Math 52H Homework 3 Solutions

February 3, 2012

1. There are $\binom{4}{3} = 4$ distinct 3-dimensional faces, one for each choice of 3 vectors which span the face. To compute the volumes of the faces, we apply Proposition 3.3 of §3.4 of the notes. Let P_{v_1,v_2,v_3} be the face spanned by the vectors v_1, v_2, v_3. We have that $(Vol_k P_{v_1,v_2,v_3})^2 = \det G(v_1, v_2, v_3)$, where G is the Gram matrix

$$G(v_1, v_2, v_3) = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \langle v_1, v_3\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \langle v_2, v_3\rangle \\ \langle v_3, v_1\rangle & \langle v_3, v_2\rangle & \langle v_3, v_3\rangle \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

So a little computation yields

$$Vol_k P_{v_1,v_2,v_3} = 4\sqrt{2}.$$

By symmetry, we have that

$$Vol_k P_{v_1,v_2,v_3} = Vol_k P_{v_1,v_2,v_4} = Vol_k P_{v_1,v_3,v_4},$$

and lastly,

$$G(v_2, v_3, v_4) = \begin{pmatrix} \langle v_2, v_2\rangle & \langle v_2, v_3\rangle & \langle v_2, v_4\rangle \\ \langle v_3, v_2\rangle & \langle v_3, v_3\rangle & \langle v_3, v_4\rangle \\ \langle v_4, v_2\rangle & \langle v_4, v_3\rangle & \langle v_4, v_4\rangle \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

and so computing the determinant,

$$Vol_k P_{v_2,v_3,v_4} = 8.$$

Lastly, the change of basis matrix from e_1, e_2, e_3, e_4 to v_1, v_2, v_3, v_4 is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

which has determinant $= -8$, so the basis defined by v_1, v_2, v_3, v_4 is oppositely oriented to the standard basis.

2. First of all, we assume that $c \neq 0$. I claim that

Lemma 1. *Let $\omega \neq 0$ be a k-form on V. Let*

$$L_\omega := \{v \in V | \omega(v, x_2, \dots, x_k) = 0 \ \ \forall x_2, \dots, x_k \in V\}.$$

If $\ell_1, \dots, \ell_k$ are linearly independent 1-forms, and we set $\omega = \ell_1 \wedge \cdots \wedge \ell_k$, then $\dim L_\omega = n - k$, and moreover,

$$L_\omega = \{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\}.$$

Proof. We show that for this choice of ω we have $\dim L_\omega \leq n - k$. Because $\omega \neq 0$, we have that for fixed $x_2, \dots, x_k$, $\omega(v, x_2, \dots, x_k)$ is a nonzero linear function of v, hence, has kernel dimension $n - 1$. I claim that if x_2' is linearly independent from then $x_2, \dots, x_k$, then $\ker \omega(-, x_2', \dots, x_k)$ is independent from $\ker \omega(-, x_2, \dots, x_k)$. Indeed this is true because $\omega(v, x_2, \dots, x_k)$ and $\omega(v, x_2', \dots, x_k)$ are linearly independent as linear functions of v. Then we have that

$$\dim\{v \in V | \omega(v, x_2, \dots, x_k) = 0 \ \ \forall x_2, \dots, x_k \in V\} \leq n - 2.$$

Repeat this process for each entry to find that in fact the dimension is $\leq n - k$.

On the other hand, $\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\}$ is clearly $n - k$ dimensional because the $\ell_1, \dots, \ell_k$ are linearly independent, and

$$\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\} \subset L_\omega,$$

hence $\dim L_\omega \geq n - k$ and the two sets are in fact equal. □

We have that by assumption

$$\mathcal{A}^*\ell_1 \wedge \cdots \wedge \mathcal{A}^*\ell_k = \mathcal{A}^*(\ell_1 \wedge \cdots \wedge \ell_k) = c\ell_1 \wedge \cdots \wedge \ell_k,$$

so that by the lemma we have

$$\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\} = L_{\ell_1 \wedge \cdots \wedge \ell_k} = L_{\mathcal{A}^*\ell_1 \wedge \cdots \wedge \mathcal{A}^*\ell_k} = \{v \in V | \mathcal{A}^*\ell_1(v) = \cdots = \mathcal{A}^*\ell_k(v) = 0\}.$$

Suppose now that for some i, $\mathcal{A}^*\ell_i \notin \operatorname{span}(\ell_1, \dots, \ell_k)$. For any nonzero $\lambda \notin \operatorname{span}(\ell_1, \dots, \ell_k)$, we have that $\{v \in V | \ell_1(v) = \cdots = \ell_k(v) = 0\} \not\subset \{v \in V | \lambda(v) = 0\}$. On the other hand, we also have that $\{v \in V | \mathcal{A}^*\ell_1(v) = \cdots = \mathcal{A}^*\ell_k(v) = 0\} \subset \{v \in V | \mathcal{A}^*\ell_i(v) = 0\}$. Thus we have reached a contradiction, and we must have $\mathcal{A}^*\ell_i \in \operatorname{span}(\ell_1, \dots, \ell_k)$ for all $i = 1, \dots, k$.

3. I claim that

Lemma 2. *For every 2-form η on V there exists a basis $v_1, \ldots, v_n$ so that η can be written as*

$$x_1 \wedge x_2 + x_3 \wedge x_4 + \cdots + x_{k-1} \wedge x_k$$

for some $k \leq n$ even.

Proof. Let L be the subspace of V given by the kernel of η, i.e. let

$$L := \{v \in V | \eta(v, x) = 0 \ \ \forall x \in V\}.$$

Pick basis vectors $v_{k+1}, \ldots v_n$ for this space, and take vectors $\tilde{v_1}, \ldots, \tilde{v_k}$ which extend these to a basis of V. We will change the tilde vectors later. Then the matrix for η in the basis we are constructing has all 0s in the last $k+1, \ldots, n$ rows and columns. Let M be the k-dimensional subspace of V generated by the $\tilde{v_1}, \ldots, \tilde{v_k}$. Then pick some nonzero $v_k \in M$. Then because $M \cap L = \{0\}$, we have that there exists $v_{k-1} \in M$ such that $\eta(v_k, v_{k-1}) = 1$. Now, replacing $\tilde{v_{k-1}}$ and $\tilde{v_k}$ by v_{k-1} and v_k, we have that the matrix of η with respect to this basis has a $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ block on the diagonal.

We now show that it is 0 in the two columns above this block. Take

$$M_1 := \{v \in V | \eta(x, v) = 0 \ \ \forall x \in \text{span}(v_{k-1}, v_k)\}.$$

I claim that that M_1 had dimension $n-2$. Indeed, $\eta(-, v_{k_1})$ is a linear function on V, and because $v_{k-1} \notin L$, we have that this function is nonzero. Thus (by, say, the rank-nullity theorem) it has kernel of dimension $n-1$. Then $U_{v_{k-1}} := \{v \in V | \eta(v, w) = 0 \ \ \forall w \in \text{span}(v_{k-1})\}$ has dimension $n-1$. By the same argument, we have that

$$\{v \in U_{v_{k-1}} | \eta(v, w) = 0 \ \ \forall w \in \text{span}(v_k)\} = M_1$$

has dimension $n-2$. Thus all the other entries in the two columns above and below the 2×2 block are zero. Now the matrix is as-yet-undetermined in the upper left $(k-2) \times (k-2)$ block, has $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal $k-1, k$ place, and 0 elsewhere. Repeat this process. The process must terminate because V is finite-dimensional. In the last step, we must be left with an undetermined 0 dimensional space and not a 1-dimensional space. Indeed, if the last undetermined entry in the matrix is a 1×1 in the upper-left, then it must be 0 by skew-symmetry, and hence it was in L to begin with. Thus k was even to begin with, and we have produced a basis with the desired property. □

Now, observe that, in fact, the condition $\eta \wedge \eta$ actually forces $k = 2$. Indeed, this can be seen by squaring-out the 2-form η explicitly in these coordinates. Thus $\eta = x_1 \wedge x_2$.

4. First, we check that

$$df = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2.$$

Indeed, $d_y f \in V_y^*$ is a linear function on the tangent space, so for any $h = h_1 v_1 + h_2 v_2 \in V_y$, we have

$$d_y f(h) = h_1 d_y f(v_1) + h_2 d_y f(v_2) = \frac{\partial f}{\partial y_1}(y) h_1 + \frac{\partial f}{\partial y_2}(y) h_2$$

by the second displayed equation of page 46 of the notes, §5.1. Also, if y_1, y_2 are the standard coordinate functions on V, which we now think of as smooth functions on V, then we have by linearity

$$dy_i = \lim_{t \to 0} \frac{y_i(y + th) - y_i(y)}{t} = y_i(h) = h_i,$$

so we get the formula for the differential we wanted.

Now we find the formula for the gradient. We have $\nabla f(y) = \mathcal{D}^{-1}(d_y f)$. Hence for any $h = h_1 v_1 + h_2 v_2 \in V_y$

$$h_1 \frac{\partial f}{\partial y_1}(y) + h_2 \frac{\partial f}{\partial y_2}(y) = \langle \nabla f(y), h \rangle = \langle g_1(y) v_1 + g_2(y) v_2, h_1 v_1 + h_2 v_2 \rangle,$$

or using symmetry and linearity of inner products,

$$\left[\frac{\partial f}{\partial y_1}(y), \frac{\partial f}{\partial y_2}(y) \right] \left[\begin{array}{c} h_1 \\ h_2 \end{array} \right] = [g_1(y), g_2(y)] \left[\begin{array}{cc} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle \end{array} \right] \left[\begin{array}{c} h_1 \\ h_2 \end{array} \right],$$

so, since h was arbitrary,

$$[g_1(y), g_2(y)] = \left[\frac{\partial f}{\partial y_1}(y), \frac{\partial f}{\partial y_2}(y) \right] \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right]^{-1} = \left[2\frac{\partial f}{\partial y_1}(y) - \frac{\partial f}{\partial y_2}(y), -\frac{\partial f}{\partial y_1}(y) + \frac{\partial f}{\partial y_2}(y) \right],$$

so we have

$$\nabla f(y) = \left(2\frac{\partial f}{\partial y_1}(y) - \frac{\partial f}{\partial y_2}(y) \right) \frac{\partial}{\partial y_1} + \left(-\frac{\partial f}{\partial y_1}(y) + \frac{\partial f}{\partial y_2}(y) \right) \frac{\partial}{\partial y_2}.$$