Math 52H Homework 4 Solutions

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1. We have

$$\begin{aligned} dx &= \sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta \\ dy &= \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta \\ dz &= \cos\phi d\rho - \rho\sin\phi d\phi. \end{aligned}$$

Hence the 1-form is equal to

$$\begin{aligned} &\cos\phi d\rho - \rho\sin\phi d\phi + \frac{1}{2}\Big(\rho\sin\phi\cos\theta\left[\sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta\right] \\ &\qquad - \rho\sin\phi\sin\theta\left[\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta\right]\Big) \\ &= \cos\phi d\rho - \rho\sin\phi d\phi + \frac{1}{2}\rho^2\sin^2\phi d\theta. \end{aligned}$$

2. Since ω is closed,

$$d\omega = (\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1)dx\wedge dy + (\frac{\partial}{\partial z}F_1 - \frac{\partial}{\partial x}F_3)dz\wedge dx + (\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2)dy\wedge dz = 0$$

so we have the three equalities

$$\frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1, \quad \frac{\partial}{\partial z}F_1 = \frac{\partial}{\partial x}F_3, \quad \frac{\partial}{\partial y}F_3 = \frac{\partial}{\partial z}F_2. \tag{1}$$

Differentiating the homogeneity equation with respect to t, and then setting $t = 1$ we obtain

$$F_k(x,y,z) = x\frac{\partial}{\partial x}F_k(x,y,z) + y\frac{\partial}{\partial y}F_k(x,y,z) + z\frac{\partial}{\partial z}F_k(x,y,z). \tag{2}$$

Hence

$$
\begin{aligned}
df &= \frac{1}{2}\Big((F_1 + x\frac{\partial}{\partial x}F_1 + y\frac{\partial}{\partial x}F_2 + z\frac{\partial}{\partial x}F_3)dx + (x\frac{\partial}{\partial y}F_1 + F_2 + y\frac{\partial}{\partial y}F_2 + z\frac{\partial}{\partial y}F_3)dy \\
&\qquad + (x\frac{\partial}{\partial z}F_1 + y\frac{\partial}{\partial z}F_2 + F_3 + z\frac{\partial}{\partial z}F_3)dz\Big) \\
&= \frac{1}{2}\omega + \frac{1}{2}\Big((x\frac{\partial}{\partial x}F_1 + y\frac{\partial}{\partial y}F_1 + z\frac{\partial}{\partial z}F_1)dx + (x\frac{\partial}{\partial x}F_2 + y\frac{\partial}{\partial y}F_2 + z\frac{\partial}{\partial z}F_2)dy \\
&\qquad + (x\frac{\partial}{\partial x}F_3 + y\frac{\partial}{\partial y}F_3 + z\frac{\partial}{\partial z}F_3)dz\Big) \\
&= \omega.
\end{aligned}
$$

3. Let $f = \sum_{i=1}^{n} x_i x_{i+n}$ and let $\mathcal{V} = dx_1 \wedge ... \wedge dx_{2n}$ denote the volume form. Then

$$
\begin{aligned}
d\Omega &= \frac{d(\omega \wedge \theta)}{f^n} - n\sum_{1}^{n}(x_{i+n}dx_i + x_i dx_{i+n}) \wedge \frac{\omega \wedge \theta}{f^{n+1}} \\
&= \frac{n}{f^n}\left(\mathcal{V} - \sum_{1}^{n}(x_{i+n}dx_i + x_i dx_{i+n}) \wedge \frac{\omega \wedge \theta}{f}\right) \\
&= \frac{n}{f^n}\left(\mathcal{V} - \sum_{1}^{n} x_i x_{i+n}\frac{\mathcal{V}}{f}\right) \\
&= 0,
\end{aligned}
$$

as desired.

4. a. We have $\Delta = \partial^2 + \partial d + d\partial + d^2$. Now

$$\partial^2 = \star^{-1} d \star \star^{-1} d\star = \star^{-1} dd\star = 0$$

so $\Delta = \partial d + d\partial$ and this is a map $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$.

b. We compute $\Delta(\alpha)$ for an arbitrary 1-form $\alpha(x) = \alpha_1(x)dx_1 + ... + \alpha_n(x)dx_n$. So that we don't confuse the ∂ operator with plain-old partial derivatives $\frac{\partial}{\partial x_i}$ we'll use the alternative notation $D_i = \frac{\partial}{\partial x_i}$ in this problem. We have

$$\begin{aligned}
\partial d\alpha &= \partial \left(\sum_{i \neq j} D_j \alpha_i dx_j \wedge dx_i \right) \\
&= \partial \left[\sum_{i<j} \left(D_i \alpha_j - D_j \alpha_i \right) dx_i \wedge dx_j \right] \\
&= (-1)^{3n+1} \star d \left[\sum_{i<j} \left(D_i \alpha_j - D_j \alpha_i \right) \star \left(dx_i \wedge dx_j \right) \right] \\
&= (-1)^{n+1} \star \left[\sum_{i<j} \left\{ \left(D_i^2 \alpha_j - D_i D_j \alpha_i \right) dx_i \wedge \star (dx_i \wedge dx_j) + \left(D_i D_j \alpha_j - D_j^2 \alpha_i \right) dx_j \wedge \star (dx_i \wedge dx_j) \right\} \right]
\end{aligned}$$

Now $dx_i \wedge \star(dx_i \wedge dx_j) = \pm \star dx_j$. We can determine the sign by noting (ω is the volume form)

$$dx_i \wedge dx_j \wedge \star(dx_i \wedge dx_j) = \omega = dx_j \wedge \star dx_j.$$

Hence $dx_j \wedge dx_i \wedge \star(dx_i \wedge dx_j) = -\omega$ so $dx_i \wedge \star(dx_i \wedge dx_j) = - \star dx_j$. Similarly we find $dx_j \wedge \star(dx_i \wedge dx_j) = \star dx_i$. Plugging this in,

$$\begin{aligned}
\partial d\alpha &= (-1)^{n+1} \star \left[\sum_{i<j} \left\{ \left(D_i D_j \alpha_i - D_i^2 \alpha_j \right) \star dx_j + \left(D_i D_j \alpha_j - D_j^2 \alpha_i \right) \star dx_i \right\} \right] \\
&= (-1)^{n+1+n-1} \sum_{i<j} \left\{ \left(D_i D_j \alpha_i - D_i^2 \alpha_j \right) dx_j + \left(D_i D_j \alpha_j - D_j^2 \alpha_i \right) dx_i \right\} \\
&= \sum_{i \neq j} (D_i D_j \alpha_j - D_j^2 \alpha_i) dx_i.
\end{aligned}$$

We also have

$$\begin{aligned}
d\partial\alpha &= (-1)^{2n+1} d \star d \left(\sum_{i=1}^{n} \alpha_i \star dx_i \right) = -d \star \left(\sum_{i=1}^{n} D_i \alpha_i dx_i \wedge \star dx_i \right) \\
&= -d \star \left(\sum_{i=1}^{n} D_i \alpha_i \omega \right) = -d \left(\sum_{i=1}^{n} D_i \alpha_i \right) \\
&= - \sum_{i,j=1}^{n} D_i D_j \alpha_i dx_j
\end{aligned}$$

Hence

$$\Delta\alpha = (d\partial + \partial d)\alpha = \sum_{i \neq j} (D_i D_j \alpha_j - D_j^2 \alpha_i) dx_i - \sum_{i,j} D_i D_j \alpha_j dx_i = - \sum_{i,j} D_j^2 \alpha_i dx_i.$$

It follows that

$$\Delta(f\alpha) = -\sum_{i,j} D_j^2(f\alpha_i)dx_i = -\sum_{i,j}(\alpha_i D_j^2 f + 2D_j f D_j\alpha_i + f D_j^2\alpha_i)dx_i.$$

c. Let's agree to write ∇^2 for the second order differential operator $\nabla^2 = \sum_{j=1}^n D_j^2$. Thus our calculation from part b shows that for 1-form $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3$ we have

$$\Delta\alpha = -(\nabla^2\alpha_1 dx_1 + \nabla^2\alpha_2 dx_2 + \nabla^2\alpha_3 dx_3).$$

For α a zero form, $\partial\alpha = \pm\star d\star\alpha = 0$ since $\star\alpha$ is an n-form. Hence

$$\begin{aligned}\Delta\alpha = (d\partial + \partial d)\alpha = \partial d\alpha &= (-1)^{2n+1}\star d(D_1\alpha dx_2\wedge dx_3 + D_2\alpha dx_3\wedge dx_1 + D_3\alpha dx_1\wedge dx_2)\\ &= -(D_1^2\alpha + D_2^2\alpha + D_3^2\alpha) = -\nabla^2\alpha.\end{aligned}$$

Now the trick to doing the $k = 2, 3$ cases without more horrible computation is the following

Lemma 1. Δ *commutes with the Hodge star operator operator* $\star$*, i.e.* $\star\Delta\alpha = \Delta\star\alpha$.

Proof. Indeed, $\star\Delta\alpha = \star(d\partial + \partial d) = -(\star d\star^{-1} d\star + d\star d)$ while $\Delta\star\alpha = (d\partial + \partial d)\star = -(d\star^{-1} d\star\star + \star^{-1} d\star d\star)$. The two are equal since $\star^{-1} = (-1)^{k(n-k)}\star$, and $\star\star = (-1)^{k(n-k)}\mathrm{Id}$. $\square$

This lemma allows us to reduce the cases of $k = 2$ and $k = 3$ to 1-forms and 0-forms, respectively. Essentially, we can use the $\star$ operator to flip a 2-form into a 1-form, apply the Laplace-de Rham operator there, and then use $\star^{-1}$ to flip back to a 2-form. We get that for α the 2-form

$$\alpha = \alpha_1 dx_2\wedge dx_3 + \alpha_2 dx_3\wedge dx_1 + \alpha_3 dx_1\wedge dx_2,$$

$$\Delta\alpha = -(\nabla^2\alpha_1 dx_2\wedge dx_3 + \nabla^2\alpha_2 dx_3\wedge dx_1 + \nabla^2\alpha_3 dx_1\wedge dx_2).$$

Lastly, for 3-forms we have that

$$\Delta\alpha(x)dx_1\wedge dx_2\wedge dx_3 = -\nabla^2\alpha dx_1\wedge dx_2\wedge dx_3.$$

d. We note that $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial\theta}$ give an orthonormal basis for each tangent plane at every point $x\in\mathbb{R}^2$. The dual forms are dr and $rd\theta$, from which we infer that $\star dr = rd\theta$ and

$\star d\theta = -\frac{1}{r}dr$. Say that $k = 0$ and fix a 0-form f. Then $d\partial f = 0$ and

$$\begin{aligned}
\partial df &= \partial \left(D_r f dr + D_\theta f d\theta\right) \\
&= -\star d \left(D_r f r d\theta - D_\theta f \frac{1}{r} dr\right) \\
&= -\star \left(D_r f dr \wedge d\theta + r D_r D_r f dr \wedge d\theta - D_\theta D_\theta f \frac{1}{r} dr \wedge dr\right) \\
&= -\left(\frac{1}{r} D_r f + D_r D_r f + \frac{1}{r^2} D_\theta D_\theta f\right).
\end{aligned}$$

For the case $k = 2$, we may write

$$\begin{aligned}
\Delta f dr \wedge d\theta = \Delta \star \frac{f}{r} &= \star \Delta \frac{f}{r} \\
&= -\star \left(\frac{1}{r} D_r(f/r) + D_r D_r(f/r) + \frac{1}{r^3} D_\theta D_\theta f\right) \\
&= -\left(\frac{1}{r} D_r(f/r) + D_r D_r(f/r) + \frac{1}{r^3} D_\theta D_\theta f\right) r dr \wedge d\theta \\
&= -\left(D_r D_r f - \frac{D_r f}{r} + \frac{f}{r^2} + \frac{1}{r^2} D_\theta D_\theta f\right) dr \wedge d\theta.
\end{aligned}$$

When $k = 1$ write the arbitrary 1-form $\alpha = \alpha_r dr + \alpha_\theta d\theta$. $\Delta\alpha = \partial d\alpha + d\partial\alpha$, and $\Delta(\alpha_r dr) = \partial d(\alpha_r dr) + d\partial(\alpha_r dr)$, so that we need to compute

$$\begin{aligned}
\partial d(\alpha_r dr) &= \partial((D_\theta \alpha_r) d\theta \wedge dr) \\
&= -\star d \star ((D_\theta \alpha_r) d\theta \wedge dr) \\
&= \star d \left(\frac{D_\theta \alpha_r}{r}\right) \\
&= \star \left(\frac{D_\theta^2 \alpha_r}{r} d\theta + \left(\frac{D_\theta D_r \alpha_r}{r} - \frac{D_\theta \alpha_r}{r^2}\right) dr\right) \\
&= -\frac{D_\theta^2 \alpha_r}{r^2} + \left(D_\theta D_r \alpha_r - \frac{D_\theta \alpha_r}{r}\right) d\theta
\end{aligned}$$

and

$$\begin{aligned}
d\partial(\alpha_r dr) &= -d \star d \star (\alpha_r dr) \\
&= -d \star d(r \alpha_r d\theta) \\
&= -d \star ((\alpha_r + r D_r \alpha_r) dr \wedge d\theta) \\
&= -d \left(\frac{\alpha_r + r D_r \alpha_r}{r}\right) \\
&= \left(-\frac{D_r \alpha_r}{r} + \frac{\alpha_r}{r^2} - D_r^2 \alpha_r\right) dr - \left(\frac{D_\theta \alpha_r}{r} + D_\theta D_r \alpha_r\right) d\theta.
\end{aligned}$$

Therefore

$$\Delta(\alpha_r dr) = \left(\frac{\alpha_r}{r^2} - \frac{D_r\alpha_r}{r} - D_r^2\alpha_r - \frac{D_\theta^2\alpha_r}{r^2}\right) dr - \frac{2D_\theta\alpha_r}{r} d\theta.$$

Now we do the same computation for $\alpha_\theta d\theta$. Recall $\Delta(\alpha_\theta d\theta) = \partial d(\alpha_\theta d\theta) + d\partial(\alpha_\theta d\theta)$, so that we compute

$$\begin{aligned}
\partial d(\alpha_\theta d\theta) &= \partial\left((D_r\alpha_\theta) dr \wedge d\theta\right) \\
&= -\star d \star \left((D_r\alpha_\theta) dr \wedge d\theta\right) \\
&= -\star d\left(\frac{D_r\alpha_\theta}{r}\right) \\
&= -\star\left(\left(\frac{D_r^2\alpha_\theta}{r} = \frac{D_r\alpha_\theta}{r^2}\right) dr + \frac{D_r D_\theta\alpha_\theta}{r} d\theta\right) \\
&= \left(\frac{D_r\alpha_\theta}{r} - D_r^2\alpha_\theta\right) d\theta + \frac{D_r D_\theta\alpha_\theta}{r^2} dr
\end{aligned}$$

and

$$\begin{aligned}
d\partial(\alpha_\theta d\theta) &= -d \star d \star (\alpha_\theta d\theta) \\
&= d \star d\left(\frac{\alpha_\theta}{r} dr\right) \\
&= d \star\left(\frac{D_\theta\alpha_\theta}{r} d\theta \wedge dr\right) \\
&= -d\left(\frac{D_\theta\alpha_\theta}{r^2}\right) \\
&= -\left(\frac{D_\theta^2\alpha_\theta}{r^2} d\theta - \frac{2D_\theta\alpha_\theta}{r^2} dr\right) - \frac{D_r D_\theta\alpha_\theta}{r^2} dr \\
&= \left(\frac{2D_\theta\alpha_\theta}{r^3} - \frac{D_r D_\theta\alpha_\theta}{r^2}\right) dr - \frac{D_\theta^2\alpha_\theta}{r^2} d\theta
\end{aligned}$$

so that, finally,

$$\Delta\alpha = \left(\frac{\alpha_r}{r^2} - \frac{D_r\alpha_r}{r} - D_r^2\alpha_r - \frac{D_\theta^2\alpha_r}{r^2} + \frac{2D_\theta\alpha_\theta}{r^3}\right) dr + \left(-\frac{2D_\theta\alpha_r}{r} + \frac{D_r\alpha_\theta}{r} - D_r^2\alpha_\theta - \frac{D_\theta^2\alpha_\theta}{r^2}\right) d\theta.$$