Math 52H: Solutions to Midterm Exam

February 2, 2011

1. In $\mathbb{R}^{2n}$ with coordinates $x_1, x_2, \dots, x_{2n}$ consider an exterior 2-form

$$\eta = \sum_{k=1}^{n} x_{2k-1} \wedge x_{2k}.$$

Given a 1-form $\alpha = \sum_1^{2n} a_i x_i$ find the 1-form

$$\beta = \star \left(\alpha \wedge \underbrace{\eta \wedge \dots \wedge \eta}_{n-1} \right).$$

We have

$$\eta^{n-1} = (n-1)! \sum_1^n x_1 \wedge \overset{2j-1}{\check{\dots}} \overset{2j}{\check{\dots}} \wedge x_{2n} \;\; (x_{2j-1} \wedge x_{2j} \;\text{ is missing}).$$

Then

$$x_{2j-1} \wedge \eta^{n-1} = (n-1)! \sum_1^n x_1 \wedge \overset{2j}{\check{\dots}} \wedge x_{2n} \;\; (x_{2j} \;\text{ is missing})$$

and

$$x_{2j} \wedge \eta^{n-1} = (n-1)! \sum_1^n x_1 \wedge \overset{2j-1}{\check{\dots}} \wedge x_{2n} \;\; (x_{2j-1} \;\text{ is missing}).$$

Hence, $\star(x_{2j-1} \wedge \eta^{n-1}) = (n-1)! x_{2j}$ and $\star(x_{2j} \wedge \eta^{n-1}) = -(n-1)! x_{2j-1}$.

Therefore

$$\beta = \star(\alpha \wedge \eta^{n-1}) = \sum_1^n \left(a_{2j-1} \star (x_{2j-1} \wedge \eta^{n-1}) + a_{2j} \star (x_{2j} \wedge \eta^{n-1})\right)$$
$$= (n-1)! \sum_1^n (-a_{2j}x_{2j-1} + a_{2j-1}x_{2j}).$$

Note that these formulas also holds for $n = 1$. In this case, $\star(a_1x_1 + a_2x_2) = a_1x_2 - a_2x_1$.

2. Consider a differential 1-form β which in cylindrical coordinates (r, ϕ, z) has the form

$$\beta = f(r)dz + g(r)d\phi, \quad \text{where } \; g'(0) = 0.$$

Find a condition when $\beta \wedge d\beta$ is a volume form, i.e. it does not vanish anywhere. Interpret this condition *geometrically* in terms of the properties of the curve given in $\mathbb{R}^2$ with Cartesian coordinates (u, v) by parametric equations

$$u = f(r), v = g(r) \;\text{ for }\; r \in [0, \infty).$$

We have

$$d\beta = f'(r)dr \wedge dz + g'(r)dr \wedge d\theta,$$

and

$$\beta \wedge d\beta = (f'g - g'f)dr \wedge dz \wedge d\theta.$$

Hence the required condition reads

$$f'(r)g(r) - g'(r)f(r) \neq 0$$

for all $r \neq 0$.

Remark. *Note that if $r = 0$ we cannot make computations in cylindrical coordinates. The condition $g'(0) = g(0) = 0$ together with the condition $f'(0) = 0$ allows us to extend the form smoothly to $r = 0$. The condition that $\beta \wedge d\beta \neq 0$ along z-axis then reads: $f(0) \neq 0$, $g''(0) \neq 0$.*

The condition $f'g - gf' \neq 0$ means that the velocity vector (f', g') of the curve

$$u = f(r), v = g(r) \text{ for } r \in [0, \infty)$$

is never collinear with the radius-vector (f, g) of the curve. If we re-express this condition in polar coordinates (ρ, ϕ) in the (u, v)-plane, it then reads that $\phi' \neq 0$, i.e. when $r \to \infty$ the point $(f(r), g(r))$ keeps rotating around the origin in the same direction.

3. Consider a differential 1-form $\alpha = dx_3 + x_2 dx_1$ on $\mathbb{R}^3$. Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that $f^*\alpha = h\alpha$ for some positive function $h : \mathbb{R}^3 \to \mathbb{R}$. Find a function $g : \mathbb{R}^3 \to \mathbb{R}$ such that the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by the formula

$$F(x_1, x_2, x_3, x_4) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3), x_4 g(x_1, x_2, x_3))$$

satisfies the equation $F^*(x_4\alpha) = x_4\alpha$.

We have

$$F^*(x_4\alpha) = (x_4 g) f^*\alpha = (x_4 gh)\alpha.$$

Hence, the equation $F^*(x_4\alpha) = x_4\alpha$ is equivalent to $gh = 1$, i.e.

$$g(x_1, x_2, x_3) = \frac{1}{h(x_1, x_2, x_3)}.$$

4. Let $k \in \mathbb{R}$ be any real number. Consider on $\mathbb{R}^n \setminus 0$ a differential $(n-1)$-form

$$\theta_k = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{r^k} dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n \quad (dx_i \text{ is missing}),$$

where $r = \sqrt{\sum_1^n x_j^2}$. For which values of the parameter k the form θ_k is closed? (Recall that a form θ is called *closed* if $d\theta = 0$.)

We have

$$d\left(\frac{x_i}{r^k}\right) = \frac{dx_i}{r^k} - kx_i \frac{\sum_1^n x_j dx_j}{r^{k+2}}.$$

Hence

$$d\theta_k = \sum_{i=1}^{n} (-1)^{i-1} d\left(\frac{x_i}{r^k}\right) dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n =$$

$$\left(\frac{n}{r^k} - k \frac{\sum_1^n x_i^2}{r^{k+2}} \right) dx_1 \wedge \cdots \wedge dx_n$$

$$= \frac{n-k}{r^k} dx_1 \wedge \cdots \wedge dx_n.$$

Hence, θ_k is closed if (and only if) $k = n$.