Math 52H Homework 8 Solutions

March 8, 2012

1. Write F for the parametrization $F(x_1, ..., x_n) = \left(x_1, ..., x_n, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$ and $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$. Then

$$\int_{L_f} \omega^{\wedge k} = \int_{F^{-1}(L_f)} F^*(\omega^{\wedge k}) = \int_{F^{-1}(L_f)} (F^*\omega)^{\wedge k}.$$

But we calculate

$$F^*\omega = \sum_{i=1}^n dx_i \wedge \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j\right) = \sum_{i \neq j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j = \sum_{i<j} \frac{\partial^2 f}{\partial x_i \partial x_j} (dx_i \wedge dx_j + dx_j \wedge dx_i) = 0.$$

Hence the integral vanishes.

2. Recall $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. Hence, since ∂A is empty,

$$0 = \int_{\partial A} \omega \wedge \eta = \int_A d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

It follows

$$\int_A \omega \wedge d\eta = (-1)^{k+1} \int_A d\omega \wedge \eta = (-1)^{k+1+(k+1)l} \int_A \eta \wedge d\omega.$$

The constant C may thus be taken to be $(-1)^{(k+1)(l+1)}$.

3. We parametrize X by $F : [0, 2\pi) \times [0, 2\pi) \to X$,

$$F(\theta, \phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi).$$

The point $x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is covered by $\theta = \phi = 0$ and at $\theta = \phi = 0$ we have

$$\frac{dF}{d\theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \frac{dF}{d\phi} = \begin{bmatrix} 0 \\ 0 \\ -\sin\phi \\ \cos\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the orientation $\{\hat{\theta}, \hat{\phi}\}$ of the parametrization space induces an orientation of the tangent space of X given at the point a by $\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}$. Since

$$x_2 \wedge x_4(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}) = 1 > 0$$

the parametrization orientation agrees with the specified orientation of X.

Note

$$F^*(x_1) = \cos\theta, F^*(x_2) = \sin\theta, F^*(x_3) = \cos\phi, F^*(x_4) = \sin\phi$$

$$F^*(dx_1) = -\sin\theta d\theta, F^*(dx_2) = \cos\theta d\theta, F^*(dx_3) = -\sin\phi d\phi, F^*(dx_4) = \cos\phi d\phi.$$

a.

$$\int_X dx_1 \wedge dx_2 + dx_3 \wedge dx_4 = \int_0^{2\pi} \int_0^{2\pi} -\sin\theta\cos\theta - \sin\phi\cos\phi \; d\phi \; d\theta = 0.$$

b.

$$\int_X dx_1 \wedge dx_3 + dx_2 \wedge dx_4 = \int_0^{2\pi} \int_0^{2\pi} \sin\theta\sin\phi + \cos\theta\cos\phi \; d\phi \; d\theta = 0.$$

c.

$$\int_X x_2 x_4 dx_1 \wedge dx_3 = \int_0^{2\pi} \int_0^{2\pi} \sin^2\theta \sin^2\phi \; d\phi \; d\theta = \pi^2.$$

4. Let V denote the submanifold given by $x^2 + y^2 + z^2 \leq 1$ with $z \geq 0$, and let D denote the disc of radius 1 centered at the origin and lying on the xy plane oriented with upwards

pointing normal. Then $\partial V = S \cup D$. Let $\omega = dx \wedge dy + z dz \wedge dx$. Note that $d\omega = 0$, and so by Stoke's Theorem

$$= \int_V d\omega = \int_S \omega - \int_D \omega.$$

On D, $z = 0$ so $\int_S \omega = \int_D dx \wedge dy = \pi$, the latter being just the area of the disc D.

To compute directly, parametrize S by $P(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ for $(x, y) \in D$. Note that the tangent plane is spanned by $T_x = (1, 0, \text{something})$ and $T_y = (1, 0, \text{something})$ so $T_x \times T_y$ always has positive z component.

Now

$$dz = \frac{2}{\sqrt{1 - x^2 - y^2}}(-2x dx - 2y dy),$$

so

$$z dz \wedge dx = z \frac{y}{\sqrt{1 - x^2 - y^2}} dx \wedge dy = y dx \wedge dy.$$

Thus $\int_S \omega = \int_D (1 + y) dx \wedge dy = \pi$ since the first term gives the area of the the disc, and the second is odd in y while D is symmetric in y.

6. Let $\lambda = \omega - d\alpha$. We want to show that $\lambda = 0$. We claim that $\lambda = \beta \wedge \alpha$ for some 1-form β. Fix any point a, and if necesasry, change the basis of $\mathbb{R}^5_a$ to $v_1, ..., v_5$ so that $\alpha_a = x_5$, where $x_1, ..., x_5$ denote the coordinate functions. Then let $\lambda_a = \sum_{i<j} a_{i,j} x_i \wedge x_j$. Since $\lambda_a(x_i, x_j) = 0$ for $1 \leq i < j \leq 4$, we conclude that $\lambda_a = (\sum_{i<5} a_{i,5} x_i) \wedge \alpha_a$ (recall that $\alpha_a = x_5$). Thus, there exists some 1-form β such that $\lambda = \beta \wedge \alpha$. [1]

Now $0 = d\omega = d(a\alpha + \lambda) = d\lambda = d\beta \wedge \alpha - \beta \wedge d\alpha$. Let $\mathcal{H}$ denote the hyperplane field defined by $\alpha = 0$. Then the above gives $\beta \wedge d\alpha|_{\mathcal{H}} = 0$. It now suffices to show that $\beta|_{\mathcal{H}} = 0$, since this implies that $\beta = c\alpha$ for some constant c, which in turn implies that $\lambda = 0$.

Before proceeding, let us note that the last condition $\alpha \wedge \omega \wedge \omega \neq 0$ is equivalent to $\alpha \wedge d\alpha \wedge d\alpha \neq 0$, upon subsituting $\omega = \beta \wedge \alpha + d\alpha$. We thus conclude that $d\alpha \wedge d\alpha \neq 0$.

Lemma 1. *Fix any exterior 2-form γ on $\mathbb{R}^4$ such that $\gamma \wedge \gamma \neq 0$. We claim that the map $f : \Lambda^1((\mathbb{R}^4)^*) \to \Lambda^1((\mathbb{R}^4)^*)$ defined by $f(\delta) = \delta \wedge \gamma$ is an isomorphism.*

Proof. Again, fix a basis $v_1, ..., v_4$ with dual basis $x_1, ..., x_4$. Let $\gamma = \sum_{i<j} a_{i,j} x_i \wedge x_j$. The matrix for γ is a skew-symmetric matrix consisting of the $a_{i,j}$ on the upper half and $-a_{i,j}$ on the lower half with 0 on the diagonal. Since all skew-symmetric matrices of even dimension can be block diagonalized, we may change basis so that

$$\gamma = l_1 x_1 \wedge x_2 + l_2 x_3 \wedge x_4.$$

Then the condition that $\gamma \wedge \gamma \neq 0$ is equivalent to $l_1 l_2 \neq 0$. [2]

[1] Note that $\alpha_a = x_5$ is only a local condition at a. For instance, it is not possible to assume that $\alpha = dx_5$ globally.

[2] Incidentally, $\pm i l_1, \pm i l_2$ are the (totally imaginery) eigenvalues of the matrix.

Now, examine the linear map f on the basis $x_1, ..., x_4$ of $\Lambda^1((\mathbb{R}^4)^*)$. We have that $f(x_1) = l_2 x_1 \wedge x_3 \wedge x_4$, $f(x_2) = l_2 x_2 \wedge x_3 \wedge x_4$, $f(x_3) = l_1 x_3 \wedge x_1 \wedge x_2 = l_1 x_1 \wedge x_2 \wedge x_3$, and $f(x_4) = l_1 x_1 \wedge x_2 \wedge x_4$. Since f sends the basis elements of $\Lambda^1((\mathbb{R}^4)^*)$ to basis elements of $\Lambda^3((\mathbb{R}^4)^*)$, it is an isomorphism. $\square$

Now, we are nearly done. Apply the above lemma with $\gamma = d\alpha_a|_{\mathcal{H}}$ to see that $(\beta \wedge d\alpha)|_{\mathcal{H}} = 0 \Rightarrow \beta_{\mathcal{H}} = 0$, as desired.