}_\Omega) \qquad (\forall\alpha)((\forall\beta < \alpha)\varphi(\beta) \rightarrow \varphi(\alpha)) \;\rightarrow\; (\forall\alpha)\varphi(\alpha).$$

It follows from these axioms that $\widehat{\mathbf{E\Omega}}$ contains the theory $\mathbf{PA}_\Omega^w$ of [9]. However, the following theorem, which is proved in [13], shows that $\widehat{\mathbf{E\Omega}}$ is significantly stronger than $\mathbf{PA}_\Omega^w$.

Theorem 9. $\widehat{\mathbf{E\Omega}}$ *is proof-theoretically equivalent to* $(\mathbf{\Pi}^0_\infty\textbf{-CA})_{<\varepsilon_{\varepsilon_0}}$.

6.3. *Upper bounds*

In this section we concentrate on a proof of the fact that $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$ can be embedded into $\widehat{\mathbf{E\Omega}}$. As already mentioned, the theory $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N})$ is studied in [14], and the following result is proved there.

Proposition 10. $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N})$ *can be embedded into* $(\mathbf{\Pi}^0_\infty\textbf{-CA})$.

The main difference in the analysis of $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N})$ and $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$ is the way in which their applicative parts are modeled. The types and the type representation machinery are then treated in more or less the same manner. In case of the theory $\mathbf{EET} + (\mathbf{F}\text{-}\mathbf{I_N})$ term application is interpreted in the sense of ordinary recursion theory by translating the $\mathbb{L}_p$ term $(a \cdot b)$ into $\{a\}(b)$, where $\{e\}$ for $e = 0, 1, 2, \ldots$ is the standard enumeration of the partial recursive functions.

In the presence of the unbounded minimum operator a more elaborate approach is necessary, and the theories with ordinals turn out to be an adequate tool for handling term application. In order to deal with the first-order part of $\mathbf{EET}(\mu) + (\mathbf{F}\text{-}\mathbf{I_N})$ we will now follow Feferman and Jäger [9] and make use of the methods developed there.

First some standard notations of first- and second-order arithmetic: $\langle \cdots \rangle$ is a standard primitive recursive function for forming n-tuples $\langle t_0, \ldots, t_{n-1} \rangle$; Seq is the primitive recursive set of sequence numbers; $lh(t)$ denotes the length of the sequence number t; Seq_n is the set of sequence numbers of length n; $(t)_i$ is the ith component of the sequence number t, i.e. $t = \langle (t)_0, \ldots, (t)_{lh(t)-1} \rangle$; $s \in (X)_t$ stands for $\langle s, t \rangle \in X$.

Then we choose suitable numerals $\hat{\mathbf{k}}, \hat{\mathbf{s}}, \hat{\mathbf{p}}, \hat{\mathbf{p}}_0, \hat{\mathbf{p}}_1, \hat{\mathbf{s}}_N, \hat{\mathbf{p}}_N, \hat{\mathbf{d}}_N, \hat{\mathbf{r}}_N$ and $\hat{\mu}$ as interpretations of the corresponding individual constants of L_p. A further numeral $\hat{\mathbf{c}}$ is used to interpret the individuals constants $\mathbf{c}_e$ of $\mathbb{L}_p$ uniformly as $\langle \hat{\mathbf{c}}, e \rangle$. We also assume that the individual variables and type variables of $\mathbb{L}_p$ are properly mapped on (or simply identified with) the number variables and set variables of $\mathscr{L}_\Omega$.

Since $\widehat{\mathbf{E\Omega}}$ contains $\mathbf{PA}^w_\Omega$, we know from [9] that there is a Σ^Ω formula $App(x, y, z)$ of $\mathscr{L}_\Omega$ which is tailored for taking care of term application. Based on this formula, we associate to each term t of $\mathbb{L}_p$ a Σ^Ω formula $Val_t(x)$, which is inductively defined as follows:

1. If t is an individual variable or an individual constant of $\mathbb{L}_p$ and $\hat{t}$ the corresponding term of $\mathscr{L}_\Omega$ in the sense of the previous paragraph, then $Val_t(x)$ is simply the formula $(\hat{t} = x)$.
2. If t is the individual term (rs), then $Val_t(x)$ is the formula

$$(\exists y)(\exists z)(Val_r(y) \wedge Val_s(z) \wedge App(y, z, x)).$$

The translations φ^* of elementary $\mathbb{L}_p$ formulas φ are then inductively defined as follows:

1. For atomic formulas of $\mathbb{L}_p$ which do not contain type variables we put

$$(t\downarrow)^* := (\exists x)Val_t(x), \qquad (s = t)^* := (\exists x)(Val_s(x) \wedge Val_t(x)),$$

$$N(t)^* := (\exists x)Val_t(x), \qquad (t \in X)^* := (\exists x)(Val_t(x) \wedge x \in X).$$

2. If φ is the formula $\neg\psi$, then φ^* is $\neg\psi^*$; if φ is the formula $(\psi \vee \chi)$, then φ^* is $(\psi^* \vee \chi^*)$; if φ is the formula $(\exists x)\psi$, then φ^* is $(\exists x)\psi^*$.

It follows immediately from the results of [9] that this translation of the first-order part of $\mathbb{L}_p$ provides a sound interpretation of the first-order axioms of $\mathbf{EET}(\mu)$ into $\widehat{\mathbf{E\Omega}}$.

Lemma 11. *If φ is one of the axioms* (1)–(11) *of* **EET** *or one of the axioms of the unbounded minimum operator, then* $\widehat{\mathbf{E\Omega}}$ *proves* φ^*.

Now we turn to the second-order part of $\mathbf{EET}(\mu)$, which comprises types and the explicit representation of types. The basic idea is to take the elementary definable subsets of the natural numbers as types and to use the Gödel numbers of their definitions as representations. This has to be done carefully to ensure the strong uniformity which is required in **EET** by elementary comprehension.

It follows from [9] that the formula $App(x, y, z)$ can be chosen so that the individual terms $\mathbf{c}_e(\vec{a})$ of $\mathbb{L}_p$, which serve as explicit representation of types, are translated in $\widehat{\mathbf{E\Omega}}$ as $\langle\hat{\mathbf{c}}, e, \langle\vec{a}\rangle\rangle$; i.e. we may assume that $\widehat{\mathbf{E\Omega}}$ proves

$$(s = \mathbf{c}_e(t_0, \ldots, t_{n-1}))^* \wedge Val_s(x) \wedge \bigwedge_{i=0}^{n-1} Val_{t_i}(y_i) \rightarrow x = \langle\hat{\mathbf{c}}, e, \langle y_0, \ldots, y_{n-1}\rangle\rangle.$$

The $*$-translation of an elementary $\mathbb{L}_p$ formula φ yields an elementary $\mathscr{L}_\Omega$ formula φ^*. Hence standard arguments show that there is a truth definition in $\widehat{\mathbf{E\Omega}}$ for the elementary formulas of $\mathbb{L}_p$. However, this truth definition is not elementary.

Proposition 12. *There exists an $\mathscr{L}_\Omega$ formula $\mathsf{Tr}(x, y, Z)$ so that $\widehat{\mathbf{E\Omega}}$ proves for every elementary $\mathbb{L}_p$ formula $\varphi[x_0, \ldots, x_{m-1}, Y_0, \ldots, Y_{n-1}]$ and its Gödel number e:*

$$\mathsf{Tr}(e, x, Y) \leftrightarrow \varphi^*[(x)_0, \ldots, (x)_{m-1}, (Y)_0, \ldots, (Y)_{n-1}].$$

In the following we write $\mathsf{Ele}(e, m, n)$ for the primitive-recursive relation which expresses that e is the Gödel number of an elementary formula of $\mathbb{L}_p$ which has at most m free individual variables and n free type variables. Following the pattern of the proof of Lemma 4 we begin with introducing codes for the types. Obviously this can be done in a primitive-recursive way.

Proposition 13. *There exists a primitive-recursive relation* Code *such that one can prove in* $\widehat{\mathbf{E\Omega}}$:

$$\mathsf{Code}(x) \leftrightarrow \begin{cases} (\exists e, s, m, n)[\mathsf{Ele}(e, m+1, n) \wedge Seq_{m+n}(s) \\ \quad \wedge (\forall i < n)\mathsf{Code}((s)_{m+i}) \wedge x = \langle \hat{\mathbf{c}}, e, s \rangle]. \end{cases}$$

Finally the Gödel numbers of elementary formulas without type variables and at most one free individual variable will determine our types. However, the comprehension principles of **EET** and the codes permit parameters so that an additional consideration is necessary. The basic idea is to associate Gödel numbers e which satisfy $\mathsf{Ele}(e, 1, 0)$ to the above defined type codes.

Let f be an arbitrary unary primitive-recursive function. Then we introduce the following abbreviation, which is convenient for formulating Proposition 14 below:

$$\mathscr{A}_f(e, s, X) := \begin{cases} (\exists m, n)[\mathsf{Ele}(e, m+1, n) \wedge Seq_{m+n}(s) \wedge (\forall i < n)\mathsf{Code}((s)_{m+i}) \\ \quad \wedge (\forall i < n)(\forall x)(x \in (X)_i \leftrightarrow \mathsf{Tr}(f((s)_{m+i}), \langle x \rangle, \emptyset))]. \end{cases}$$

If f is a function which assigns Gödel numbers of formulas to codes, then $\mathscr{A}_f(e, s, X)$ means: (i) e is the Gödel number of an elementary formula with $m+1$ individual and n type variables; (ii) s represents a sequence of m number parameters and n type codes; (iii) the set X contains the information about the sets which are defined by the formulas which correspond to these n type codes via f.

By some straightforward but tedious manipulations with Gödel numbers and some coding arguments it is possible to find primitive-recursive translations of type codes into Gödel numbers of appropriate formulas. More precisely:

Proposition 14. *There exists a primitive-recursive function* Φ *so that one can prove in* $\widehat{\mathbf{E\Omega}}$:

1. $\mathsf{Code}(x) \rightarrow \mathsf{Ele}(\Phi(x), 1, 0)$.
2. $\mathscr{A}_\Phi(e, s, X) \rightarrow [\mathsf{Tr}(\Phi(\langle \hat{\mathbf{c}}, e, s \rangle), \langle x \rangle, \emptyset) \leftrightarrow \mathsf{Tr}(e, \langle x \rangle * s, X)]$.

Now we are ready to tackle the second-order part of **EET** as well. First we introduce two more definitions:

$$Rep(e, X) := \mathsf{Code}(e) \wedge (\forall x)(x \in X \leftrightarrow \mathsf{Tr}(\Phi(e), \langle x \rangle, \emptyset)),$$

$$Type(X) := (\exists e) Rep(e, X).$$

The last step is now to extend the $*$-translation from the elementary formulas of $\mathbb{L}_p$ to all formulas of $\mathbb{L}_p$, which is inductively done as follows:

1. If φ is an elementary $\mathbb{L}_p$ formula, then φ^* is already defined.
2. $\mathfrak{R}(t, X)^* := (\exists x)(Val_t(x) \wedge Rep(x, X))$.
3. If φ is a non-elementary formula of the form $\neg\psi$, $(\psi \vee \chi)$ or $(\exists x)\psi$, then φ^* is $\neg\psi^*$, $(\psi^* \vee \chi^*)$ or $(\exists x)\psi^*$, respectively.
4. If φ is the formula $(\exists X)\psi$, then φ^* is the formula $(\exists X)(Type(X) \wedge \varphi^*)$.

In view of Propositions 12–14 it follows that the translations of the explicit representation and elementary comprehension axioms are provable in $\widehat{\mathbf{E\Omega}}$. Since full induction on the natural numbers is available in $\widehat{\mathbf{E\Omega}}$, it is also clear that the translations of all instances of formula induction of **EET** do not create problems. Together with Lemma 11 we therefore obtain the following theorem.

Theorem 15. *Let φ be a closed formula of $\mathbb{L}_p$. Then we have*

$$\mathbf{EET}(\mu) + (\mathbf{F\text{-}I_N}) \vdash \varphi \;\Rightarrow\; \widehat{\mathbf{E\Omega}} \vdash \varphi^*.$$

This was the last step which was missing for providing the proof-theoretic analysis of the theories $\mathbf{EET} + (\mathbf{F\text{-}I_N})$ and $\mathbf{EET}(\mu) + (\mathbf{F\text{-}I_N})$. The following result is an immediate consequence of the previous theorem and Propositions 7, 8, 10 and Theorem 9.

Corollary 16. *We have the following proof-theoretic equivalences:*
1. $\mathbf{EET} + (\mathbf{F\text{-}I_N}) \equiv (\mathbf{\Pi^0_\infty\text{-}CA})$.
2. $\mathbf{EET}(\mu) + (\mathbf{F\text{-}I_N}) \equiv \widehat{\mathbf{E\Omega}} \equiv (\mathbf{\Pi^0_\infty\text{-}CA})_{<\varepsilon_{\varepsilon_0}}$.

Appendix

Now we present a finite axiomatization $\mathbf{EET}^f$ of **EET**. To this end we only have to show that the scheme (ECA), which consists of two parts (ECA.1) and (ECA.2), can be replaced by a finite number of its instances.

The language $\mathbb{L}(\mathbf{EET}^f)$ of $\mathbf{EET}^f$ is like the language of **EET** with the only difference that the infinitely many constants $\mathbf{c}_e$ $(e < \omega)$ are replaced by the following new individual constants: $\underline{\mathrm{nat}}$, $\underline{\mathrm{id}}$, $\underline{\mathrm{co}}$, $\underline{\mathrm{int}}$, $\underline{\mathrm{inv}}$ and $\underline{\mathrm{dom}}$.

The axioms of $\mathbf{EET}^f$ are the axioms of **EET** except the instances of (ECA) which are replaced by the universal closures of the following twelve axioms:

Natural numbers

(i) $(\exists X)(\forall x)(x \in X \leftrightarrow N(x))$,

(ii) $(\forall x)(x \in A \leftrightarrow N(x)) \rightarrow \Re(\underline{\mathrm{nat}}, A)$,

Identity

(iii) $(\exists X)(\forall x)(x \in X \leftrightarrow (\exists y)((x = (y, y)))$

(iv) $(\forall x)(x \in A \leftrightarrow (\exists y)(x = (y, y))) \rightarrow \Re(\underline{\mathrm{id}}, A)$,

Complements

(v) $(\exists X)(\forall x)(x \in X \leftrightarrow x \notin B)$,

(vi) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow x \notin B) \rightarrow \Re(\underline{\mathrm{co}}\, b, A)$,

Intersections

(vii) $(\exists X)(\forall x)(x \in X \leftrightarrow x \in B \wedge x \in C)$,

(viii) $\Re(b, B) \wedge \Re(c, C) \wedge (\forall x)(x \in A \leftrightarrow x \in B \wedge x \in C) \rightarrow \Re(\underline{\mathrm{int}}\,(b, c), A)$, *Domains*

(ix) $(\exists X)(\forall x)(x \in X \leftrightarrow (\exists y)((x, y) \in B))$,

(x) $\Re(b, B) \wedge (\forall x)(x \in X \leftrightarrow (\exists y)((x, y) \in B)) \rightarrow \Re(\underline{\mathrm{dom}}\, b, A)$,

Inverse images

(xi) $(\exists X)(\forall x)(x \in X \leftrightarrow ax \in B)$,

(xii) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow ax \in B) \rightarrow \Re(\underline{\mathrm{inv}}(a, b), A)$.

It is obvious that $\mathbf{EET}^f$ can be embedded into **EET**. On the other hand it is also possible to interpret **EET** into $\mathbf{EET}^f$. This is a consequence of the following proposition, whose proof is left to the reader as an exercise.

Proposition. *Let $\varphi[x, \vec{y}, \vec{Z}]$ be an elementary formula of $\mathbb{L}(\mathbf{EET}^f)$ and e its Gödel number. Then there exists a term t of $\mathbb{L}(\mathbf{EET}^f)$, depending on e, so that we have:*

1. $(\exists X)(\forall x)(x \in X \leftrightarrow \varphi[x, \vec{a}, \vec{B}])$,
2. $\Re(\vec{b}, \vec{B}) \wedge (\forall x)(x \in A \leftrightarrow \varphi[x, \vec{a}, \vec{B}]) \rightarrow \Re(t(\vec{a}, \vec{b}), A)$.

References

[1] M.J. Beeson, Foundations of Constructive Mathematics (Springer, Berlin, 1985).

[2] S. Feferman, Formal theories for transfinite iterations of generalized inductive definitions and some subsystems of analysis, in: A. Kino, J. Myhill and R.E. Vesley, eds., Intuitionism and Proof Theory, Proceedings of the Summer Conference at Buffalo, New York, 1968 (North-Holland, Amsterdam, 1970) 303–326.

[3] S. Feferman, A language and axioms for explicit mathematics, in: J. Crossley, ed., Algebra and Logic, Lecture Notes in Math. 450 (Springer, Berlin, 1975) 87–139.

[4] S. Feferman, Constructive theories of functions and classes, in: M. Boffa, D. van Dalen and K. McAloon, eds., Logic Colloquium '78 (North-Holland, Amsterdam, 1979) 159–224

[5] S. Feferman, A theory of variable types, Rev. Colombiana Mat. 19 (1981) 95–105.

[6] S. Feferman, Weyl vindicated: "Das Kontinuum" 70 years later, in: Temi e prospettive della logica e della filosofia della scienza contemporanee, CLUEB, Bologna, 1988, 59–93.

[7] S. Feferman, Hilbert's program relativized: proof-theoretical and foundational studies, J. Symbolic Logic 53 (1988) 364–384.

[8] S. Feferman, Logics for termination and correctness of functional programs, in: Y. Moschovakis, ed., Logic from Computer Science, MSRI Publ. 21 (Springer, New York, 1992) 95–127.

[9] S. Feferman and G. Jäger, Systems of explicit mathematics with non-constructive μ-operator. Part I, Ann. Pure Appl. Logic 65 (1993) 243–263.

[10] H. Friedman, Iterated inductive definitions and Σ^1_2–AC, in: A. Kino, J. Myhill and R.E. Vesley, eds., Intuitionism and Proof Theory, Proceedings of the Summer Conference at Buffalo, New York, 1968 (North-Holland, Amsterdam, 1970) 435–442.

[11] G. Jäger, Induction in the elementary theory of types and names, in: E. Börger, H. Kleine Büning and M.M. Richter, eds., Proceedings of CSL '87, Lecture Notes in Computer Science 329 (Springer, Berlin, 1988) 118–128.

[12] G. Jäger and T. Strahm, Totality in applicative theories, Ann. Pure Appl. Logic 74 (1995) 105–120.

[13] G. Jäger and T. Strahm, Second order theories with elementary comprehension and ordinals, in preparation.

[14] M. Marzetta, Predicative theories of types and names, Dissertation, Universität Bern, 1993.

[15] K. Schütte, Proof Theory (Springer, Berlin, 1977).

[16] A.S. Troelstra and D. van Dalen, Constructivism in Mathematics, Vol. I (North-Holland, Amsterdam, 1988).