Math 52H: Homework N4

Due to Friday, February 10

1. Spherical coordinates $\rho \in [0,\infty)$, $\varphi \in [0,\pi]$, $\theta \in [0,2\pi)$, are introduced in $\mathbb{R}^3$ by the formulas:

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi.$$

Express the 1-form $dz + \frac{1}{2}(xdy - ydx)$ in spherical coordinates.

2. Consider a closed differential 1-form $\omega = F_1 dx + F_2 dy + F_3 dz$ in $\mathbb{R}^3$. Suppose that each function F_k, $k = 1,2,3$, satisfies the homogeneity equation

$$F_k(tx, ty, tz) = tF_k(x,y,z),$$

for any $t \in \mathbb{R}$. Prove that $\omega = df$, where

$$f(x,y,z) = \frac{1}{2}\left(xF_1(x,y,z) + yF_2(x,y,z) + zF_3(x,y,z)\right).$$

3. Consider in $\mathbb{R}^{2n}$ differential forms

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \text{ and } \theta = \sum_{n+1}^{2n} (-1)^{j-1} x_j dx_{n+1} \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{2n}.$$

Prove that the $(2n-1)$-form

$$\Omega = \frac{\omega \wedge \theta}{\left(\sum_1^n x_i x_{i+n}\right)^n}$$

is closed.

4. Consider an operator $\partial = (-1)^k \star^{-1} d\star : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$. Note that we equivalently can write $\partial = (-1)^{nk+n+1} \star d\star$. Denote $\Delta := (\partial + d)^2$ (this is *Laplace-de Rham operator*).

a) Verify that Δ is an operator $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$.

b) Compute $\Delta(f\alpha)$ for a function $f : \mathbb{R}^n \to \mathbb{R}$ and a differential 1-form α.

c) Find explicit formulas for $\Delta\alpha$ for the cases when $n = 3$ and $k = 0, 1, 2, 3$.

d) For $n = 2$ find explicit formulas for $\Delta\alpha$ in polar coordinates for $k = 0, 1, 2$.

Each problem (including subproblems in Problem 10) is 10 points.