Math 52H: Practice problems for the Final Exam

1. Prove that if S is a closed surface in $\mathbb{R}^3$, $\mathbf{n}$ its unit normal vector field and $\mathbf{l}$ any fixed vector then

$$\iint_S \langle \mathbf{n}, \mathbf{l} \rangle dS = 0.$$

2. Given a function $u : U \to \mathbb{R}$, where U is an open domain in $\mathbb{R}^n$ we denote by Δu the *Laplace operator*

$$\Delta u = \sum_1^n \frac{\partial^2 u}{\partial x_j^2}.$$

A function u is called *harmonic* in U if $\Delta u = 0$. Suppose that $n = 2$, i.e. U is a planar domain.

a) Prove that u is harmonic in U if and if for any closed 1-dimensional submanifold $\Gamma \subset U$ one has

$$\oint_\Gamma \frac{\partial u}{\partial \mathbf{n}} ds = 0,$$

where n is a unit normal vector field to Γ and $\frac{\partial u}{\partial \mathbf{n}} = du(\mathbf{n})$ is the directional derivative.

b) Prove that for any C^2-smooth function $u : U \to \mathbb{R}$ one has

$$\iint_S \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right) dx_1 dx_2 = - \int_S u \Delta u dx_1 dx_2 + \oint_\Gamma u \frac{\partial u}{\partial \mathbf{n}} ds,$$

where $S \subset U$ is any compact domain with boundary Γ.

c) Let S and Γ be as in the previous problem. Prove that for any two C^2-functions $u, v : U \to \mathbb{R}$ one has the following identity:

$$\iint\limits_S \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx_1 dx_2 = \oint\limits_\Gamma \begin{vmatrix} \frac{\partial u}{\partial \mathbf{n}} & \frac{\partial v}{\partial \mathbf{n}} \\ u & v \end{vmatrix} ds\,.$$

3. Compute the integral

$$\iint\limits_S (x^2 + y^2) dS,$$

where S is the boundary of the domain $\{\sqrt{x^2+y^2} \le z \le 1\}$.

4. Compute

$$\int\limits_S \frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y} + \frac{dx \wedge du}{z},$$

where S is the ellipsoid

$$S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

co-oriented by the outward normal to the domain which it bounds. .

5. Consider a differential form $\omega = \sum\limits_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$.

a) Find a vector field $\mathbf{v}$ on $\mathbb{R}^{2n}$ such that

$$d(\mathbf{v} \lrcorner \omega) = \omega.$$

(This problem has infinitely many solutions. Find any of them.)

b) Compute $\mathrm{Flux}_S\, \mathbf{v}$, where S is an ellipsoid

$$\left\{ \sum_1^n \frac{x_i^2 + y_i^2}{a_i^2} = 1 \right\}$$

cooriented by the outward normal vector field. Explain why the answer is independent of the choice of $\mathbf{v}$ in Part a).

6. Consider a 4-dimensional submanifold with boundary in $\mathbb{R}^8$:

$$\begin{aligned}\Gamma =&\big\{(x_1,\dots,x_8)\in\mathbb{R}^8;\ x_5=x_1\cos\alpha+x_2\sin\alpha, x_6=-x_1\sin\alpha+x_2\cos\alpha,\\ &x_7=2x_3-x_4, x_8=-x_3+x_4,\ x_1^2+x_2^2+x_3^2+x_4^4\le 1\big\}.\end{aligned}$$

Suppose that Γ is oriented by its parameterization by coordinates (x_1,x_2,x_3,x_4). Compute

$$\int\limits_\Gamma dx_5\wedge dx_6\wedge dx_7\wedge dx_8.$$

7. Consider a vector field

$$\mathbf{v}=\frac{1}{r^3}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right),$$

$r=\sqrt{x^2+y^2+z^2}$ in $\mathbb{R}^3\setminus 0$. Let us denote

$$S:=\left\{(x,y,z)\in\mathbb{R}^3;\ z=e^{x^2+y^2-\frac{1}{2}},\ x^2+y^2+z^2\le\frac{3}{2}\right\}$$

and co-orient this surface by a normal vector field which is equal to $(0,0,1)$ at the point $(0,0,\frac{1}{\sqrt{e}})\in S$. Compute $\mathrm{Flux}_S\,\mathbf{v}$.

8. Suppose that a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P,Q,R) satisfies $\operatorname{curl}\mathbf{v}=0$. Find an explicit expression for a function F such that $\mathbf{v}=\nabla F$.

9. Let C be the intersection of the sphere $S=\{x^2+y^2+z^2=1\}$ and the plane $P=\{x+y+z=0\}$. We orient C counter-clockwise when looking from the point $(0,0,100)$.

Compute $\int\limits_C z^3dx$.

10. Let M be an oriented closed n-dimensional manifold, and ω be a differential $(n-1)$-form on M. Prove that there exists a point $a\in M$ such that $(d\omega)_a=0$.

11. Let us view the space $\mathbb{R}^4$ with coordinates (x_1,y_1,x_2,y_2) as a complex vector space $\mathbb{C}^2$ with coordinates $(z_1=x_1+iy_1, z_2=x_2+iy_2)$. Consider a surface

$$S=\{(z_1,z_2)\in\mathbb{C}^2;\ z_2=z_1^2, |z_1|\le 1.\}$$

Compute $\mathrm{Area}(S)$.