(1) Consider a system of two quasi-homogeneous equations

$$xy\frac{dy}{dx} = z^2,$$
$$z\frac{dz}{dx} = xy, \qquad x, y, z > 0.$$

Find a change of variables which reduces it to a first order system.

Solution: We first determine the weights for which the system is quasi-homogeneous. There are a couple of (ultimately equivalent) ways of doing this.

Method 1. Let $g^s(x, y, z) = (e^{\alpha s}x, e^{\beta s}y, e^{\gamma s}z)$. Write the system of equations as

$$dy = \frac{z^2}{xy}\,dx$$
$$dz = \frac{xy}{z}\,dx.$$

If g^s is a symmetry of the system, then $(g^s)^*(dy) = (g^s)^*(\frac{z^2}{xy}\,dx)$ and $(g^s)^*(dz) = (g^s)^*(\frac{xy}{z}\,dx)$. Expanding these equations ultimately yields $\beta = \gamma$ and $2\alpha + \beta = 2\gamma$, respectively, which together give $2\alpha = \beta = \gamma$.

Alternatively, one may prefer to use $G^t(x, y, z) = (t^\alpha x, t^\beta y, t^\gamma z)$ instead of g^s.

Method 2. Let $g^s(x, y, z) = (e^{\alpha s}x, e^{\beta s}y, e^{\gamma s}z)$. Let λ denote the line field in $\mathbb{R}^3$ given by

$$\lambda|_{(x,y,z)} = \operatorname{span}\left\{\left(1, \tfrac{z^2}{xy}, \tfrac{xy}{z}\right)\right\}.$$

For the system to be quasi-homogeneous, we must have $(g^s)_*\lambda|_{(x,y,z)} = \lambda|_{g^s(x,y,z)}$. One can calculate that

$$(g^s)_*\lambda|_{(x,y,z)} = \operatorname{span}\left\{\left(1, e^{(2\gamma-\alpha-\beta)s}\tfrac{z^2}{xy}, e^{(\alpha+\beta-\gamma)s}\tfrac{xy}{z}\right)\right\}$$
$$\lambda|_{g^s(x,y,z)} = \operatorname{span}\left\{\left(1, e^{(\beta-\alpha)s}\tfrac{z^2}{xy}, e^{(\gamma-\alpha)x}\tfrac{xy}{z}\right)\right\},$$

which forces $2\gamma - \alpha - \beta = \beta - \alpha$ and $\alpha + \beta - \gamma = \gamma - \alpha$. This implies $2\alpha = \beta = \gamma$, as above.

At any rate, we may take $\alpha = 1$ and $\beta = \gamma = 2$. Take the plane $\Sigma = \{(x, y, z) : x = 1\} \subset \mathbb{R}^3$ as the transverse surface. Then our coordinate change amounts to

$$\begin{aligned} x &= e^s & &\iff & s &= \log(x) \\ y &= e^{2s}u_1 & &\iff & u_1 &= yx^{-2} \\ z &= e^{2s}u_2 & &\iff & u_2 &= zx^{-2}. \end{aligned}$$

In these coordinates – that is, via pullback – our equations take the form

$$\frac{du_1}{ds} = \frac{u_2 - 2u_1^2}{u_1}, \qquad \frac{du_2}{ds} = \frac{u_1 - 2u_2^2}{u_2},$$

which together yield a first-order equation $\dfrac{du_2}{du_1} = \dfrac{u_1 - 2u_2^2}{u_2 - 2u_1^2} \cdot \dfrac{u_1}{u_2}$. $\Diamond$

(3) Consider a pendulum $\ddot{x} = -\sin x$. Find the limit of its period when the amplitude goes to zero.

Solution: Defining $y = \dot{x}$, we can rewrite our second-order equation as a first-order system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x. \end{aligned}$$

Note that $(0,0)$ is an equilibrium point of the system. We will assume without proof that the phase curves in a deleted neighborhood of $(0,0)$ are all closed.

It is a theorem (*cf.* Arnold §2D: pages 13-14) that as the amplitude approaches zero, the period of oscillation in a neighborhood of $(0,0)$ tends to the period of oscillation of the linearized system. So, we have to find the period of the linearized system.

Linearizing at the point $(0,0)$ gives the linearized system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x. \end{aligned}$$

In this linearized system, we have $\ddot{x} = -x$, whose general solution is $x(t) = A\cos t + B\sin t$, for any $A, B \in \mathbb{R}$. For $(A, B) \neq (0,0)$, the period of $x(t)$ is $\boxed{2\pi}$. $\diamondsuit$

(4) Find the solutions, the criminant, and the discriminant of the equation

$$(y')^3 + 3xy' - 3y = 0.$$

Solution: Let $F(x,y,p) = p^3 + 3xp - 3y$, where $p = y'$. Let us consider the surface $M = \{(x,y,p)\colon F(x,y,p) = 0\}$ in the space of 1-jets. Solutions of the differential equation correspond to curves in M whose tangent vectors lie on the contact planes $dy - p\,dx = 0$.

So, we need to find the integral curves of the following system

$$\begin{array}{rcrr} F(x,y,p) = 0 & \Longrightarrow & p^3 + 3xp - 3y = 0 & \text{(belonging to } M) \\ dF = 0 & \Longrightarrow & 3(p^2 + x)\,dp + 3p\,dx - 3\,dy = 0 & \text{(tangent to } M) \\ dy - p\,dx = 0 & \Longrightarrow & dy - p\,dx = 0 & \text{(tangent to contact planes)} \end{array}$$

Using $dy = p\,dx$, the second equation becomes

$$(p^2 + x)\,dp = 0.$$

This gives two cases.

Case One: $dp = 0$. Then $p = C$, a constant. Plugging this into $p^3 + 3xp - 3y = 0$ gives the solutions

$$\boxed{y = Cx + \frac{1}{3}C^3.}$$

Case Two: $p^2 + x = 0$. Then $x = -p^2$. Plugging this into $p^3 + 3xp - 3y$ gives $y = -\frac{2}{3}p^3$, hence $y = \pm\frac{2}{3}(p^2)^{3/2}$, and we obtain the solutions

$$\boxed{y = \pm\frac{2}{3}(-x)^{3/2}.}$$

The **criminant** is the set of points in the surface $M = \{(x,y,p)\colon F(x,y,p) = 0\}$ such that $\frac{\partial F}{\partial p} = 0$. That is, the set of points in the 1-jet space for which

$$\begin{aligned} p^3 + 3xp - 3y &= 0 \\ 3p^2 + 3x &= 0. \end{aligned}$$

The second equation gives $x = -p^2$. Plugging this into the first equation gives $y = -\frac{2}{3}p^3$. Thus, the criminant can be described by the parametric curve in the 1-jet space given by

$$\begin{aligned} x(t) &= -t^2 \\ y(t) &= -\tfrac{2}{3}t^3 \\ p(t) &= t. \end{aligned}$$

The **discriminant** is the projection of the criminant onto the xy-plane via $(x,y,p) \mapsto (x,y)$. This results in the parametric curve in the xy-plane given by

$$\begin{aligned} x(t) &= -t^2 \\ y(t) &= -\tfrac{2}{3}t^3. \end{aligned}$$

One can also describe this curve as the graphs $y = \pm\frac{2}{3}(-x)^{3/2}$. $\Diamond$

(5) Let $f = \sum_1^n a_{ij} x_i x_j$ be a quadratic form on $\mathbb{R}^n$. Show that its Legendre transform $g(p)$ is again a quadratic form $g(p) = \sum_1^n b_{ij} p_i p_j$, and the value of these forms and the points corresponding to each other under the Legendre map coincide.

Solution: Let $A = (a_{ij})$. Since f is a quadratic form, the matrix A is symmetric. We will assume for this problem that A is positive-definite, hence invertible.

We let δ_{ij} denote the Kronecker delta symbol. That is, $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. Said another way, we can write the identity matrix as $I = (\delta_{ij})$.

By definition, the Legendre transform $g(p)$ is given by

$$g(p) = \sup_x \left(\sum_i p_i x_i - \sum_i \sum_j a_{ij} x_i x_j \right).$$

Fix $p \in \mathbb{R}^n$. For a point $x \in \mathbb{R}^n$ at which the supremum is attained, we have

$$\begin{aligned}
0 &= \frac{\partial}{\partial x_k} \left(\sum_i p_i x_i - \sum_i \sum_j a_{ij} x_i x_j \right) \\
&= p_k - \sum_i \sum_j a_{ij} \frac{\partial}{\partial x_k}(x_i x_j) \\
&= p_k - \sum_i \sum_j a_{ij} x_i \delta_{jk} - \sum_i \sum_j a_{ij} \delta_{ik} x_j \\
&= p_k - \sum_i a_{ik} x_i - \sum_j a_{kj} x_j \\
&= p_k - \sum_i 2 a_{ik} x_i,
\end{aligned}$$

where in the last line we used that A is a symmetric matrix. In coordinate-free notation, we have determined that $p = 2Ax$, and so $x = \frac{1}{2} A^{-1} p$.

Let $B = (b_{ij})$ denote the inverse matrix $B = A^{-1}$. Then $x_i = \sum \frac{1}{2} b_{ij} p_j$, so that (for this specific choice of x)

$$\begin{aligned}
g(p) &= \sum_i p_i x_i - \sum_i \sum_j a_{ij} x_i x_j \\
&= \sum_i p_i x_i - \sum_j \frac{1}{2} p_j x_j \\
&= \frac{1}{2} \sum_i p_i x_i \\
&= \frac{1}{4} \sum_i \sum_j b_{ij} p_i p_j,
\end{aligned}$$

which shows that g is a quadratic form.

Thus, our Legendre map is $L(x) = 2Ax$, and its inverse is $L^{-1}(p) = \frac{1}{2}Bp$. That our quadratic forms f and g coincide on corresponding points follows from

$$
\begin{aligned}
g(L(x)) &= \frac{1}{4}\sum_i \sum_j b_{ij} \left(\sum_k 2a_{ki}x_k\right)\left(\sum_\ell 2a_{\ell j}x_\ell\right) \\
&= \sum_i \sum_j \sum_k \sum_\ell b_{ij}a_{ki}a_{\ell j}x_k x_\ell \\
&= \sum_j \sum_k \sum_\ell \delta_{jk}a_{\ell j}x_k x_\ell \\
&= \sum_k \sum_\ell a_{\ell k}x_k x_\ell \\
&= f(x),
\end{aligned}
$$

where in the last line we used that A is symmetric. $\Diamond$

Remark: It is, of course, possible to do all of the above calculations in a coordinate-free manner as well.