(1) Consider the differential equation

$$2xy' - y = \ln y'.$$

Find all solutions and the discriminant.

Solution: Let $F(x, y, p) = 2xp - y - \ln p$, where $p = y'$. Let us consider the surface $M = \{(x, y, p) \colon F(x, y, p) = 0\}$ in the space of 1-jets. Solutions of the differential equation correspond to curves in M whose tangent vectors lie on the contact planes $dy - p\,dx = 0$.

So, we need to find the integral curves of the following system:

$$\begin{aligned} F(x, y, p) = 0 \quad &\Longrightarrow \quad 2xp - y - \ln p = 0 \\ dF = 0 \quad &\Longrightarrow \quad 2p\,dx - dy - (2x - \tfrac{1}{p})\,dp = 0 \\ dy - p\,dx = 0 \quad &\Longrightarrow \quad dy - p\,dx = 0. \end{aligned}$$

Using $dy = p\,dx$, the second equation becomes $p\,dx + (2x - \frac{1}{p})\,dx = 0$. Multiplying by p gives

$$p^2\,dx + (2xp - 1)\,dx = 0.$$

That is,

$$d(p^2 x - p) = 0.$$

Thus, $p^2 x - p \equiv \frac{1}{4}C$, where C is a constant. Solving for p gives

$$p = \frac{1 \pm \sqrt{1 + Cx}}{2x}.$$

Using the first equation, $y = 2xp - \ln p$, we find

$$y = 1 \pm \sqrt{1 + Cx} - \ln\left(\frac{1 \pm \sqrt{1 + Cx}}{2x}\right).$$

The **criminant** is the set of points in the surface $M = \{(x, y, p) \colon F(x, y, p) = 0\}$ such that $\frac{\partial F}{\partial p} = 0$. That is, the set of points in the 1-jet space such that

$$\begin{aligned} 2xp - y &= \ln p \\ 2x - \tfrac{1}{p} &= 0. \end{aligned}$$

The second equation gives $x = \frac{1}{2p}$. Plugging this into the first equation gives $y = 1 - \ln p$. Thus, the criminant can be described by the parametric curve in the 1-jet space given by

$$\begin{aligned} x(t) &= \tfrac{1}{2t} \\ y(t) &= 1 - \ln t \\ z(t) &= t. \end{aligned}$$

The **discriminant** is the projection of the criminant onto the xy-plane via $(x, y, p) \mapsto (x, y)$. This results in the parametric curve in the xy-plane given by

$$\begin{aligned} x(t) &= \tfrac{1}{2t} \\ y(t) &= 1 - \ln t. \end{aligned}$$

One can also describe this curve as the graph of $y = 1 + \ln(2x)$. $\Diamond$

(2) Find a curve on the plane whose tangent lines form with the coordinate axis triangles of area $2a^2$.

Remark: The following two solutions are ultimately equivalent.

Solution 1: The area of the triangle formed by the coordinate axes and the line $Y = mX + b$ is $A = \frac{1}{2}\left|\frac{b^2}{m}\right|$. The tangent line to a curve $Y = f(X)$ at a point (x, y) is given by $Y = f'(x)X + (f(x) - xf'(x))$. Thus, we require that

$$\frac{1}{2}\left|\frac{(f(x) - xf'(x))^2}{f'(x)}\right| = 2a^2,$$

or equivalently,

$$(f(x) - xf'(x))^2 = (2a)^2|f'(x)|.$$

Letting $y = f(x)$ and $p = f'(x)$, we have $(y - px)^2 = (2a)^2|p|$, and hence

$$y - px = \pm 2a\sqrt{|p|}.$$

Let $F(x, y, p) := y - px \pm 2a\sqrt{|p|}$. Solutions to the differential equation correspond to curves in $M = \{(x, y, p) \colon F(x, y, p) = 0\}$ whose tangent vectors lie on the contact planes $dy - p\,dx = 0$. Thus, we have

$$\begin{aligned} 0 = dF &= -p\,dx + dy + (-x \pm \tfrac{a}{\sqrt{|p|}}\operatorname{sgn}(p))\,dx. \\ &= (-x \pm \tfrac{a}{\sqrt{|p|}}\operatorname{sgn}(p))\,dx. \end{aligned}$$

This implies that $x = \pm\frac{a}{\sqrt{|p|}}\operatorname{sgn}(p)$, so that $x^2 = \frac{a^2}{|p|}$, so $|p| = \frac{a^2}{x^2}$. In other words, $f'(x) = \pm\frac{a^2}{x^2}$, so that

$$\boxed{f(x) = \pm\frac{a^2}{x}.} \quad \Diamond$$

Solution 2: We will use the fact that every curve $y = f(x)$ is the envelope of its family of tangent lines.

Note first (*cf* Arnold: page 20) that the envelope of a family of lines $\{y = px - g(p)\}_{p\in\mathbb{R}}$ is the curve $y = f(x)$, where f is the Legendre transform of g. Second, note that the family of lines $\{y = px \pm 2a\sqrt{|p|}\}_{p\in\mathbb{R}}$ form triangles of area $2a^2$ with the coordinate axes. Thus, the required curve is the Legendre transform of $g(p) = \pm 2a\sqrt{|p|}$.

To calculate the Legendre transform of g, we set $\frac{\partial}{\partial p}(px \pm 2a\sqrt{|p|}) = 0$, which yields $x = \mp\frac{a}{\sqrt{|p|}}$, so $|p| = \frac{a^2}{x^2}$, so $p = \pm\frac{a^2}{x^2}$. Thus,

$$f(x) = px \pm 2a\sqrt{|p|} = \pm\frac{a^2}{x} \mp \frac{2a^2}{x} = \boxed{\mp\frac{a^2}{x}.} \quad \Diamond$$

(3) Prove that the rank of any skew-symmetric bilinear form is even.

Solution 1 (Sketch): Show that the eigenvalues of any skew-symmetric bilinear form are either zero or pure imaginary, and that the eigenvalues come in complex-conjugate pairs. Thus, the number of non-zero eigenvalues – i.e., the rank of the bilinear form – is even. $\Diamond$

Solution 2: Let ω be a skew-symmetric bilinear form on a finite-dimensional vector space V. The result will follow from the following structure theorem.

Theorem: There exists a basis for V, denoted $\{u_1, \ldots, u_k, e_1, \ldots, e_m, f_1, \ldots, f_m\}$, for which

$$\begin{aligned} \omega(u_i, v) &= 0, \quad \forall v \in V \\ \omega(e_i, e_j) &= 0 \\ \omega(f_i, f_j) &= 0 \\ \omega(e_i, f_j) &= \delta_{ij}. \end{aligned}$$

Such a basis is called a *symplectic basis.*

Proof of Theorem: Let $U := \{u \in V : \omega(u, v) = 0, \ \forall v \in V\}$ denote the nullspace of ω. Let $\{u_1, \ldots, u_k\}$ be a basis of U. Write

$$V = U \oplus W.$$

We will (inductively) decompose W into a direct sum of 2-dimensional subspaces. If $W = 0$, we're done, so assume $W \neq 0$.

Let $e_1 \in W$, $e_1 \neq 0$. Since $e_1 \in W$, there exists $f_1 \in W$ with $\omega(e_1, f_1) \neq 0$. By rescaling, we may assume that $\omega(e_1, f_1) = 1$.

Let $W_1 := \operatorname{span}\{e_1, f_1\}$. Let $W_1^\omega := \{w \in W : \omega(w, v) = 0, \forall v \in W_1\}$ denote the symplectic complement to W_1 in W, so that

$$W = W_1 \oplus W_1^\omega.$$

Let $e_2 \in W_1^\omega$, $e_2 \neq 0$. Then there exists $f_2 \in W_1^\omega$ with $\omega(e_2, f_2) \neq 0$. By rescaling, we may assume that $\omega(e_2, f_2) = 1$.

Repeat this process. This process eventually terminates because $\dim V < \infty$. Thus, we have a direct sum decomposition

$$V = U \oplus \operatorname{span}\{e_1, f_1\} \oplus \cdots \oplus \operatorname{span}\{e_m, f_m\}. \qquad \Diamond$$

With respect to a symplectic basis for ω, the skew-symmetric bilinear form ω can be written as

$$\omega(u, v) = u^T \begin{pmatrix} 0_k & & \\ & 0 & \mathrm{Id}_m \\ & -\mathrm{Id}_m & 0 \end{pmatrix} v.$$

That is, with respect to a symplectic basis, the form ω is represented by the matrix

$$\begin{pmatrix} 0_k & & \\ & 0 & \mathrm{Id}_m \\ & -\mathrm{Id}_m & 0 \end{pmatrix},$$

which has rank $2m$. In other words, $\operatorname{rank}(\omega) = 2m$ is even. $\Diamond$

(4) Let us view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ with the operation of multiplication by i. Prove that

$$\mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R}) = \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{SO}(2n) = \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{Sp}(2n;\mathbb{R}) = \mathrm{U}(n).$$

Notation: Let g denote the standard inner product, let ω denote the standard symplectic form, and let h denote the standard Hermitian form.

We let $z, w \in \mathbb{C}^n$ denote

$$z = (x^1, \ldots, x^n, y^1, \ldots, y^n) = \sum x^k + iy^k$$
$$w = (u^1, \ldots, u^n, v^1, \ldots, v^n) = \sum u^k + iv^k.$$

Solution 1 (Sketch): Recall from basic linear algebra that

$$\mathrm{SO}(2n) = \{A \in \mathrm{GL}(2n;\mathbb{R})\colon A^T A = \mathrm{Id} \text{ and } \det_{\mathbb{R}}(A) = 1\}$$
$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n;\mathbb{C})\colon A^* A = \mathrm{Id}\}.$$

Let J denote the matrix

$$J = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

We observe that the matrix J corresponds to multiplication by i. That is,

$$Jz = J\begin{pmatrix} x^1 \\ \vdots \\ x^n \\ y^1 \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ \vdots \\ x^n \\ y^1 \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} -y_1 \\ \vdots \\ -y^n \\ x_1 \\ \vdots \\ x^n \end{pmatrix} = i\sum x^k + iy^k = iz.$$

We observe also that $J^T = -J$ is the matrix representation of ω in the standard basis. That is,

$$\omega(z, w) = z^T J^T w.$$

From these facts, it follows that

$$\mathrm{GL}(n;\mathbb{C}) = \{A \in \mathrm{GL}(2n;\mathbb{R})\colon AJ = JA\}$$
$$\mathrm{Sp}(2n;\mathbb{R}) = \{A \in \mathrm{GL}(2n;\mathbb{R})\colon A^T J A = J\}.$$

Using these four descriptions of the matrix groups $\mathrm{SO}(2n)$, $\mathrm{U}(n)$, $\mathrm{GL}(n;\mathbb{C})$, $\mathrm{Sp}(2n;\mathbb{R})$, the first three desired intersections are now fairly immediate. For example:

If $A \in \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{O}(2n)$, then $JA = AJ$ and $A^T A = \mathrm{Id}$, so $A^T J A = A^T A J = J$, so $A \in \mathrm{Sp}(2n;\mathbb{R})$. Thus, $\mathrm{O}(2n) \cap \mathrm{GL}(2n;\mathbb{C}) \subset \mathrm{Sp}(n;\mathbb{R})$. Analogous arguments hold for the other two.

If $A \in \mathrm{U}(n)$, then $A \in \mathrm{GL}(n;\mathbb{C})$ and $g(Az, Az) = h(Az, Az) = h(z, z) = g(z, z)$, so $g \in \mathrm{O}(2n)$, so $A \in \mathrm{Sp}(2n;\mathbb{R})$ by the preceding paragraph, and so A lies in any of these pairwise intersections.

Conversely, if $A \in \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{SO}(2n)$, then $h(Az, Az) = g(Az, Az) = g(z, z) = h(z, z)$, hence (by polarization) $h(Az, Aw) = h(z, w)$, so $A \in \mathrm{U}(n)$. $\Diamond$

(4) Let us view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ with the operation of multiplication by i. Prove that

$$\mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R}) = \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{SO}(2n) = \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{Sp}(2n;\mathbb{R}) = \mathrm{U}(n).$$

Notation: Let g denote the standard inner product, let ω denote the standard symplectic form, and let h denote the standard Hermitian form.

We let $z, w \in \mathbb{C}^n$ denote

$$\begin{aligned} z &= (x^1, \ldots, x^n, y^1, \ldots, y^n) = \sum x^k + iy^k \\ w &= (u^1, \ldots, u^n, v^1, \ldots, v^n) = \sum u^k + iv^k. \end{aligned}$$

Solution 2 (Sketch): Let us observe first that

$$\begin{aligned} h(z,w) = \sum z_k \overline{w}_k &= \sum (x_k + iy_k)(u_k - iv_k) \\ &= \sum (x_k u_k + y_k v_k) + i(y_k v_k - x_k u_k) \\ &= g(z,w) - i\omega(z,w). \end{aligned} \tag{1}$$

From this, we observe secondly that

$$\omega(z,w) + ig(z,w) = ih(z,w) = h(iz,w) = g(iz,w) - i\omega(iz,w),$$

and so by equating real parts, we obtain

$$g(iz,w) = \omega(z,w) \tag{2}$$

Using equations (1) and (2), the result is now straightforward to check. $\Diamond$

If $A \in \mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R})$, then $h(Az,Aw) = g(Az,Aw) - i\omega(Az,Aw) = g(z,w) - i\omega(z,w) = h(z,w)$, so $A \in \mathrm{U}(n)$. Thus, $\mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R}) \subset \mathrm{U}(n)$.

If $A \in \mathrm{U}(n)$, then $g(Az,Aw) - i\omega(Az,Aw) = h(Az,Aw) = h(z,w) = g(z,w) - i\omega(z,w)$, so equating real and imaginary parts shows that $A \in \mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R})$. Thus, $U(n) \subset \mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R})$.

This shows that $\mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R})$.

If $A \in \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{SO}(2n)$, then $\omega(Az,Aw) = g(iAz,Aw) = g(A(iz),Aw) = g(iz,w) = \omega(z,w)$, so $\omega \in \mathrm{Sp}(2n;\mathbb{R})$. Thus, $\mathrm{GL}(n;\mathbb{C}) \cap \mathrm{SO}(2n) \subset \mathrm{U}(n)$.

If $A \in \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{Sp}(2n;\mathbb{R})$, then $g(Az,Aw) = -\omega(iAz,Aw) = -\omega(A(iz),Aw) = -\omega(iz,w) = g(z,w)$, so $A \in \mathrm{SO}(2n)$. Thus, $\mathrm{GL}(n;\mathbb{C}) \cap \mathrm{Sp}(2n;\mathbb{R}) \subset \mathrm{U}(n)$.

Finally, if $A \in \mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{Sp}(2n;\mathbb{R})$, then

$$g(iAz,Aw) = \omega(Az,Aw) = \omega(z,w) = g(iz,w) = g(A(iz),Aw),$$

so $g(iAz - Aiz, Aw) = 0$. Since g is non-degenerate, this forces $A(iz) = i(Az)$, so $A \in \mathrm{GL}(n;\mathbb{C})$. $\Diamond$

(5) Prove that the plane fields given by equations $dz - y\,dx = 0$ and $dz - \frac{1}{2}(x\,dy - y\,dx) = 0$ are diffeomorphic, but the plane fields $dz - \frac{1}{2}(x\,dy - y\,dx) = 0$ and $dz - \frac{1}{2}(x\,dy + y\,dx) = 0$ are not.

Solution: Let $\alpha = dz - y\,dx$ and $\beta = dz - \frac{1}{2}(x\,dy - y\,dx)$. Consider the normal vector fields

$$\begin{aligned} \alpha^{\#} &= -y\frac{\partial}{\partial x} + \frac{\partial}{\partial z} \\ \beta^{\#} &= -\frac{1}{2}y\,\frac{\partial}{\partial x} + \frac{1}{2}x\,\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \end{aligned}$$

Let $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Note that $\Phi^*\alpha = \beta$ if and only if $(\Phi_*)^T(\alpha^{\#}) = \beta^{\#}$. This fact suggests a way of finding Φ. Namely, by observing that

$$\begin{pmatrix} 1 & 0 & \frac{1}{2}y \\ 0 & 1 & \frac{1}{2}x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -y \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}y \\ \frac{1}{2}x \\ 1 \end{pmatrix},$$

we are led to search for bijections $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\Phi_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix}.$$

One possibility is $\Phi(x, y, z) = (x, y, z + \frac{1}{2}xy)$.

It is easy to check that Φ is bijective and Φ_* is invertible, so that Φ is indeed a diffeomorphism. One can also verify that $\Phi^*\alpha = \beta$.

To see that the plane fields $dz - \frac{1}{2}(x\,dy - y\,dx) = 0$ and $dz - \frac{1}{2}(x\,dy + y\,dx) = 0$ are not diffemorphic, observe that the first is not integrable, whereas the second one is. $\Diamond$

(6) Consider the PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u - xy.$$

Solve the Cauchy problem for the initial data $u(2, y) = 1 + y^2$.

Solution 1: Let A denote the vector field in $\mathbb{R}^2$ given by

$$A = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

If $\gamma(t) = (x(t), y(t))$ is an integral curve of A, then $\gamma'(t) = A|_{\gamma(t)}$, so that

$$\begin{aligned} x'(t) &= x(t) & &\Longrightarrow & x(t) &= x_0 e^t \\ y'(t) &= y(t) & &\Longrightarrow & y(t) &= y_0 e^t. \end{aligned}$$

Thus, the *flow* of A is given by $\theta_t(x, y) = (xe^t, ye^t)$.

Let $S = \{(2, s) : s \in \mathbb{R}\}$ denote the initial hypersurface in $\mathbb{R}^2$. The *flowout* of A *along* the line $S \subset \mathbb{R}^2$ is defined by

$$\Psi(t, s) := \theta_t(2, s) = (2e^t, se^t).$$

One can check the following properties of Ψ (which are generally true of flowouts):

$$\begin{aligned} \Psi(0, s) &= (2, s) \\ \Psi_*(\tfrac{\partial}{\partial t}) &= A. \end{aligned}$$

We remark that $\Psi^{-1}(x, y) = (\ln(x/2), 2y/x)$.

Pulling back our Cauchy problem to the (t, s)-plane via Ψ results in the new problem

$$\begin{aligned} \frac{\partial \widehat{u}}{\partial t} &= \widehat{u} - 2se^{2t}, \\ \widehat{u}(0, s) &= 1 + s^2, \end{aligned}$$

where $\widehat{u} := \Psi^* u = u \circ \Psi$. Regarding s as fixed, we may view this problem as an ODE for $\widehat{u}(\cdot, s)$, namely $\widehat{u}' = \widehat{u} - 2se^{2t}$. Multiplying by the integrating factor e^{-t} gives

$$\frac{\partial}{\partial t}(e^{-t}\widehat{u}) = -2se^t.$$

Integrating yields $e^{-t}\widehat{u} = -2se^t + h(s)$ for some function $h(s)$. Invoking the initial condition $\widehat{u}(0, s) = 1 + s^2$ shows that

$$\widehat{u}(t, s) = -2se^{2t} + (s + 1)^2 e^t.$$

Pulling back to the (x, y)-plane via Ψ^{-1}, we find

$$\boxed{u(x, y) = \widehat{u}\left(\ln\left(\frac{x}{2}\right), \frac{2y}{x}\right) = \frac{1}{2}x + 2y - 2xy + \frac{2y^2}{x}.}$$

$\Diamond$

(6) Consider the PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u - xy.$$

Solve the Cauchy problem for the initial data $u(2, y) = 1 + y^2$.

Solution 2: Let $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ have coordinates $(x, y; z)$. Let ξ be the characteristic vector field in $\mathbb{R}^3$, i.e.:

$$\xi := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (z - xy)\frac{\partial}{\partial z}.$$

If $\gamma(t) = (x(t), y(t), z(t))$ is an integral curve of ξ, then $\gamma'(t) = \xi|_{\gamma(t)}$, so that

$$\begin{aligned} x'(t) &= x(t) & &\Longrightarrow & x(t) &= x_0e^t \\ y'(t) &= y(t) & &\Longrightarrow & y(t) &= y_0e^t \\ z'(t) &= z(t) - x(t)y(t) & &\Longrightarrow & z(t) &= -x_0y_0e^{2t} + (z_0 + x_0y_0)e^t. \end{aligned}$$

Thus, the *flow* of ξ is given by $\theta_t(x, y, z) = (xe^t, ye^t, -xye^{2t} + (z + xy)e^t)$.

Let $S = \{(2, s) \colon s \in \mathbb{R}\}$ be denote the initial hypersurface in $\mathbb{R}^2$. Let $\varphi \colon S \to \mathbb{R}$ denote $\varphi(2, s) = 1 + s^2$. Then $\text{Graph}(\varphi)$ can be parametrized as $((2, s), \varphi(2, s)) = (2, s; 1 + s^2)$.

The *flowout* of ξ *along* the curve $\text{Graph}(\varphi) \subset \mathbb{R}^3$ is defined by

$$\Psi(t, s) := \theta_t(2, s, 1 + s^2) = (2e^t, se^t, -2se^{2t} + (s + 1)^2e^{2t}).$$

A solution to our Cauchy problem is the function u whose graph is the flowout above – i.e.: $\text{Image}(\Psi) = \text{Graph}(u)$. Thus, setting

$$(2e^t, se^t, -2se^{2t} + (s + 1)^2e^{2t}) = (x, y, u(x, y)),$$

we find that

$$\begin{aligned} x &= 2e^t & \qquad t &= \ln(x/2) \\ y &= se^t, & \qquad s &= 2y/x, \end{aligned}$$

so that

$$\boxed{u(x, y) = \widehat{u}\left(\ln\left(\frac{x}{2}\right), \frac{2y}{x}\right) = \frac{1}{2}x + 2y - 2xy + \frac{2y^2}{x}.}$$

$\Diamond$