Math 52H: Homework N2

Due to Friday, January 27

1. Do Exercise 4.5 from the online text:

   (a) For any special orthogonal operator $\mathcal{A} : V \to V$ the operators $\mathcal{A}^* : \Lambda^k(V^*) \to \Lambda^k(V^*)$ and $\star$ commute, i.e.

   $$\mathcal{A}^* \circ \star = \star \circ \mathcal{A}^*.$$

   (b) Let A be an orthogonal matrix of order n with $\det A = 1$. Prove that the absolute value of each k-minor M of A is equal to the absolute value of its complementary minor of order $(n-k)$. Here k is any integer between 1 and n. (*Hint:* Apply (a) to the form $x_{i_1} \wedge \dots \wedge x_{i_k}$).

2. Denote coordinates in $\mathbb{R}^{2n}$ by $x_1, y_1, \dots, x_n, y_n$. Let ω denote the 2-form $\sum\limits_1^n x_i \wedge y_i$.

   A linear operator $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *symplectic* if $\mathcal{A}^*\omega = \omega$.

   Denote

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Note that $J^2 = -I$. Let $\mathcal{J} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the operator with the matrix J.

An operator $\mathcal{U} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *unitary* if it is orthogonal and commutes with $\mathcal{J}$, i.e. $\mathcal{U} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{U}$. Respectively, a matrix U is called unitary, if it is orthogonal and $UJ = JU$.

(b) Prove that an operator $\mathcal{A}$ is symplectic if and only if its matrix A (in the standard basis of $\mathbb{R}^{2n}$) satisfies the equation $A^T JA = J$.

(c) Prove that an orthogonal operator $\mathcal{U} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is unitary if and only if it is symplectic.

(d) An n-dimensional vector subspace $L \subset \mathbb{R}^{2n}$ is called *Lagrangian* if $\omega|_L = 0$. Prove that L is Lagrangian if and only if $\mathcal{J}(L) = L^\perp$.

3. A 2-form β on $\mathbb{R}^4$ is called self-dual if $\star\beta = \beta$. What is the dimension of the space of self-dual 2-forms on $\mathbb{R}^4$. Find a basis of this space.

4. Consider $V = \mathbb{R}^3$ with the dot-product. Show that the formula

$$X \times Y = \mathcal{D}^{-1} \left(\star(\mathcal{D}(X) \wedge \mathcal{D}(Y)) \right)$$

defines the cross-product on $\mathbb{R}^3$.

Recall that the cross-product $X \times Y$ is defined as the vector Z orthogonal to $\mathrm{Span}(X, Y)$ which has length equal to the area of the parallelogram $P(X, Y)$ and (assuming that X, Y are linearly independent) directed in such a way that the basis (X, Y, Z) defines the standard orientation of $\mathbb{R}^3$. We also recall that the isomorphism $\mathcal{D} : V \to V^*$ is defined by the formula $\mathcal{D}(v)(X) = \langle v, X \rangle, \;\; v, X \in V$.

5. Define

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots; \quad \text{here } I \text{ denotes the unit matrix.}$$

Let A be an skew-symmetric $n \times n$ matrix, i.e. $A^T = -A$. Prove that the matrix e^A is orthogonal. Conversely, if $\exp(tA)$ is orthogonal for all t then A is skew-symmetric.

All problems and their subproblems are 10 points each.