Math 52H: Homework N1

Due on Friday, January 20

1. Recall that the *trace* TrA of a square matrix A is the sum of its diagonal elements. Let M_k denote the vector space of symmetric $k \times k$ matrices with real entries. Show that the formula

$$\langle X, Y \rangle = \mathrm{Tr}(XY)$$

defines an inner product on M_k. For the case $k = 2$ find the matrix of this bilinear form in the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. Suppose that the inner product on $V = \mathbb{R}^n$ is defined by the formula

$$\langle X, Y \rangle = \sum_{k=1}^{n} k x_k y_k,$$

for $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n)$. Find the matrix of the map $\mathcal{D} : V \to V^*$ corresponding to this inner product with respect to the standard basis $e_1, \ldots, e_n$ of $V = \mathbb{R}^n$ and the dual basis $x_1, \ldots, x_n$ of V^*.

3. Let f and g be bilinear functions on $\mathbb{R}^n$ with matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$, respectively. Find the expression of the 4-tensor $f \otimes g$ in the basis

$$x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}, \quad 1 \le i_1, i_2, i_3, i_4 \le n\,.$$

4. Let α be an exterior 2-form, and $\beta \neq 0$ is a 1-form on a 3-dimensional space. Suppose that $\alpha \wedge \beta = 0$. Prove that there exists a 1-form γ such that $\alpha = \beta \wedge \gamma$.

5. Let

$$\theta = \sum_{i=1}^{n-1} x_i \wedge x_{i+1}$$

be an exterior 2-form on $\mathbb{R}^n$, and $A, B \in \mathbb{R}^n$ are vectors

$$A = (1, 1, 1, \ldots, 1),\ B = (-1, 1, -1, \ldots, (-1)^n)\,.$$

Compute $\theta(A, B)$.

6. Denote coordinates in the space $V = \mathbb{R}^{2n}$ by $(x_1, x_2, \ldots, x_n, y_1, \ldots, y_n)$. Consider a 2-form

$$\omega = \sum_{i=1}^{n} x_i \wedge y_i.$$

a) Let A be a $n \times n$ matrix. Consider a map $L_A : \mathbb{R}^n \to \mathbb{R}^{2n}$, given by the formula $L_A(x) = (x, Ax)$, i.e. $L_A(x) = (x, y)$, where $y = Ax$, where we view the vector $x \in \mathbb{R}^n$ as a column-matrix. Compute $(L_A)^*\omega$ and show that $(L_A)^*\omega = 0$ if and only if the matrix A is symmetric.

b) Consider the map $\mathcal{C}_\omega$ of the space $V = \mathbb{R}^{2n}$ to the dual space V^* given by the formula

$$\mathcal{C}_\omega(v)(Z) = \omega(v, Z)\,, \quad v, Z \in V\,.$$

Find the matrix C of the map $\mathcal{C}_\omega$ with respect to the standard basis in $V = \mathbb{R}^{2n}$ an its dual basis in V^*.

c) Let $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map, such that $F^*\omega = \omega$. Prove that $\det F = 1$.

Each problem (including subproblems 6a) 6b) and 6c)) is 10 points.