Math 52H: Homework N8

Due to Friday, March 9

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Consider a graphical n-dimensional submanifold with boundary

$$L_f = \left\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n};\ \sum_1^n x_j^2 \le 1;\ \ y_j = \frac{\partial f}{\partial x_j}(x_1, \dots, x_n),\ j = 1, \dots, n \right\} \subset \mathbb{R}^{2n}.$$

Suppose that L_f is oriented by the parameteriztion

$$(x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) \right).$$

Suppose that $n = 2k$. Compute

$$\int\limits_{L_f} (dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n)^{\wedge k}.$$

2. Let $A \subset V$ be a closed n-dimensional, $n = k + l + 1$, submanifold, ω a differential k-form and η a differential l-form defined on a neighborhood of A in V. Prove that

$$\int\limits_A \omega \wedge d\eta = C \int\limits_A \eta \wedge d\omega$$

for some constant C, and find C.

3. Consider the torus

$$T = \{x_1^2 + x_2^2 = 1,\ x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4.$$

This is a 2-dimensional submanifold of $\mathbb{R}^4$. Orient T in such a way that the tangent plane at the point $a = (1, 0, 1, 0)$ is oriented by the vectors $e = (0, 1, 0, 0), f = (0, 0, 0, 1) \in T_a T \subset \mathbb{R}^4_a$. Compute $\int\limits_T \omega$ for the following differential 2-forms:

(i) $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$;

(ii) $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$;

(iii) $\omega = x_2 x_4 dx_1 \wedge dx_3$.

4. Let $S = \{x^2 + y^2 + z^2 = 1; z \geq 0\} \subset \mathbb{R}^3$. We co-orient the sphere by the upward pointing vector at the point $(0, 0, 1)$. Compute $\int\limits_S dx \wedge dy + 2z dz \wedge dx$ directly and using Stokes' theorem.

6*. Let α be a differential 1-form and ω a differential 2-form on $\mathbb{R}^5$. Suppose that

- ω is closed;
- The form $\alpha \wedge \omega \wedge \omega$ does not vanish, i.e. $(\alpha \wedge \omega \wedge \omega)_a \neq 0$ for any $a \in \mathbb{R}^5$;
- for any point $a \in \mathbb{R}^5$ and any two vectors $X, Y \in \mathbb{R}^5_a$ such that $\alpha_a(X) = \alpha_a(Y) = 0$ we have $\omega_a(X, Y) = (d\alpha)_a(X, Y)$; in other words, the forms ω and $d\alpha$ coincide on the hyperplane field ξ defined by a Pfaffian equation $\alpha = 0$.

Prove that $\omega = d\alpha$.

Problem 6 is an extra-credit. Each problem is 10 points.