Math 52H: Solutions to practice problems for the midterm

1. Consider the Euclidean space $V = \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the standard dot-product. The space $\Lambda(V^*)$ of all exterior k-forms for all $k = 0, \dots, 2n$ is also an Euclidean space with the scalar product of a k-form α and an l-form β defined by the formula

$$\langle\langle \alpha, \beta \rangle\rangle = \begin{cases} \star^{-1}(\alpha \wedge \star\beta), & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}$$

Consider a linear operator $\Omega : \Lambda(V^*) \to \Lambda(V^*)$ defined by the formula $\Omega(\alpha) = \alpha \wedge \omega$, where $\omega = \sum\limits_1^n x_i \wedge y_1$. Find the adjoint linear operator Ω^*, i.e. the operator $\Omega^* : \Lambda(V^*) \to \Lambda(V^*)$ such that

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \langle\langle \alpha, \Omega^*(\beta) \rangle\rangle$$

for any forms $\alpha, \beta \in \Lambda(V^*)$.

We have

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \star^{-1}(\alpha \wedge \omega \wedge \star\beta) = \star^{-1}(\alpha \wedge \star(\star^{-1}(\omega \wedge \star\beta)) = \langle\langle \alpha, \star^{-1}(\omega \wedge \star\beta) \rangle\rangle.$$

Therefore,

$$\langle\langle \alpha, \star^{-1}(\omega \wedge \star\beta) - \Omega^*(\beta) \rangle\rangle = 0$$

for any k=forms α, β, which means that

$$\Omega^*(\beta) = \star^{-1}(\omega \wedge \star\beta)$$

2. Let $v_1, \dots v_k$ be a basis of V and $x_1, \dots, x_k$ be the dual basis of V^*. Let $l_i = \sum_{j=1}^{n} a_{ij} x_j \in V^*$, $i = 1, \dots, k$, be any linear functions. Prove that $\sum_1^k x_i \wedge l_i = 0$ if and only if the matrix $A = (a_{ij})$ is symmetric.

We have
$$\sum_1^k x_i \wedge l_i = \sum_1^k x_i \wedge \sum_{j=1}^k a_{ij} x_j = \sum_{1 \le i < j \le k} (a_{ij} - a_{ji}) x_i \wedge x_j .$$
The forms $x_i \wedge x_j$, $1 \le i < j \le k$ form a basis of the space of 2-forms. Hence,
$$\sum_1^k x_i \wedge l_i = 0 \iff a_{ij} = a_{ji}$$
for all $i < j$.

3. Consider two differential 1-forms in $\mathbb{R}^3$:
$$\alpha = dx + ydz \quad \text{and} \quad \beta = xdy\,.$$
Prove that there is no map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f^*(\beta) = \alpha$.

We have $d\alpha = dy \wedge dz$ and $d\beta = dx \wedge dy$. Thus, $\alpha \wedge \beta = dx \wedge dy \wedge dz$ and $\beta \wedge d\beta = 0$. If $f^*(\beta) = \alpha$ then
$$f^*(\beta \wedge d\beta) = f^*\beta \wedge df^*\beta = \alpha \wedge d\alpha.$$
But $\beta \wedge d\beta = 0$, and this would imply that $\alpha \wedge d\alpha = 0$, which is a contradiction.

4. The cylindrical coordinates
$$r \in [0, \infty),\ \varphi \in [0, 2\pi),\ z \in \mathbb{R}\,,$$
are introduced in $\mathbb{R}^3$ by the formulas
$$x = r\cos\varphi, y = r\sin\varphi, z\,,$$
where (x, y, z) are Cartesian coordinates. Consider a differential 1-form
$$\alpha = \cos r dz + \frac{r \sin r}{\pi} d\varphi\,.$$

Suppose that a curve $\Gamma \subset \mathbb{R}^3$ be given by the parametric equations

$$r = \frac{\pi}{4},\ z = h(t),\ \varphi = 2t,\ t \in [0, \pi]\,.$$

Find the function h such that $\alpha|_\Gamma = 0$ and $h(0) = 1$.

The restriction of α to Γ is equal $\left(\frac{\sqrt{2}}{2}h' + \frac{\sqrt{2}}{4}\right) dt$. Hence we get $h' = -\frac{1}{2}$, and thus $h(t) = -\frac{t}{2} + C$. The constant C is equal to $h(0) = 1$. Hence, $h(t) = -\frac{t}{2} + 1$.

5. Consider a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Let S_f be a surface in $\mathbb{R}^4$ given by equations

$$x_3 = \frac{\partial f}{\partial x_1}(x_1, x_2), \quad x_4 = \frac{\partial f}{\partial x_2}(x_1, x_2) \tag{1}$$

Suppose that this system of equations can is solved with respect to the coordinates x_2 and x_4, i.e. there exist smooth functions $x_2 = g(x_1, x_3)$ and $x_4 = h(x_1, x_3)$ such that

$$\begin{aligned} x_3 &\equiv \frac{\partial f}{\partial x_1}(x_1, g(x_1, x_3)), \\ h(x_1, x_3) &\equiv \frac{\partial f}{\partial x_2}(x_1, g(x_1, x_3)). \end{aligned} \tag{2}$$

Prove that the Jacobian of the map $(h, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is equal to -1, i.e. that

$$\begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_3} \end{vmatrix} = -1.$$

One can check that $\omega|_{S_f} = 0$. Let F be the map described in the Hint. Then we have

$$0 = F^*\omega = dx_1 \wedge dx_3 + dg \wedge dh = (1 + J) dx_1 \wedge dx_3,$$

where $J = \begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_3} \end{vmatrix}$. Hence, $J = -1$.

6. Consider a smooth differential k-form

$$\alpha = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \ldots i_k}(x) dx_{i_1} \wedge \ldots dx_{i_k}$$

in $\mathbb{R}^n$ such that $f_{i_1\dots i_k}(0) = 0$ (i.e. all coefficients of the form α are equal to 0 at the origin). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ denote the dilatation $x \mapsto 2x$. Suppose that $F^*\alpha = \alpha$. Prove that $\alpha \equiv 0$.

We have for any point $a \in \mathbb{R}^n$ and vectors $X_1, \dots, X_n \in \mathbb{R}^n_a$

$$\alpha_a(X_1, \dots, X_n) = \left(F^{-1}\right)^* \alpha_a(X_1, \dots, X_k) = \alpha_{\frac{a}{2}} \left(\frac{1}{2}X_1, \dots, \frac{1}{2}X_n\right).$$

Iterating this formula, we get

$$\alpha_a(X_1, \dots, X_n) = \left(f^{-n}\right)^* \alpha_a(X_1, \dots, X_k) = \alpha_{\frac{a}{2^n}} \left(\frac{1}{2^n}X_1, \dots, \frac{1}{2^n}X_n\right) \underset{n\to\infty}{\longrightarrow} 0.$$

7. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, consider a map $F : \mathbb{R}^n \to \mathbb{R}^{2n+1}$ defined by the formula

$$F(x_1, \dots, x_n) = \left(x_1, \dots, x_n, \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n), f(x_1, \dots, x_n)\right).$$

Compute $F^*(\alpha)$, where

$$\alpha = dx_{2n+1} - \sum_{i=1}^{n} x_{i+n} dx_i \, .$$

We have

$$F^*\alpha = df - \sum_1^n \frac{\partial f}{\partial x_i} dx_i = 0.$$