Math 53H: Additional chapters

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Chapter 1

What is a differential equation?

1.1 Preliminaries

Differential equations and system of equations are equations or system of equations involving derivatives of unknown functions. If all the unknown functions are of the same one variable then the differential equations are called *ordinary*. In the case of functions of more than one variable one speaks of *partial* differential equations.

Thus any system of ordinary differential equations can be written as

$$F(t, u(t), u'(t), u''(t), \ldots, u^{(k)}(t)) = 0, \tag{1.1.1}$$

$t \in [a, b]$, where $u : [a, b] \to \mathbb{R}^m$ is a vector-valued function, and F is a map of a domain U in the space $\mathbb{R}^N$, $N = km + 2$ to $\mathbb{R}^l$ for some integer l.

An important observation is that it is always possible to equivalently rewrite the system (1.1.1) to involve only the *first derivatives* of the unknown functions.

Indeed, the system

$$\begin{aligned}
&F(t, u(t), v_1(t), v_2(t), \ldots, v_{k-1}(t), v'_{k-1}(t)) = 0,\\
&u'(t) = v_1(t),\\
&v'_1(t) = v_2(t),\\
&\ldots\\
&v'_{k-2}(t) = v_{k-1}(t),
\end{aligned}$$

$t \in [a, b], \;\; u, v_1, \ldots, v_{k-1} : [a, b] \to \mathbb{R}^m$, is equivalent to the system (1.1.1).

Let us stress the point that when dealing with concrete equations this transformation is not always the best way of action. However, in many cases it is, and also for theoretical purposes considering the systems of first order differential equations is sufficient and we will usually do that . In other words, we will be studying the systems

$$F(t, u(t), u'(t)) = 0, \tag{1.1.2}$$

$t \in [a, b]$, $u : [a, b] \to \mathbb{R}^m$, $F : U \to \mathbb{R}^l$, where U is a domain in $\mathbb{R}^{2k+1}$.

1.2 Differential equations as vector fields

If $m = l$, i.e. the number of equations is equal to the number of unknown functions the system is called *determined.* If $l > m$ it is called *over-dertermined* and if $l < m$ *under-determined.* We will be dealing in this class exclusively with determined systems.

More precisely, for determined system one usually imposes an additional condition, that the minor of the Jacobi matrix of the map $F : U \to \mathbb{R}^l$ corresponding to the last m coordinates does not vanish at every points $(t, u, y) \in U \subset \mathbb{R}^{2m+1} = \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ for which $F(t, u, y) = 0$. Then according to the implicit functions locally near each such point the system (1.1.2) can be solved with respect to the derivatives, i.e. written in the form

$$u'(t) = v(t, u(t)), \tag{1.2.1}$$

$t \in [a, b]$, $u : \mathbb{R}^m \to \mathbb{R}, v : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$.

Let us consider first the case when v is independent of t, i.e. the system has the form

$$u'(t) = v(u(t)), \tag{1.2.2}$$

$t \in [a, b]$, $u, v : \mathbb{R}^m \to \mathbb{R}$. A system of this type is called *autonomous*. It is useful to think about v as a vector field on $\mathbb{R}^m$, or on a domain $\Omega \subset \mathbb{R}^m$. In other word, if coordinates in $\mathbb{R}^m$ are denoted by $(u_1, \dots, u_m)$ and the coordinate functions of v are $(v_1, \dots v_m)$ then we can think of v as a vector field $v = \sum\limits_1^m v_i(u) \frac{\partial}{\partial u_i}$. Then the problem of solving the ODE (1.2.2) can be interpreted as finding a path

$$u : [a, b] \to \mathbb{R}^m \tag{1.2.3}$$

such that its velocity vector $u'(t)$ at each point $t \in [a, b]$ coincides with the vector field v at the point $u(t)$, i.e. with the vector $v(u(t))$. Usually one also impose an *initial condition* on the solution: $u(a) = A \in \mathbb{R}^m$.

The space $\mathbb{R}^m$ on which the vector field v lives is called the *phase space* of the system (1.2.2), and the solutions (1.2.3) are called *phase curves* or *integral curves* of the system (1.2.2). The dimension of the phase space is called the *order* of the system.

If one thinks about the vector field v as a velocity vector field of a motion of some fluid then phase curves are trajectories of the individual particles. In the mechanical context when we think about the parameter t as the time, it is customary to denote the derivative by the dot, i.e. to write $\dot{u}$ instead of u'.

Let us point out, however, that usually for problems arising from Mechanics the phase space is not the space in which the motion takes place. Indeed, consider, for instance, the so-called, 3-body problem when, three bodies (say, the Sun, the Earth and the Moon) move in the 3-space according to the law of gravity, The motion of this system can be described by Newton equations of the form

$$\ddot{u}_1 = f_1(u_1, u_2, u_3),$$
$$\ddot{u}_2 = f_2(u_1, u_2, u_3),$$
$$\ddot{u}_3 = f_3(u_1, u_2, u_3),$$

where $u_1, u_2, u_3 \in \mathbb{R}^3$ are positions of (the centers of mass) of the bodies. After transforming this into a system of first order equations we get a vector field in $\mathbb{R}^{18}$. *This is the phase space of our system.*

Thus a motion of a the 3-body system corresponds to a phase trajectory of the corresponding point in its 18-dimensional phase space.

A *non-autonomous* system (1.2.1) can be viewed as a *time-dependent* vector field $v_t(u) = v(t,u)$. For instance, one encounters this situation when studying a non-steady flow of a fluid. Note that any non-autonomous system of order m can be viewed as an autonomous system of order $m+1$:

$$\dot{u} = v(\tau(t), u(t)),$$
$$\dot{\tau} = 1.$$

The space $\mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ of variables (u, τ) is called the *extended phase space* of the original non-autonomous system (1.2.1). In the extended phase space we can write the system as

$$\dot{\widehat{u}} = \widehat{v}(\widehat{u}(t)), \tag{1.2.4}$$

where $\widehat{u} = (u, \tau) \in \mathbb{R}^{m+1}$,

$$\widehat{v} = \sum v_i(\widehat{u}) \frac{\partial}{\partial u_i} + \frac{\partial}{\partial \tau}.$$

1.3 Line (direction) fields and Pfaffian equations

Let us denote by λ the *line field* $\lambda := \mathrm{Span}(\widehat{v})$ generated by the vector field $\widehat{v}$. We note that the vector field $\widehat{v}$ can be uniquely reconstructed from λ, and hence the system (1.2.4) can be equivalently viewed as the line field λ. [1]

More generally, given any line field λ in a domain $U \subset \mathbb{R}^n$ one can consider the problem of its integration as finding integral curves for this line field, i.e. paths $u : [a,b] \to U$ such that $\dot{u}(t) \in \lambda_{u(t)}$ for any $t \in [a,b]$. Note that in this case while the direction of the velocity vector is prescribed at any point, its length is not. Hence, one can reparameterize γ by composing it with a diffeomorphism $\phi : [c,d] \to [a,b]$ and get a different integral path which corresponds to the same integral curve viewed as a submanifold of U.

Note that in our original example of the line field λ generated by the vector field when the line field λ has a non-singular projection to one of the coordinates lines (namely, τ). Hence, any integral curve is graphical with respect to this projection, and therefore we can choose τ as the parameter

[1] In Arnold's book is used the term *direction field* for the line field λ.

on them. In fact any line field, in a neighborhood of each point projects non-singularly to one of coordinate axis, and hence the corresponding coordinate can be chosen as a parameter for integral curves near that point.

Consider now the case when $n = 2$, i.e. when λ is a line field on a domain $U \subset \mathbb{R}^2$. Then, if the line field λ is co-orientable it can be defined by a Pfaffian equation

$$\alpha = 0$$

for a 1-form $\alpha = Pdx + Qdy$ on U.

A solution of this equation, or which is the same, an integral curve of the line field $\lambda = \{\alpha = 0\}$. Hence, if it is given parametrically by $x = x(t), y = y(t)$, $t \in [a, b]$, then we get

$$(P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t))\, dt = 0$$

or

$$P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t) = 0.$$

Near a point where $(x_0, y_0) \in U$ where $Q(x_0, y_0) \neq 0$ (i.e. near a point where the projection of the line field λ to the x-axis is non-singular, we can equivalently write the equation $Pdx + Qdy = 0$ as $dy = -\frac{P}{Q}dx$, and hence look for solutions $y = f(x)$ of the equation

$$f'(x) = -\frac{P(x, f(x))}{Q(x, f(x)},$$

and similarly if $P(x_0, y_0) \neq 0$ we can write the equation in the form $dx = -\frac{Q}{P}dy$ and look for solutions $x = g(y)$ of the equation

$$g'(y) = -\frac{Q(g(y), y)}{P(g(y), y)}.$$

Example 1.1. Vector field on the line. Consider a vector field $v(x) = f(x)\frac{\partial}{\partial x}$ on $\mathbb{R}$ where $f(x) \neq 0$ for all $x \in \mathbb{R}$. Consider the corresponding differential equation

$$\dot{x} = v(x).$$

Passing to the extended phase space $\mathbb{R}^2$ with coordinates (x, t) this equivalent to a Pafaffian equation

$$dx = f(x)dt,$$

which in turn can be rewritten as

$$dt = \frac{dx}{f(x)},$$

because by our assumption $f(x) \neq 0$. Suppose we are looking for an integral curve passing through a point (t_0, x_0). Then integrating this equation we get

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{f(x)}.$$

Chapter 2

Phase flow

In this chapter we denote by U, V domains in $\mathbb{R}^n$. However, everything can be generalized to the case when U and V are any two n-dimensional manifolds.

2.1 Action of a diffeomorphism on a vector field

Let $f : U \to V$ be a diffeomorphism. Let us denote by $\mathrm{Vect}(U)$ and $\mathrm{Vect}(V)$ the spaces of vector fields on U and V, respectively.

Given a diffeomorphism $f : U \to V$ one can define the *push-forward map* $f_* : \mathrm{Vect}(U) \to \mathrm{Vect}(V)$ as follows. Let $X \in \mathrm{Vect}(U)$ be a vector field on U. Then we define the vector field $Y = f_* X$ by the formula

$$Y(v) = d_x f(X(u)), \ \text{ where } \ u = f^{-1}(v).$$

Let us point out that unlike the pull-back operator f^* on differential forms which defined for any smooth maps and not, necessarily for diffeomorphisms, the push-forward operator f_* on vector fields is defined *only for diffeomorphisms* (why?).

We can similarly define the push-forward operator on *line* fields. If X is a vector field and $\lambda = \mathrm{Span}(X)$ the line field which it generates then $f_* \lambda = \mathrm{Span}(f_* v)$.

Exercise 2.1. 1. Suppose $n = 2$ and a line field λ on U is defined by a Pfaffian equation $\alpha = 0$, where α is a 1-form on U. Show that given a diffeomorphism $f : U \to V$ the line field $f_* \lambda$ on V can be defined by a Pfaffian equation $\beta = 0$, where $\beta := \left(f^{-1}\right)^* \alpha = (f^*)^{-1} \alpha)$.

2. Let $P : U \to V$ be the map introducing polar coordinates. In other word $U = \{0 < r < \infty,\ 0 < \phi < 2\pi\}$ be a domain in $\mathbb{R}^2$ with Cartesian coordinates (r, ϕ), $V = \mathbb{R}^2 \setminus \{y = 0, x \geq 0\}$ in $\mathbb{R}^2$ with Cartesian coordinates (x, y) and P is defined by the formula

$$P(r, \phi) = (r \cos \phi, r \sin \phi).$$

Let $X = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial \phi}$ be a vector field on U. Find $Y := P_* X = A\frac{\partial}{\partial x} + B\frac{p}{\partial y}$. This can also be equivalently formulated as relating the expressions of a given vector field Y on $\mathbb{R}^2$ in two different bases, the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}\right)$.

2.2 Isotopy and diffeotopy

Let us denote by $\Delta \subset \mathbb{R}$ an interval in $\mathbb{R}$. This interval can be closed, open, semi-open, and even concides with the whole $\mathbb{R}$ or the rays (a, ∞) or $(-\infty, a)$.

Let us recall that a homotopy $f_t : U \to V$, $t \in \Delta$, is just a continuous family of continuous maps $U \to V$, which depends continuously on the parameter Δ. Equivalently, one can think of a homotopy as a continuous map $F : U \times \Delta \to V$. The relation to the first definition is given by the formula

$$F(x, t) = f_t(x), \quad \text{for } \ x \in U, \ \ t \in \Delta.$$

In this class we will always assume *all homotopies to be smooth*, i.e. $F : U \times \Delta \to V$ is at least a C^1-smooth map.

We will also need two special cases of a homotopy, called an *isotopy* and a *diffeotopy.*

A homotopy $f_t : U \to V$, $t \in \Delta$, is called a *diffeotopy* if $f_t : U \to V$ is a diffeomorphism for each $t \in U$. A homotopy $f_t : U \to V$, $t \in \Delta$, is called a *isotopy* if for each $t \in U$ the map $f_t : U \to V$ is an *embedding*, i.e. a diffeomorphism onto its image $f_t(U)$. Thus, an embedding need not to be *onto*, and the image $f_t(U)$ can move during an isotopy. Of course, a diffeotopy is a special case of an isotopy.

Let $f_t : U \to U$ (note that the source and the target are the same!) be a diffeotopy. Then we can define a family of vector fields X_t on U by the formula

$$X_t(x) = \frac{df_t}{dt}(f_t^{-1}(x)), \quad x \in U, t \in \Delta. \tag{2.2.1}$$

Equivalently, one can write

$$X_t(f_t(x)) = \frac{df_t}{dt}(x), \quad x \in U, t \in \Delta,$$

which means that *for every* $x_0 \in U$ *the path* $t \mapsto f_t(x_0)$, $t \in \Delta$, *is a solution of the equation*

$$\dot{x} = X_t(x) \tag{2.2.2}$$

For any $t_0 \in \Delta$ this solution satisfies the initial condition $x(t_0) = f_t(x_0)$.

2.3 Rectification theorems

Theorem 2.2. *Let X be a C^1-smooth vector field in a domain $\Omega \subset \mathbb{R}^n$. Then for any point $x_0 \in \Omega$ there exists $\epsilon > 0$ and a neighborhood $U \ni x_0, U \subset \Omega$, such that there exists an isotopy $f_t : U \to \Omega$, $t \in (-\epsilon, \epsilon)$ such that*

- $f_0(x) = x$ *for all* $x \in U$;
- $\frac{df_t(x)}{dt} = X(f_t(x))$.

We will prove this theorem later on.

The isotopy f_t is called the *local phase flow.* If f_t defined globally, i.e. it is a diffeotopy $U \to U$, even defined for small interval of time $(-\epsilon, \epsilon)$ then it is automatically defined for all $t \in \mathbb{R}$, see the next section.

Theorem 2.2 have several corollaries, most of which are essentially equivalent to the theorem itself.

First, we note that by the standard trick of reducing the non-autonomous case to an autonomous one in a space of a bigger dimension, Theorem 2.2 implies its own generalization:

Theorem 2.3. *Let X_t, $t \in \Delta$ be a C^1-smooth family of vector fields in a domain $\Omega \subset \mathbb{R}^n$. Then for any points $x_0 \in \Omega$ and $t_0 \in \Delta$ there exists $\epsilon > 0$ and a neighborhood $U \ni x_0, U \subset \Omega$, such that there exists an isotopy $f_t : U \to \Omega$, $t \in (t_0 - \epsilon, t_0 + \epsilon)$ which satisfies*

- $f_{t_0}(x) = x$ *for all* $x \in U$;

- $\frac{df_t(x)}{dt} = X_t(f_t(x)), x \in U, t \in (t_0 - \epsilon, t_0 + \epsilon)$

The next theorem shows that two *non-vanishing* smooth vector fields are *locally diffeomorphic*. More precisely,

Theorem 2.4. *Let X be a C^1-smooth vector field in a domain $\Omega \subset \mathbb{R}^n$. Suppose that that $X(a) \neq 0$ for some point $a \in \Omega$. Then there exists a local coordinate system $(y_1, \dots, y_n)$ on a neighborhood $U \ni a$, $U \subset \Omega$, centered at the point a such that the vector field X on U is equal to $\frac{\partial}{\partial y_1}$.*

In particular,

Theorem 2.5. *Let λ be a C^1-smooth line field in a domain $\Omega \subset \mathbb{R}^n$. Then for any point $a \in \Omega$ there exists a neighborhood $U \ni a, U \subset \Omega$ and a local coordinate system $(y_1, \dots, y_n)$ on U, centered at the point a such that the line field Y on U is spanned by the vector field $\frac{\partial}{\partial y_1}$.*

Proof of Theorem 2.4. We can assume without loss of generality that a is the origin of the Cartesian coordinate system, and the vector $X(a)$ coincides with the vector $\frac{\partial}{\partial x_1}$ at the point a. This could be achieved by rotating and scaling the original Cartesian system of coordinates. Let

$$D_\delta^{n-1} := \{x_1 = 0; \sum_2^n x_j^2 \leq \delta^2\}.$$

Suppose that ϵ is chosen so small that $D_\delta^{n-1} \subset U$, where U is the neighborhood provided by Theorem 2.2. Let $f_t : U \to \Omega, t \in (-\epsilon, \epsilon)$ be the local phase flow constructed in Theorem 2.2. Denote

$$H := \{|x_1| \leq \epsilon, \sum_2^n x_j^2 \leq \delta^2\}$$

and define a map $F : H \to \Omega$ given by the formula $F(x_1, x_2, \dots, x_n) = f_{x_1}(0, x_2, \dots, x_n))$.

The map F is an embedding, provided that ϵ, δ are small enough. Indeed, the differential of F at the origin is the identity map (why?), and hence by the implicit function theorem it is an embedding in a sufficiently small neighborhood of 0. But $F_*(\frac{\partial}{\partial x_1}) = X$, and hence, assuming that ϵ, δ are small enough, the coordinate system introduced on the neighborhood $U' = F(H)$ by the diffeomorphism $F^{-1} : U' \to H$ is the required one. ■

This theorem, in particular implies existence of the solution of a system $\dot{x} = X(x)$ for any initial data $x(t_0) = x_0$ on an interval $(t - \epsilon, t + \epsilon)$, provided that the vector field X is C^1-smooth. It also implies the uniqueness of solution with given initial data and its smooth dependence on the initial data.

2.4 Phase flow

Let X be a smooth vector field in a domain $\Omega \subset \mathbb{R}^n$. Choose $a \in \Omega$. Recall that according to Theorem 2.3 there exists a neighborhood $U \ni a$ in Ω and $\epsilon > 0$ such that there exists a local phase flow for the equation

$$\dot{x} = X(x),\ x \in \Omega, \tag{2.4.1}$$

i.e. an isotopy $f_t : U \to \Omega$, $t \in (-\epsilon, \epsilon)$, such that

- $f_0(x) = x$ for all $x \in U$;
- $\frac{df_t(x)}{dt} = X_t(f_t(x)), x \in U, t \in (-\epsilon, \epsilon)$.

Let us observe that that the interval $(-\epsilon, \epsilon)$ depends on the choice of an initial point $a \in \Omega$ and its neighborhood U. However, if the flow is defined on the whole Ω, i.e. it is a *diffeotopy* $f_t : \Omega \to \Omega$ then the flow is defined for all $t \in \mathbb{R}$.

Indeed, let $E = \sup \epsilon$ such that the flow is defined on $(-\epsilon, \epsilon)$. Suppose that $E < \infty$. Then the flow is defined on $(-E + \delta, E + \delta)$ for $\delta < \frac{\epsilon_0}{2}$ but then we can define it on $(-E', E')$, where $E' = E - \delta + \frac{3\epsilon_0}{4} > E$ by the formula $f_t := f_{\frac{3\epsilon_0}{4}} \circ f_{t - \frac{3\epsilon_0}{4}}$ for $t \in (E - \delta, E')$. This contradiction shows that $E = \infty$, i.e. the flow is defined for all $t \in \mathbb{R}$. The following lemma follows from the definition of the flow.

Lemma 2.6. *Suppose the flow $f_t : \Omega \to \Omega$ for a vector field X is defined for all $t \in \mathbb{R}$. Then*

1. $f_t \circ f_u = f_{t+u}$ *for all* $t, u \in \mathbb{R}$;
2. $f_0 = \mathrm{Id}$;
3. $f_{-t} = f_t^{-1}$.

One may express this lemma by saying that the *flow of an autonomous system which is defined for all $t \in \mathbb{R}$ forms a 1-parametric group of diffeomorphisms.*

Often for the flow f_t generated by a vector field X we will use the notation X^t instead of f_t.

Conversely, any 1-parametric group of diffeomorphisms $f_t : \Omega \to \Omega$ corresponds to a vector field X on Ω. Indeed, according to the formula (2.2.1) the isotopy f_t defines a *family* of vector fields $X_t(x) = \frac{df_t}{dt}(f_t^{-1}(x))$, $x \in \Omega, t \in \mathbb{R}$. But in this case, denoting $y = f_t^{-1}(x)$

$$X_t(x) = \frac{df_t}{dt}(y) = \lim_{u \to 0} \frac{f_{t+u}(y) - f_t(y)}{u} = \lim_{u \to 0} \frac{f_u(x) - f_t(x)}{u} = X_0(x),$$

i.e. X_t is independent of t.

Proposition 2.7. *Suppose that a vector field X on Ω integrates to a flow $X^t : \Omega \to \Omega$, $t \in \mathbb{R}$, and $f : \Omega \to \widetilde{\Omega}$ a diffeomorphism. Denote $\widetilde{X} := f_* X$. Then the vector field $\widetilde{X}$ integrates to a flow $\widetilde{X}^t$, $t \in \mathbb{R}$, on $\widetilde{\Omega}$ and*

$$\widetilde{X}^t = f \circ X^t \circ f^{-1}, \; t \in \mathbb{R}.$$

Proof. For any point $y = f(x) \in \widetilde{\Omega}$ we have

$$\begin{aligned}
\frac{d}{dt}(\widetilde{X}^t(y))\big|_{t=0} &= \frac{d}{dt}(f \circ X^t \circ f^{-1}(y))\big|_{t=0} \\
&= \frac{d}{dt}(f \circ X^t(x))\big|_{t=0} = d_x f(\frac{d}{dt}(X^t(x)\big|_{t=0}) = d_x f(X(x)) \\
&= f_* X(y) = \widetilde{X}(y).
\end{aligned}$$

■

2.5 Symmetries

Let λ be a line field in $\Omega \subset \mathbb{R}^n$. A diffeomorphism $f : \Omega \to \Omega$ is called a *symmetry* of the line field λ if $f_* \lambda = \lambda$.

Lemma 2.8. *All symmetries of the line field λ form a group.*

Indeed, Id is a symmetry, if f, g are symmetries then $f \circ g$ is a symmetry and if f is a symmetry then f^{-1} is a symmetry.

Consider a differential equation

$$\dot{x} = X_t(x), \; x \in \Omega, \; t \in \Delta. \tag{2.5.1}$$

with the phase space $\Omega \subset \mathbb{R}^n$. Let λ be the corresponding line field on its extended phase space $\Omega \times \Delta$. Then any symmetry $f : \Omega \times \Delta \to \Omega \times \Delta$ of the line field λ is called the *symmetry of the equation* (2.5.1).

Let us stress the point that a symmetry is a diffeomorphism of an *extended* phase space, i.e. it acts on space-time domain, even in the case of an autonomous system. Of course, in the case of an autonomous system $\dot{x} = X(x)$, $x \in \Omega$, one can consider also more restricted class of symmetries, namely diffeomorphisms $h : \Omega \to \Omega$ preserving the vector field X, i.e. $h_* X = X$, as for instance, in the following

Proposition 2.9. *Consider an autonomous system $\dot{x} = X(x)$ on $\Omega \subset \mathbb{R}^n$. Suppose that it integrates to a phase flow $X^t : \Omega \to \Omega$. Then for each $s \in \mathbb{R}$ the diffeomorphism X^s is a symmetry of the equation.*

Proof. Let us compute $Y := X^s_*(X)$. By definition of the phase flow,

$$X(x) = \frac{d}{dt} X^t(x)\big|_{t=0}.$$

On the other hand, by the chain rule for any path $\gamma : (-\epsilon, \epsilon) \to \Omega$ such that $\gamma(0) = x$ and $\gamma'(0) = X(x)$ we have $\frac{d}{dt} f(\gamma(t))\big|_{t=0} = df_x(X(x)) = f_* X(f(x))$. Denote $f := X^s$. Then

$$f_* X(f(x)) = \frac{d}{dt} f \circ X^t(x)\big|_{t=0} = \frac{d}{dt} X^{s+t}(x)\big|_{t=0} = X(X^s(x)).$$

In other words, $f_* X(f(x)) = X(f(X))$, i.e. $f_* X = X$. ■

Theorem 2.10. *Let Y and λ be a vector field and a line field in Ω.*

- *Y integrates to a flow $Y^s : \Omega \to \Omega$;*
- *Y admits a transverse hypersurface Σ such that $\bigcup_{s \in \mathbb{R}} Y^s(\Sigma) = \Omega$ and either*
  - *(a) $Y^s(\Sigma) \neq Y^{s'}(\Sigma)$ for $s \neq s'$, or*
  - *(b) the flow Y^s is periodic with period T, i.e. $Y^{s+T} = Y^s$ for all $s \in \mathbb{R}$ and $Y^s(\Sigma) \neq Y^{s'}(\Sigma)$ if $|s - s'| < T$.*

Suppose that Y^s is a symmetry of λ for all $s \in \mathbb{R}$. Then the order of the differential equation corresponding to λ can be reduced by 1. In particular, if $\dim \Omega = 2$ then the Pfaffian equation *corresponding to λ can be reduced to an equation with separable variables, and hence solved in quadratures.*

Proof. We consider below only the case $n = 2$. The proof in the general case follows a similar scheme. In this case Σ is a 1-dimensional manifold, and hence it is diffeomorphic either to $\mathbb{R}$ or to S^1. We will concentrate below on the case of $\mathbb{R}$. Consider a parameterization $\phi : \mathbb{R} \to \Sigma$. Define a map $\Phi : \mathbb{R}^2 \to \Omega$ by the formula

$$\Phi(u, v) = Y^v(\phi(u)).$$

We can think about (u, v) as curvilinear coordinates in Ω. The flow Y^s in these coordinates look like translation along the v-direction:

$$(u, v) \mapsto (u, v + s).$$

The line field λ in these coordinates can be defined by a 1-form $\alpha = P(u, v)du + Q(u, v)dv$. Let us assume that $P \neq 0$. In fact, at every point (u, v) either $P(u, v) \neq 0$ or $Q(u, v) \neq 0$. The case when $Q \neq 0$ can be considered similarly. Then we can define the line field λ by a Pfaffian equation $du + R(u, v)dv = 0$, where $R = \frac{Q}{P}$.

The fact that the line field λ is preserved by the flow Y^s means that

$$(Y^s)^*(du + R(u, v)dv) = f_s(u, v)(du + R(u, v)dv).$$

But $(Y^s)^*(du + R(u, v)dv) = du + R(u, v + s)dv$. Hence, $f_s(u, v) \equiv 1$ and $R(u, v + s) = R(u, v)$, i.e. the function R is independent of V, so we will just write $R(u)$.

Thus in coordinates (u, v) the equation takes the form

$$du + R(u)dv = 0$$

which is an equation with separable variables. ∎

Let us notice that if we change the variables (u, v) to (u, V) where $v = h(V)$ then the variables will separate anyway. Indeed, the form $du + R(u)dv$ in coordinates (u, V) takes the form $du + R(u)h'(V)dV$. And thus the variables in the equation $du + R(u)h'(V)dV = 0$ separate as well.

Hence, it is not so important that the coordinate v along trajectories of Y coincides with the time-parameter, but what is crucial is that v is constant on translates of Σ under the flow Y^s.

2.6 Quasi-homogeneous equations

Consider in $\mathbb{R}^n$ the vector field

$$Y = \sum \alpha_i x_i \frac{\partial}{\partial x_i},$$

where $\alpha_1, \ldots, \alpha_n$. It is called an *Euler field* with weights $\alpha_1, \ldots, \alpha_n$, or just an Euler field, if all weights are equal to 1.

The vector field Y integrates to a 1-parametric group of linear transformations $Y^s : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$Y^s(x_1, \ldots, x_n) = (e^{\alpha_1 s} x_1, \ldots, e^{\alpha_n s} x_n).$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *quasi-homogeneous of degree d with weights* $\alpha_1, \ldots, \alpha_n$ if $f(Y^s(x)) = e^{ds} F(x)$ for all $x \in \mathbb{R}^n, s \in \mathbb{R}$.

A line field λ in a domain Ω is called *quasi-homogeneous of degree d with weights* $\alpha_1, \ldots, \alpha_n$) if $Y^s_* \lambda = \lambda$ for all s, i.e. transformations Y^s are symmetries of λ.[1]

A differential equation is called *quasi-homogeneous* if the corresponding line field in the extended phase space is quasi-homogeneous.

Exercise 2.11. 1. Consider a system of equations $\dot{x} = f(x),\ x \in \mathbb{R}^n$. Suppose that the coordinate functions f_i are quasi-homogeneous of degrees d_i with the same weights $\alpha_1, \ldots, \alpha_n$. The corresponding line field λ in the extended phase space (x, t) is given by the system of Pfaffian equations

$$dx_1 = f_1(x_1, \ldots, x_n)dt;$$

$$\cdots$$

$$dx_n = f_n(x_1, \ldots, x_n)dt.$$

Suppose $d_1 - \alpha_1 = \cdots = d_n - \alpha_n$. Prove that the line field λ is quasi-homogeneous and find the weights. Let Y^s be the quasi-homogeneous flow $Y^s(x_1, \ldots, x_n) = (e^{\alpha_1 s} x_1, \ldots, e^{\alpha_n s} x_n)$. Compute the push-forward by Y^s of the vector field $X = \sum\limits_1^n f_i \frac{\partial}{\partial x_i}$.

[1] Note that the above definition implies, among other things that domain Ω itself is *invariant* with respect to Y^s, i.e. $Y^s(\Omega) = \Omega$ for all $s \in \mathbb{R}$.

2. Consider equation of k-th order with respect to 1 unknown function:

$$\frac{d^k y}{dx^k} = f(x, y).$$

Suppose that $f(x, y)$ is a quasi-homogeneous function of degree d with weights α, β. Find a relation between α, β and d which ensures that the line field representing the system in its extended $(k+1)$-dimensional phase space is quasi-homogeneous (and find weights).

2.7 Directional derivative revisited

Let X be a smooth vector field defined on a domain $U \subset \mathbb{R}^n$ (more generally we can assume that U is any n-dimensional manifold). Given a function $f : U \to \mathbb{R}$ we can define the *directional derivative* $L_X f$ of f along X:

$$L_X f = \lim_{s \to 0} \frac{f(x + tX) - f(x)}{t}. \tag{2.7.1}$$

The directional derivative has many other notation: $D_X(f)$, $\frac{\partial f}{\partial X}$, $df(X), \dots$.

Let us denote by $X^t : U' \to U$, $t \in (-\epsilon, \epsilon)$, the local phase flow of X^t defined on a neighborhood $U' \subset U$ of a point $a \in U$.

Let us observe that the directional derivative can be also defined by the formula

$$L_X f(a) = \frac{d}{ds} f \circ X^s \Big|_{s=0}(a). \tag{2.7.2}$$

It turns out that formula (2.7.2) can be generalized to define an analog of directional derivatives for differential forms and vector fields, which is the *Lie derivative*.

2.8 Lie derivative of a differential form

Let ω be a differential k-form. We define the Lie derivative $L_X\omega$ of ω along a vector field X as

$$L_X \omega = \frac{d}{ds} (X^s)^* \omega \Big|_{s=0}. \tag{2.8.1}$$

Note that if ω is a 0-form, i.e. a function f, then $(X^s)^* f = f \circ X^s$, and hence, in this case definitions (2.7.2) and (2.8.1) coincide, i.e. for functions the Lie derivative is the same as the directional derivative.

Proposition 2.12. *The following identities hold*

1. $L_X(\omega_1 \wedge \omega_2) = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge L_X\omega_2$.

2. $L_X(d\omega) = d(L_X\omega)$.

Proof.

1. $$L_X(\omega_1 \wedge \omega_2) = \frac{d}{ds}(X^s)^*(\omega_1 \wedge \omega_2)\Big|_{s=0} = \frac{d}{ds}\big((X^s)^*\omega_1 \wedge (X^s)^*\omega_2\big)\Big|_{s=0}$$
$$= \frac{d}{ds}\big((X^s)^*\omega_1\big)\Big|_{s=0} \wedge \omega_2 + \omega_1 \wedge \frac{d}{ds}\big((X^s)^*\omega_2\big)\Big|_{s=0} = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge L_X\omega_2.$$

2. $$L_X(d\omega) = \frac{d}{ds}\big((X^s)^*d\omega\big)\Big|_{s=0} = \frac{d}{ds}\big(d(X^s)^*\omega\big)\Big|_{s=0} = d\Big(\frac{d}{ds}(X^s)^*\omega\Big|_{s=0}\Big) = L_X(d\omega).$$

■

The following formula of Élie Cartan provides an effective way for computing the Lie derivative of a differential form.

Theorem 2.13. *Let X be a vector field and ω a differential k-form. Then*

$$L_X\omega = d(X \lrcorner\, \omega) + X \lrcorner\, d\omega. \tag{2.8.2}$$

Proof. Suppose first that $\omega = f$ is a 0-form. Then $L_Xf = df(X) = X \lrcorner\, df$, which is equivalent to formula (2.8.2), because in this case the first term in the formula is equal to 0. Then, using Proposition 2.122) we get

$$L_Xdf = dL_Xf = d(df(X)) = d(X \lrcorner\, df),$$

which is again equivalent to (2.8.2) because in this case $ddf = 0$. Next we note that if the formula

(3.8.1) holds for ω_1 and ω_2 then it holds also for $\omega_1 \wedge \omega_2$. Indeed, we have

$$
\begin{aligned}
(\star)\quad L_X(\omega_1 \wedge \omega_2) &= (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge L_X\omega_2 \\
&= (X \lrcorner\, d\omega_1 + d(X \lrcorner\, \omega_1)) \wedge \omega_2 + \omega_1 \wedge (X \lrcorner\, d\omega_2 + d(X \lrcorner\, \omega_2)) \\
&= (X \lrcorner\, d\omega_1) \wedge \omega_2 + \omega_1 \wedge (X \lrcorner\, d\omega_2) + d(X \lrcorner\, \omega_1) \wedge \omega_2 + \omega_1 \wedge d(X \lrcorner\, \omega_2)
\end{aligned}
$$

On the other hand, denoting by d_1 and d_2 the degrees of ω_1 and ω_2, we get

$$
\begin{aligned}
&(\star\star)\ X \lrcorner\, d(\omega_1 \wedge \omega_2) + d(X \lrcorner\, (\omega_1 \wedge \omega_2)) \\
&= X \lrcorner\, (d\omega_1 \wedge \omega_2 + (-1)^{d_1}\omega_1 \wedge d\omega_2) + d((X \lrcorner\, \omega_1) \wedge \omega_2 + (-1)^{d_1}\omega_1 \wedge (X \lrcorner\, \omega_2)) \\
&= (X \lrcorner\, d\omega_1) \wedge \omega_2 + (-1)^{d_1+1} d\omega_1 \wedge (X \lrcorner\, \omega_2) + (-1)^{d_1}(X \lrcorner\, \omega_1) \wedge d\omega_2 + \omega_1 \wedge (X \lrcorner\, d\omega_2) \\
&+ d(X \lrcorner\, \omega_1) \wedge \omega_2 + (-1)^{d_1+1} X \lrcorner\, \omega_1 \wedge d\omega_2 + (-1)^{d_1} d\omega_1 \wedge (X \lrcorner\, \omega_2) + \omega_1 \wedge (d(X \lrcorner\, \omega_2)) \\
&= (X \lrcorner\, d\omega_1) \wedge \omega_2 + \omega_1 \wedge (X \lrcorner\, d\omega_2) + d(X \lrcorner\, \omega_1) \wedge \omega_2 + \omega_1 \wedge d(X \lrcorner\, \omega_2)\,.
\end{aligned}
$$

Comparing the computation in $(\star)$ and $(\star\star)$ we conclude that

$$L_X(\omega_1 \wedge \omega_2) = X \lrcorner\, d(\omega_1 \wedge \omega_2) + d(X \lrcorner\, (\omega_1 \wedge \omega_2)).$$

By induction we can prove a similar formulas for an exterior product of any number of forms.

Finally we observe that any differential k-form ω can be written in coordinates as

$$\sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \dots i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

i.e. ω is a sum of exterior products of functions (0-forms) and exact 1-forms, and hence Cartan's formula follows. ∎

Proposition 2.14. *We have*

$$L_X\omega = 0 \iff (X^s)^*\,\omega = \omega \ \textit{ for all } \ s \in \mathbb{R}.$$

Proof. If $(X^s)^*\,\omega \equiv \omega$ then $L_X\omega = \frac{d}{ds}(X^s)^*\omega\Big|_{s=0} = 0$. To prove the converse we note that

$$
\begin{aligned}
&\frac{d}{ds}(X^s)^*\omega\Big|_{s=s_0} = \lim_{t\to 0}\frac{(X^{s_0+t})^*\omega - (X^{s_0})^*\omega}{t} = (X^{s_0})^*\left(\lim_{t\to 0}\frac{(X^t)^*\omega - \omega}{t}\right) \\
&= (X^{s_0})^*\,(L_X\omega)\,,
\end{aligned}
$$

and hence if $L_X\omega = 0$ then $(X^s)^*\,\omega = \omega$. ∎

2.9 Lie bracket of vector fields

Let $A, B \in \text{Vect}(U)$ be two vector fields on a domain $U \subset \mathbb{R}^n$. As it was shown in 52H, there is a vector field $C \in \text{Vect}(V)$, called the *Lie bracket* of the vector fields A and B and denoted by $C = [A, B]$, which is characterized by the following property: *for any smooth function* $\phi : U \to \mathbb{R}$ *one has*

$$L_C\phi = (L_A L_B - L_B L_A)\phi.$$

A surprising fact here is that though the right-hand side of this equation seems to be the second order differential operator, the left-hand side is the first order operator, so the second derivatives on the right side cancel each other.

Recall that the bracket $[A, B]$ has the following properties

- Lie bracket is a bilinear operation;
- $[A, B] = -[B, A]$ (skew-symmetricity);
- $[[A, B]C] + [[B, C], A] + [[C, A], B] = 0$ (Jacobi identity);
- If $A = \sum\limits_1^n a_j \frac{\partial}{\partial x_j}$ and $B = \sum\limits_1^n b_j \frac{\partial}{\partial x_j}$ then

$$[A, B] = \sum_{i=1}^n \left(\sum_{j=1}^n a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}\,. \tag{2.9.1}$$

In this section we will give a new interpretation of the Lie bracket $[A, B]$.

Recall that given a diffeomorphism $f : U \to V$ we can define the push-forward map

$$f_* : \text{Vect}(U) \to \text{Vect}(V).$$

We can also define the *pull back* map

$$f^* : \text{Vect}(V) \to \text{Vect}(U)$$

by the formula $f^* := f_*^{-1}$. Note that we also have $f^* = f_*^{-1}$.

We define the *Lie derivative* $L_A B$ of the vector field B along the vector field A in a similar way as we defined in Section 2.8 the Lie derivative of a differential form. Namely,

$$L_A B = \frac{d\,(A^s)^*\,B}{ds}\Big|_{s=0}. \tag{2.9.2}$$

More explicitly,

$$L_A B(x) = \lim_{s\to 0} \frac{d_{A^s(x)}\left(A^{-s}\right)\left(B(A^s(x)) - B(x)\right)}{s}.$$

Similarly, to Proposition 2.14 we have

Proposition 2.15.

$$L_A B = 0 \iff \left(A^s\right)^* B \equiv B \ \textit{ for all } \ s \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}
&\left.\frac{d\left(A^s\right)^* B}{ds}\right|_{s=s_0} = \lim_{s\to 0} \frac{\left(A^{s+s_0}\right)^* B - \left(A^{s_0}\right)^* B}{s} \\
&= \lim_{s\to 0} \left(A^{s_0}\right)^* \left(\frac{\left(A^s\right)^* B - B}{s}\right) = \left(A^{s_0}\right)^* \left(\lim_{s\to 0} \frac{\left(A^s\right)^* B - B}{s}\right) \\
&= \left(A^{s_0}\right)^* \left(L_A B\right).
\end{aligned}$$

Hence, if $L_A B = 0$ then $\frac{d(A^s)^* B}{ds}$ for all s and hence $\left(A^s\right)^* B = \left(A^0\right)^* B = B$. The converse is obvious. ∎

Theorem 2.16. *For any two vector fields $A, B \in \mathrm{Vect}(U)$*

$$L_A B = [A, B].$$

Proof. Note that $A^s(x) = x + sA(x) + o(s)$. Hence, we can write

$$d_y A^{-s} = \mathrm{Id} - s d_y A + o(s),$$

where we view here A as a map $\mathbb{R}^n \to \mathbb{R}^n$. Furthermore, plugging $y = A^s(x)$ we get

$$d_{A^s(x)} A^{-s} = \mathrm{Id} - s d_x A + o(s).$$

Indeed, $d_{A^s(x)} A - d_x A \underset{s\to 0}{\to} 0$ and hence $s(d_y A - d_x A) = o(s)$. We also have

$$B(A^s(x)) = B(x + sA(x) + o(x)) = B(x) + s d_x B(A(x)) + o(s).$$

Thus, ignoring $o(s)$-terms we get

$$
\begin{aligned}
L_A B &= \lim_{s\to 0} \frac{1}{s}\left(d_{A^s(x)}\left(A^{-s}\right)\left(B(A^s(x))\right) - B(x)\right) \\
&= \lim_{s\to 0} \frac{1}{s}\left((\mathrm{Id} - s d_x A)\right)\left(B(x) + s d_x B(A(x))\right) - B(x)) \\
&= \lim_{s\to 0} \frac{1}{s}\left(B(x) - s d_x A(B) + s d_x B(A) - B(x)\right) = d_x B(A) - d_x A(B).
\end{aligned}
$$

But the right-hand-side expression written in coordinates has the form

$$
d_x B(A) - d_x A(B) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}
$$

which coincides with the expression (2.9.1) for the Lie bracket. ■

Exercise 2.17. Prove that for any smooth function ϕ one has

$$
L_{[A,B]}\phi = \frac{\partial^2(\phi \circ A^s \circ B^t)}{\partial s \partial t}.
$$

If $[A,B]=0$ then one says that the vector field A and B *commute.*

Lemma 2.18. *Suppose two commuting vector fields A, B on Ω can be integrated into phase flows A^t, B^s. Then*

$$
A^t \circ B^s = B^s \circ A^t,
$$

$t, s \in \mathbb{R}$, *i.e. the flows of commuting vector fields. Conversely, if two flows A^t, B^s commute for all $t, s \in \mathbb{R}$ then $[A,B]=0$.*

Proof. We have $[A,B] = L_A B$. Then according to Proposition 2.15 we have

$$
(A^s)^* B = B. \tag{2.9.3}
$$

Recall from Proposition 2.7 that for any diffeomorphism $f : \Omega \to \Omega$ if $f^* B = C$ then

$$
C^t = f^{-1} \circ B^t \circ f, \quad t \in \mathbb{R}.
$$

Applying this to $f = A^s$ and using (2.9.3) we conclude

$$
B^t = A^{-s} \circ B^t \circ A^s,
$$

or

$$
A^s \circ B^t = B^t \circ A^s, \quad s, t \in \mathbb{R}.
$$

■

2.10 First integrals

Suppose we are given a differential equation

$$\dot{x} = A(x), \tag{2.10.1}$$

where A is a vector field on the domain $U \subset \mathbb{R}^n$ A function $\phi : U \to \mathbb{R}$ is called a *first integral*, or simply an *integral* of equation (2.10.1) if it is constant on solutions of this equation, or equivalently on integral curves of the vector field A.

Clearly, a necessary and sufficient condition for ϕ to be an integral is to satisfy the equation $L_A\phi = 0$. Here $L_A\phi$ denotes the directional derivative of ϕ along A.

If ϕ is an integral of (2.9.2) then the solutions are contained in the level sets of the function ϕ, and hence, this allows us to reduce the order of equation by 1. If (2.9.2) has two integrals ϕ_1, ϕ_2, then the solutions lie in the intersection of level sets $\{\phi_1 = c_1\}$ and $\{\phi_2 = c_2\}$, $c_1, c_2 \in \mathbb{R}$. Hence, if these level sets transverse to each other (which means that the differential $d\phi_1$ and $d\phi_2$ are linearly independent at every point of the intersection), then the solutions lie in $\{\phi_1 = c_1\} \cap \{\phi_2 = c_2\}$, which allows to further reduce the order of the system. If the order is reduced to 1 then the equation can be explicitly integrated in quadratures. Such systems are called *completely intregrable.*

Some important examples of integrals which come from Mechanics are discussed in the next section.

2.11 Hamiltonian vector fields

Consider the vector space $\mathbb{R}^{2n}$ with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and a closed differential 2-form $\omega = \sum\limits_1^n dp_i \wedge dq_i$. Note that this form is non-degenerate, i.e. its matrix is non-degenerate at every point. Therefore, the map $J : \mathrm{Vect}(\mathbb{R}^{2n}) \to \Omega^1(\mathbb{R}^{2n})$ given by the formula $X \mapsto X \lrcorner\, \omega$ is an isomorphism between the space $\mathrm{Vect}(\mathbb{R}^{2n})$ of vector fields and the space $\Omega^1(\mathbb{R}^{2n})$ of differential 1-forms on $\mathbb{R}^n$. In coordinates the map J associates with a vector field $\sum\limits_1^n P_i \frac{\partial}{\partial P_i} + \sum\limits_1^n Q_i \frac{\partial}{\partial Q_i}$ the differential form $\sum\limits_1^n P_i dq_i - Q_i dp_i$.

Lemma 2.19. *Given a vector field A on $\mathbb{R}^{2n}$ the differential 1-form $J(A) = A \lrcorner\, \omega$ is closed if and only if $L_A\omega = 0$.*

Proof. Indeed, according to Cartan's formula (2.8.2) we have $L_A\omega = d(A \lrcorner \omega) = dJ(A)$ because ω is closed. ∎

Given a function $H : \mathbb{R}^{2n} \to \mathbb{R}$ we denote by X_H the vector field $-J^{-1}(dH)$. Vector fields obtained by this construction are called *Hamiltonian.*

To find a coordinate expression for X_H we write $X_H = \sum\limits_1^n a_i \frac{\partial}{\partial p_i} + b_i \frac{\partial}{\partial q_i}$. Then

$$X_H \lrcorner \omega = \left(\sum_1^n a_i \frac{\partial}{\partial p_i} + b_i \frac{\partial}{\partial q_i} \right) \lrcorner \sum_1^n dp_i \wedge dq_i = \sum_1^n -b_i dp_i + a_i dq_i.$$

Hence, the equation

$$X_H \lrcorner \omega = -dH = -\sum_1^n \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i$$

implies $a_i = -\frac{\partial H}{\partial q_i}, b_i = \frac{\partial H}{\partial p_i}$, $i = 1, \dots, n$. Thus,

$$X_H = \sum_1^n -\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}.$$

In a shorter form, omitting indices we will write

$$X_H = -\frac{\partial H}{\partial q} \frac{\partial}{\partial p} + \frac{\partial H}{\partial p} \frac{\partial}{\partial q}.$$

Thus the system of differential equations corresponding to the vector field X_H has the form

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p}. \end{aligned} \tag{2.11.1}$$

These equations play an important role in Mechanics, and called *Hamilton canonical equations.* They describe the phase flow of a mechanical system. Here the coordinates $q = (q_1, \dots, q_n)$ determine a position of the system, or a point in the *configuration space* of the mechanical system. The coordinates $p = (p_1, \dots, p_n)$ are called *momenta* and can be viewed as vectors of the cotangent bundle to the configuration space. The function H is the full energy of the system expressed through coordinates and momenta.

Lemma 2.20. *The function H is a first integral of the equation* (2.11.1)*, i.e.* $L_{X_H} H = 0$.

Proof.

$$L_{X_H}H = dH(X_H) = -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = 0.$$

■

Example 2.21. Consider Newton equations

$$\ddot{q}_i = -\frac{\partial U}{\partial q_i}, i = 1, \dots, n,$$

or in shorter notation

$$\ddot{q} = -\frac{\partial U}{\partial q} = -\nabla U.$$

Reducing it to a system of first order equation we get

$$\dot{p} = -\frac{\partial U}{\partial q} \tag{2.11.2}$$

$$\dot{q} = p. \tag{2.11.3}$$

Consider the *full energy* $H(p,q) = \sum\limits_1^n \frac{p_i^2}{2} + U(q) = \frac{1}{2}p^2 + U(q)$. Then $\frac{\partial H}{\partial q} = \frac{\partial U}{\partial q}$ and $\frac{\partial H}{\partial p} = p$, and hence equation (2.11.2) takes the form (2.11.1) with this Hamiltonian function H. Lemma 2.20 is the law of conservation law of energy.

Lemma 2.22. *Let X_H be a Hamiltonian vector field and X_H^s the phase flow it generates. Then $(X_H^s)^*\omega = \omega$ for all $s \in \mathbb{R}$. In other words, the flow of a Hamiltonian vector field preserves the form ω.*

Proof. It is sufficient to prove that $L_{X_H}\omega = 0$. Using Theorem 2.13 we get

$$L_{X_H}\omega = d(X_H \lrcorner\, \omega) + X_H \lrcorner\, d\omega.$$

But ω is closed, and hence $d\omega = 0$, while $X_H \lrcorner\, \omega = dH$. Thus, $L_{X_H}\omega = ddH = 0$. ■

2.12 Canonical transformations

The equations (2.11.1) are called canonical because they are invariant with respect to a large group of transformation of the phase space. Let us call a diffeomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a *symplectomorphism* (or alternatively a *canonical transformation*) if it preserves the form ω. Then it preserves also the form of the equations (2.11.1). Indeed, suppose $f(p,q) = (\widetilde{p}, \widetilde{q})$. Then $f^*(\omega) = f^*(dp \wedge dq) = d\widetilde{p} \wedge d\widetilde{q} = \omega = dp \wedge dq$. Thus if we express the function $H(p,q)$ through the coordinates $\widetilde{p}, \widetilde{q}$, $H(p,q) = \widetilde{H}(\widetilde{p}, \widetilde{q})$ then the equation (2.11.1) will take the same form in coordinates $(\widetilde{p}, \widetilde{q})$:

$$
\begin{aligned}
\dot{\widetilde{p}} &= -\frac{\partial \widetilde{H}}{\partial \widetilde{q}} \\
\dot{\widetilde{q}} &= \frac{\partial \widetilde{H}}{\partial \widetilde{p}}.
\end{aligned}
\tag{2.12.1}
$$

The following proposition provides an important class of canonical transformations,

Proposition 2.23. *Consider any diffeomorphism $f : U \to V$ between two domains $U, V \subset \mathbb{R}^n$. Let Df be the Jacobi matrix of the map U. Then the map*

$$(p,q) \mapsto \left((Df)^{-1}\right)^T p, f(q))$$

is a symplectomorphism $\widehat{f}$ of the domain $\widehat{U} = \{p \in \mathbb{R}^n, q \in U\}$ to the domain $\widehat{V} = \{p \in \mathbb{R}^n, q \in V\}$. Here $\left((Df)^{-1}\right)^T$ is the matrix transpose to inverse of the Jacobi matrix Df.

In other words, *any change of q-coordinates extends to a canonical change of the (p,q)-coordinates.*

Proof. Let us denote the elements of the matrix $(Df)^{-1}$ by $g_{ij},\ i,j = 1,\dots,n$. Thus, $\sum\limits_i^n g_{ji} \frac{\partial f_i}{\partial q_k} = \delta_{jk}$, $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$.

Let us compute $\widehat{f}^*(pdq) = \widehat{f}^*\left(\sum\limits_1^n p_i dq_i\right)$. We have

$$\widehat{f}(p_1,\dots,p_n,q_1,\dots,q_n) = \left(\sum_1^n g_{j1}p_j, \dots, \sum_1^n g_{jn}p_j, f_1(q), \dots, f_n(q)\right).$$

Hence,

$$
\begin{aligned}
\widehat{f}^*(pdq) = \widehat{f}^*\left(\sum_1^n p_i dq_i\right) &= \sum_{i=1}^n \sum_{j=1}^n g_{ji} p_j df_i \\
&= \sum_{i,j,k=1}^n g_{ji} \frac{\partial f_i}{\partial q_k} p_j dq_k = \sum_{j,k=1}^n \delta_{jk} p_j dq_k \\
&= \sum_1^n p_k dq_k = pdq.
\end{aligned}
$$

Hence,

$$\widehat{f}^*\omega = \widehat{f}^* dp \wedge dq = d(\widehat{f}^*(pdq)) = d(pdq) = dp \wedge dq = \omega.$$

■

Corollary 2.24. . *Suppose that there exists a change of coordinates $\widetilde{q} = f(q)$ such that in new coordinates the Hamiltonian function H is independent of the coordinate $\widetilde{q}_1$. Then $\widetilde{p}_1 = \sum_1^n g_{j1}p_j$ is a first integral of the system* (2.11.1). *Here the notation g_{ij} stands for the elements of the matrix $(Df)^{-1}$.*

Proof. Let us extend the coordinate change $q \mapsto \widetilde{q} = f(q)$ to a canonical change of coordinates $(p, q) \mapsto (\widetilde{p}, \widetilde{q}) = \widetilde{f}(p, q)$ as in Proposition 2.23. Then the equation in the new coordinates $(\widetilde{p}, \widetilde{q})$ also has the canonical Hamiltonian form (2.12.1). Then $\dot{\widetilde{p}}_1 = \frac{\partial H}{\partial \widetilde{q}_1} = 0$ because by assumption the Hamiltonian is independent of the coordinate $\widetilde{q}_1$. Hence $\widetilde{p}_1 = \sum_1^n g_{j1}p_j$ is constant along trajectories, i.e. it is a first integral. ■

2.13 Example: angular momentum

Consider a Newton equation

$$\ddot{q} = -\nabla U(q), \quad q \in \mathbb{R}^3, \tag{2.13.1}$$

which describes the motion of a particle of mass 1 in a field with a potential energy function $U(q)$. Suppose there exists an axis l in $\mathbb{R}^3$ such that the function $U(q)$ remains invariant with respect to rotations around l.

The system (2.13.1) can be rewritten in the Hamiltonian form (2.11.1) with the Hamiltonian function $H = \frac{p^2}{2} + U(q) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{p_3^2}{2} + U(q_1, q_2, q_3)$. Let us assume for simplicity that the q_3-axis coincides with the axis l.

Let us change coordinates (q_1, q_2, q_3) to cylindrical coordinates (ϕ, r, z):

$$q_1 = r\cos\phi, q_2 = r\sin\phi, q_3 = z.$$

Equivalently,

$$\phi = \arctan\frac{q_2}{q_1}, r = \sqrt{q_1^2 + q_2^2}, z = q_3.$$

Computing the Jacobi matrix $\frac{D(\phi,r,z)}{D(q_1,q_2,q_3)}$ we get

$$\begin{pmatrix} \frac{\partial\phi}{\partial q_1} & \frac{\partial\phi}{\partial q_2} & \frac{\partial\phi}{\partial q_3} \\ \frac{\partial r}{\partial q_1} & \frac{\partial r}{\partial q_2} & \frac{\partial r}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{pmatrix} = \begin{pmatrix} -\frac{q_2}{q_1^2+q_2^2} & \frac{q_1}{q_1^2+q_2^2} & 0 \\ \frac{q_1}{\sqrt{q_1^2+q_2^2}} & \frac{q_2}{\sqrt{q_1^2+q_2^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then the inverse matrix is equal to

$$\begin{pmatrix} -q_2 & \frac{q_1}{\sqrt{q_1^2+q_2^2}} & 0 \\ q_1 & \frac{q_2}{\sqrt{q_1^2+q_2^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let us extend the coordinate change $(q_1, q_2, q_3) \mapsto (r, \phi, z)$ to a canonical coordinate change

$$(q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (\phi, r, z, p_\phi, p_r, p_z),$$

where we denoted by p_r, p_ϕ, p_z momenta variables corresponding to new coordinates (r, ϕ, z). In fact, we need only the coordinate p_ϕ which is given by $p_\phi = -p_1 q_2 + q_1 p_2$. Thus, the function $-p_1 q_2 + p_2 q_1$ is the first integral. It is called *the angular momentum* around the q_3-axis.

Recall that along trajectories we have $p_i = \dot{q}_i$, $i = 1, 2, 3$. Hence, $q_1 \dot{q}_2 - \dot{q}_1 q_2$ is constant along the trajectories. But this is exactly the projection M_3 of the cross-product $M = q \times \dot{q}$ to the q_3-axis which is the axis of rotational symmetry. Introducing cylindrical coordinates $(r, \phi, z)$with the axis q_3 as z, then we get $M_3 = r^2 \dot{\phi}$.

In particular, if $U(q)$ is invariant under *all* rotations, i.e. it depends only on the distance $r = ||q||$ from the origin, then all components of the angular momentum vector $M = q \times \dot{q}$, and hence, the angular momentum vector M is constant along trajectories. Note that $q\dot{M} = 0$, and hence the motion happens in the plane orthogonal to the vector M. In the cylindrical coordinates with M at its axis, the absolute value of the angular momentum,

$$||M|| = r^2 \dot{\phi}$$

is preserved.

Chapter 3

Simplification of the matrix of a linear operator

3.1 Linear operators and their matrices

Let $\mathcal{A} : V_1 \to V_2$ be a linear operator between n_1-dimensional vector space V_1 and n_2-dimensional manifold V_2. Given bases $\mathcal{B}^1 = (v_1^1, \dots, v_{n_1}^1)$ in V_1 and $\mathcal{B}^2 = (v_1^2, \dots, v_{n_2}^2)$ in V_2 one can associate with it an $n_1 \times n_2$-matrix $M_{\mathcal{B}_1}^{\mathcal{B}_2}(\mathcal{A}) = A$ whose columns are coordinates of the vectors $\mathcal{A}(v_1^1), \dots, \mathcal{A}(v_{n_1}^1)$ in the basis $\mathcal{B}_2$. Given a third space V_3 with a basis $\mathcal{B}^3 = (v_1^3, \dots, v_{n_3}^3)$ and a map $\mathcal{B} : V_2 \to V_3$ one can associate with it a matrix $B = M_{\mathcal{B}_1}^{\mathcal{B}_2}(\mathcal{A})$. Then the composition $\mathcal{C} = \mathcal{B} \circ \mathcal{A}$ has a matrix

$$C = M_{\mathcal{B}_1}^{\mathcal{B}_3}(\mathcal{C}) = M_{\mathcal{B}_2}^{\mathcal{B}_3}(\mathcal{B}) M_{\mathcal{B}_1}^{\mathcal{B}_2}(\mathcal{A}) = BA. \tag{3.1.1}$$

Let us apply this formula to the following situation. Suppose we are given two different bases $\widetilde{\mathcal{B}}^1 = (\widetilde{v}_1^1, \dots, \widetilde{v}_{n_1}^1)$ in V_1 and $\widetilde{\mathcal{B}}^2 = (\widetilde{v}_1^2, \dots, \widetilde{v}_{n_1}^2)$ in V_2. In order to relate the matrices $A = M_{\mathcal{B}^1}^{\mathcal{B}^2}(\mathcal{A})$ and $\widetilde{A} = M_{\widetilde{\mathcal{B}}_1}^{\widetilde{\mathcal{B}}_2}(\mathcal{A})$ let us consider the following diagram

$$\underset{\widetilde{\mathcal{B}}_1}{V_1} \xrightarrow{\mathrm{Id}} \underset{\mathcal{B}^1}{V_1} \xrightarrow{\mathcal{A}} \underset{\mathcal{B}^2}{V_2} \xrightarrow{\mathrm{Id}} \underset{\widetilde{\mathcal{B}}^2}{V_2}\,.$$

Then, using (3.1.1) we get

$$M_{\widetilde{\mathcal{B}}^2}^{\widetilde{\mathcal{B}}^2}(\mathcal{A}) = M_{\widetilde{\mathcal{B}}^2}^{\mathcal{B}^2}(\mathrm{Id}) M_{\mathcal{B}_1}^{\mathcal{B}_2}(\mathcal{A}) M_{\mathcal{B}_1}^{\widetilde{\mathcal{B}}_1}(\mathrm{Id}). \tag{3.1.2}$$

The matrix $M_{\mathcal{B}^1}^{\widetilde{\mathcal{B}}^1}(\mathrm{Id})$ is called the *transition matrix* from the basis $\mathcal{B}^1$ to $\widetilde{\mathcal{B}}^1$. Its columns are coordinates of the vectors of the basis $\widetilde{\mathcal{B}}^1$ in the basis $\mathcal{B}^1$. Note that the formula (3.1.1) also implies that

$$M_{\mathcal{B}^1}^{\widetilde{\mathcal{B}}^1} = \left(M_{\widetilde{\mathcal{B}}^1}^{\mathcal{B}^1}\right)^{-1}.$$

We will mostly consider below operators $\mathcal{A} : V \to V$ which map an n-dimensional space V into itself. Given a basis $\mathcal{B}$ in V we will write $M_{\mathcal{B}}(\mathcal{A})$ instead of $M_{\mathcal{B}}^{\mathcal{B}}(\mathcal{A})$.

Suppose we are given another basis $\widetilde{\mathcal{B}}$. Let us denote by C the transition matrix $M_{\mathcal{B}}^{\widetilde{\mathcal{B}}}$ from $\mathcal{B}$ to $\widetilde{\mathcal{B}}$. Then, using (3.1.2) we get

$$\widetilde{A} := M_{\widetilde{\mathcal{B}}}(\mathcal{A}) = C^{-1}AC, \tag{3.1.3}$$

where we denoted $A := M_{\mathcal{B}}(\mathcal{A})$.

3.2 Characteristic polynomial, eigenvectors and eigenvevalues

Let us assume that V is a *complex* vector space and $\mathcal{A} : V \to V$ is a *complex linear* operator.

A complex number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the operator $\mathcal{A}$ if there exists a non-zero vector $v \in V$ such that $\mathcal{A}(v) = \lambda v$. The vector v is called an *eigenvector corresponding to the eigenvalue* λ. The set of all eigenvectors corresponding to the eigenvalue λ (including the 0-vector) form a linear subspace of V, which is called the *eigenspace corresponding to the eigenvalue* λ and is denoted by E_λ.

Let us observe that if A is the matrix of $\mathcal{A}$ in a basis $\mathcal{B}$ then the determinant $\det(A - \lambda I)$ is independent of a choice of $\mathcal{B}$. Indeed,

$$\begin{aligned}
&\det(C^{-1}AC - \lambda I) = \det(C^{-1}AC - C^{-1}\lambda IC) \\
&= \det(C^{-1}A - \lambda I)C) = \det C^{-1} \det(A - \lambda I) \det C = \det(A - \lambda I).
\end{aligned}$$

This determinant, which is a polynomial of degree n is called the *characteristic polynomial* of the operator $\mathcal{A}$, or the matrix A. We will denote it either by $\chi_{\mathcal{A}}(\lambda)$ or $\chi_A(\lambda)$.

Lemma 3.1. *λ is an eigenvalue of an operator $\mathcal{A}$ if and only if it is a root of its characteristic polynomial, i.e. $\chi_{\mathcal{A}}(\lambda) = 0$.*

Indeed, if λ is an eigenvalue, then $\mathrm{Ker}\,(\mathcal{A}-\lambda\mathrm{Id}) \neq 0$, and hence the rank of the operator is $< n$. This is in turn equivalent to the vanishing of its determinant, i.e. $\chi_{\mathcal{A}}(\lambda) = \det(\mathcal{A}-\lambda\mathrm{Id}) = 0$. Conversely, $\det(\mathcal{A}-\lambda\mathrm{Id}) = 0$ implies that $\mathrm{Ker}\,(\mathcal{A}-\lambda\mathrm{Id}) \neq 0$, i.e. λ is an eigenvalue.

Consider an expansion of the characteristic polynomial of a matrix $A = (a_{ij})$:

$$\begin{aligned}\chi_A(\lambda) &= \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1(n-1)} & a_{1n} \\ & \dots & \dots & \dots & \\ a_{(n-1)1} & a_{(n-1)2} & \dots & a_{(n-1)n}-\lambda & a_{(n-1)n} \\ a_{n1} & a_{n2} & \dots & a_{(n-1)n} & a_{nn}-\lambda \end{vmatrix} \\ &= (-1)^n\lambda^n + (-1)^{n-1}\mathrm{Tr}A\lambda^{n-1} + \dots + \det A \\ &= (-1)^n(\lambda-\lambda_1)^{k_1}\dots(\lambda-\lambda_s)^{k_s}.\end{aligned}$$

Here we denote by $\mathrm{Tr}A$ the *trace* $\sum_1^n a_{ii}$ of the matrix A, i.e. the sum of its diagonal elements. Note that $\mathrm{Tr}A$ (as well as $\det A$ and all other coefficients of the characteristic polynomial) depends only on the operator $\mathcal{A}$ and not on its matrix A. Hence, we can also use the notation $\mathrm{Tr}\mathcal{A}$ instead of $\mathrm{Tr}A$.

Exercise 3.2. *Prove that*

$$\det e^A = e^{\mathrm{Tr}A}.$$

Hint: Replace A by At and differentiate both parts with respect to t.

Note that the decomposition $\chi_A(\lambda) = (-1)^n(\lambda-\lambda_1)^{k_1}\dots(\lambda-\lambda_s)^{k_s}$ into the product of linear terms is possible *only because we consider the complex case.*

The set of all eigenvalues of an operator $\mathcal{A}$ is called its *spectrum*.

3.3 Diagonalization of the matrix of a linear operator

The matrix of an operator $\mathcal{A}$ is diagonal in a basis $\mathcal{B}$ if and only if this basis consists of eigenvectors of $\mathcal{A}$. In this case the diagonal elements are eigenvalues of $\mathcal{A}$.

Lemma 3.3. *If $v_1,\dots,v_k$ are non-zero eigenvectors which correspond to pairwise distinct eigenvalues of an operator $\mathcal{A}$ then they are linearly independent.*

Proof. We argue by induction. Suppose the claim is already proven for the eigenvectors $v_1, \dots, v_{k-1}$. Suppose that we have

$$c_1 v_1 + \cdots + c_{k-1} v_{k-1} + c_k v_k = 0. \tag{3.3.1}$$

Then applying to both parts the operator $\mathcal{A}$ we get

$$\lambda_1 c_1 v_1 + \cdots + \lambda_{k-1} c_{k-1} v_{k-1} + \lambda_k c_k v_k = 0.$$

Subtracting from the second equality the first one multiplied by λ_k, we get

$$(\lambda_1 - \lambda_k) c_1 v_1 + \cdots + (\lambda_{k-1} - \lambda_k) c_{k-1} v_{k-1} = 0.$$

But $\lambda_j - \lambda_k \neq 0$ for all $j = 1, \dots, k-1$. Hence, $c_1 = \cdots = c_{k-1} = 0$. But then from (3.3.1) follows that $c_k = 0$ as well, and therefore, the vectors $v_1, \dots, v_k$ are linearly independent. ∎

Corollary 3.4. *Suppose that $\mathcal{A}$ has n distinct eigenvalues (i.e. its characteristic polynomial does not have multiple roots). Then there is a basis of its eigenvalues, i.e. the matrix of $\mathcal{A}$ is diagonalizable.*

The next lemma shows that a generic matrix has distinct eigenvalues, and hence diagonalizable.

Lemma 3.5. *The set of diagonalizable matrices is everywhere dense in the space M_n of all $n \times n$ complex matrices.*

Exercise 3.6. *Prove that the set of diagonalizable matrices is also open in the space M_n.*

To prove Lemma 3.5 we need the following

Lemma 3.7. *For any operator $\mathcal{A} : V \to V$ there exists a basis in which its matrix has an upper triangular (or a lower triangular) form.*

Proof. We prove it by induction over the dimension of the vector space V. For 1-dimensional spaces, the claim is obviously true. Suppose we already proved it for operators on spaces of dimension $< n$. Suppose now that $\dim V = n$. Operator $\mathcal{A}$ has at least one eigenvalue λ_1. Let $v_1 \neq 0$ be the corresponding eigenvector. Let us complete it to a basis $v_1, v_2 \dots, v_n$ of V. The matrix A of $\mathcal{A}$ in this basis has the form

$$A = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ & \dots & \dots & \dots & \\ 0 & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

■

Denote $V' := \mathrm{Span}(v_2, \dots, v_n)$ and consider an operator $\mathcal{A}' : V' \to V'$ which is given in the basis $v_2, \dots, v_n$ of V' by the matrix

$$A' = \begin{pmatrix} a_{22} & \dots & a_{2n} \\ \dots & \dots & \\ a_{n2} & \dots & a_{nn} \end{pmatrix}$$

By the inductional hypothesis, there exists a basis $\widetilde{v}_2, \dots, \widetilde{v}_n$ of the space V' such that the matrix of $\mathcal{A}'$ in this basis has an upper triangular form. Then the vectors $v_1, \widetilde{v}_2, \dots, \widetilde{v}_n$ form a basis of V, and the matrix of $\mathcal{A}$ in this basis has an upper triangular form, because for each $j = 2, \dots, n$ we have $\mathcal{A}(\widetilde{v}_j) = \mathcal{A}'(\widetilde{v}) + c_j v_1$ for some c_j. ■

Proof of Lemma 3.5 According to Lemma 3.7 any matrix can be written as $A = C^{-1}TC$, where T is a triangular matrix. But for a triangular matrix its eigenvalue concides with the diagonal elements. Hence, there exists an arbitrarily close matrix T' with all distinct eigenvalues. Then the matrix $A' = C^{-1}T'C$ is the required approximation of A. ■

Let us remark that matrix can be diagonalizable even when it has multiple eigenvalues, though in general in that case one can only get a more complicated Jordan normal form of the matrix which we discuss below.

Let us also point out that there several important classes of diagonalizable matrices. For instance, *any real symmetric matrix is diagonalizable* (specrtral theorem). More generally any complex *Hermitian matrix* A, i.e. a matrix which satisfies $A^T = \overline{A}$ is diagonalizable.

3.4 Hamilton-Cayley theorem

We consider in this section polynomial functions of linear operators, or which is equivalent, of their matrices. Given a polynomial $f(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$ we have $f(A) = c_0A^n + c_1A^{n-1} + \cdots + c_nI$ for a square matrix A.

Note that for any two polynomials $f(\lambda)$ and $g(\lambda)$ the matrices $f(A)$ and $g(A)$ always commute:

$$f(A)g(A) = g(A)f(A),$$

because two powers A^k and A^l of the same matrix commute.

The following is the main result of this section

Theorem 3.8. *For any matrix A we have*

$$\chi_A(A) = 0,$$

i.e. the matrix A is annihilated by its own characteristic polynomial.

We begin the proof this theorem of Hamilton-Cayley with its special case:

Lemma 3.9. *Theorem 3.8 holds for diagonal matrices.*

Proof.

$$\chi_A(\lambda) = (-1)^n(\lambda - \lambda_1)^{k_1} \ldots (\lambda - \lambda_s)^{k_s}.$$

Hence,

$$\begin{aligned}\chi_A(A) &= (-1)^n(A - \lambda_1 I)^{k_1} \ldots (A - \lambda_s I)^{k_s} \\ &= \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ & \ldots & \ldots & \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \quad \ldots \quad \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ & \ldots & \ldots & \\ 0 & 0 & \ldots & 0 \end{pmatrix} = 0\end{aligned}$$

■

Proof of Theorem 3.8. The function $\chi_A(A)$ on the space M_n of complex $n \times n$ matrices depends continuously on A. According to Lemma 3.9 the function vanishes on the set of diagonalizable matrices $\mathcal{D} \subset M_n$. But according to Lemma 3.5 the set $\mathcal{D}$ is everywhere dense in M_n. Hence, by continuity the function $\chi_A(A)$ is identically 0. ■

3.5 The structure of nilpotent operators

An operator $\mathcal{B} : V \to V$ is called *nilpotent* if there exists k such that $\mathcal{B}^k = 0$.

Consider an operator $\mathcal{J}_m : \mathbb{C}^m \to \mathbb{C}^m$ with the matrix

$$J_m = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

This operator is nilpotent: $\mathcal{J}_m^m = 0$. On the standard basis in $\mathbb{C}^m$ it acts as the shift $\mathcal{J}_m(e_m) = e_{m-1}, \mathcal{J}_m(e_{m-1}) = e_{m-2}, \dots, \mathcal{J}_m(e_2) = e_1$, $\mathcal{J}_m(e_1) = 0$.

Theorem 3.10. *For any nilpotent operator $\mathcal{B}$ there exists a basis in which its matrix has a block-diagonal form with matrices $J_{m_1}, J_{m_2}, \dots, J_{m_k}$ along the diagonal.*

Proof. Let $v_1, \dots, v_n$ be a basis of V. Take the first vector v_1. Let h_1 be the maximal power such that $\mathcal{B}^{h_1}(v_1) \neq 0$. We claim that the vectors $w_1^0 = v_1, w_1^1 = \mathcal{B}(v_1), \dots w_1^{h_1} = \mathcal{B}^{h_1}(v_1)$ are linearly independent. We prove this by induction. We assume that the vectors $w_1^{h_1-i+1}, \dots, w_1^{h_1}$ are linearly independent for some $i = 1, \dots, h_1$ and then show that $w_1^{h_1-i}, \dots, w_1^{h_1}$ are independent (the base of the induction $i = 1$ holds by our assumption). If we have $\sum\limits_{h_1-i}^{h_1} c_j w_1^j = 0$ the applying the operator $\mathcal{B}^i$ to both parts we get $c_{h_1-i} w_1^{h_1} = 0$ which means that $c_{h_1-i} = 0$. But then by the inductional hypothesis we have $c_j = 0$ for all $j \geq h_1 - i$, i.e. the vectors $w_1^{h_1-i}, \dots, w_1^{h_1}$ are independent.

If vectors $w_1^0, \dots w_1^{h_1}$ form a basis of V, i.e. if $h_1 = m-1$ then we are done: in the basis $w_1^0, \dots w_1^{h_1}$ the operator $\mathcal{B}$ has the matrix J_m.

If this is not the case, we continue the process. Take the first of the remaining basic vectors which is not in V_1. We can assume that this is the vector v_2: the vectors always can be reordered to achieve this, and denote by h_2 the largest power such that $\mathcal{B}^{h_2} v_2 \neq 0$. We can assume that $h_2 \leq h_1$. If this is not the case we rename the vectors v_1 into v_2 and v_2 into v_1.

As above, form the sequence $w_2^0 := v_2, \dots, w_2^1 := \mathcal{B}(v_2), \dots, w_2^{h_2} := \mathcal{B}^{h_2}(v_2)$. Consider two cases:

a) vectors $w_2^{h_2}$ and $w_1^{h_1}$ are linearly independent;

b) $w_2^{h_2} = c w_1^{h_1}$ for some $c \in \mathbb{R}$.

In case a) the vectors $w_1^0, \ldots, w_1^{h_1}, w_2^0, \ldots, w_2^{h_2}$ are linearly independent. Indeed, suppose that

$$\sum_0^{h_1} c_i w_1^i + \sum_0^{h_2} d_j w_2^j = 0, \tag{3.5.1}$$

where not all coefficients d_j are equal to 0. Let us denote $k := \min\{j; d_j \neq 0\}$. Thus $k \in \{0, \ldots, h_2\}$. If all coefficients $c_i, i = 0, \ldots, h_1$ are 0 then we get a contradiction similar to the previous step of the inductional process. Denote $l := \min\{j; c_j \neq 0\}$. Thus $l \in \{0, \ldots, h_1\}$. If $h_1 - l > h_2 - k$, then applying to both parts of (3.5.1) operator $\mathcal{B}^{h_2-k+1}$ we get

$$\sum_0^{h_1-h_2+k-1} c_i w_1^{i+h_2-k+1} = 0.$$

But we already proved that the vectors $w_1^0, \ldots w_1^{h_1}$ are linearly independent, and hence all coefficients c_i for $i < h_1 - h_2 + k$ are equal to 0. Similarly, we get a contradiction if $h_1 - l < h_2 - k$. Thus, $h_1 - l = h_2 - k$ and hence, by applying to both parts of (3.5.1) operator $\mathcal{B}^{h_2-k}$ we get

$$c_l w_1^{h_1} + d_k w_2^{h_2} = 0.$$

Hence $d_k \neq 0$ contradicts assumption a).

If the vectors $w_1^0, \ldots w_1^{h_1}, w_2^0, \ldots, w_2^{h_2}$ form a basis of V, i.e. if $h_1 + h + 2 = m$ then we are done: in the basis $w_1^0, \ldots w_1^{h_1}, w_2^0, \ldots, w_2^{h_2}$ the operator $\mathcal{B}$ has the Jordan form with the blocks matrix J_{h_1} and J_{h_2} along the diagonal. If $h_1 + h + 2 < m$ we continue the process, building asimilar tower over v_3 etc.

In the case b) we can replace in the basis $v_1, \ldots, v_n$ the vector v_2 with the vector $v_2' := v_2 - c\mathcal{B}^{h_1-h_2} v_1$. Then $\mathcal{B}^{h_2} v_1' = w_2^{h_2} - c w_1^{h_1} = 0$, i.e. the vector v_2' has a height $h_2' < h_2$. We repeat the above procedure again considering two cases a) and b) as above, and in both cases we proceed exactly as before. The process will terminate after a finite number of steps.

■

3.6 Root vectors and root spaces

Let $\chi_{\mathcal{A}}(\lambda) = (-1)^n (\lambda - \lambda_1)_1^k \ldots (\lambda - \lambda_s)^{k_s}$ be the characteristic polynomial of an operator $\mathcal{A} : V \to V$. The *root space* R_{λ_i}, $i = 1, \ldots, s$, corresponding to an eigenvalue λ_i is the set of all vectors $v \in V$ such that $(\mathcal{A} - \lambda_i \mathrm{Id})^k v = 0$ for some integer k. Thus, eigenvectors are root vectors of height 1

If v is the root vector for an eigenvalue λ_i then its *height* is defined as

$$\text{height}(v) = (\min(k; (\mathcal{A} - \lambda_i \text{Id})^k v = 0).$$

Thus, eigenvectors are root vectors of height 1. As we will see below, if v is the root vector for an eigenvalue λ_i then $\text{height}(v) \leq k_i$, i.e. the height of a root vector is bounded above by by the multiplicity of the corresponding eigenvalue in the characteristic polynomial. Let us begin with the following

Lemma 3.11. *The root space R_{λ_i} is an invariant subspace of the operator $\mathcal{A}$, i.e. if $v \in R_{\lambda_i}$ then $\mathcal{A}(v) \in R_{\lambda_i}$.*

Indeed, if $(\mathcal{A} - \lambda_i \text{Id})^k v = 0$ then

$$(\mathcal{A} - \lambda_i \text{Id})^k \mathcal{A} v = \mathcal{A}((\mathcal{A} - \lambda_i \text{Id})^k v) = 0.$$

The goal of this section is to prove the following

Proposition 3.12. *Suppose that the characteristic polynomial $\chi_{\mathcal{A}}(\lambda)$ of an operator $\mathcal{A} : V \to V$ is equal to $(-1)^n (\lambda - \lambda_1)^{k_1} \dots \lambda - \lambda_s)^{k_s}$. Then V decomposes as the direct sum of its root spaces:*

$$V = R_{\lambda_1} \oplus \dots \oplus R_{\lambda_s} = \overset{s}{\underset{1}{\oplus}} R_{\lambda_i}.$$

In other words, if one chooses a basis in each root space, then they together form a basis of V, and in this basis the matrix of $\mathcal{A}$ has a block-diagonal form.

We will need several lemmas. The first one is a general fact concerning the divisibility of polynomials.

Let f, g be two polynomial of one variable (let us call it λ) with complex coefficients. We say that g is a divisor of f if there exists a polynomial h such that $f = gh$. Suppose that $\deg(f) = n, \deg(g) = k$ and $k \geq n$. Then one can always divide f by g with a *remainder* term, i.e. present f in the form $f = hg + r$, where r is a polynomial of degree $< k$.

The greatest common divisor of two polynomials f, g is a polynomial h of maximal degree which divides simultaneously f and g.

Exercise 3.13. *Show that the greatest common divisor is unique up to multiplication by a constant.*

The greatest common divisor of f and g is denoted by (f, g). If $(f, g) = 1$ then f and g are called *mutually prime*. Equivalently this means that the polynomials f and g have no common roots.

Lemma 3.14. *If $(f, g) = 1$ then there exist polynomials p and q such that*

$$pf + qg = 1.$$

Proof. Let m be the minimal degree of a non-zero polynomial r which can be presented in the form $r = pf + qg$ for some polynomials p, q. We need to show that $m = 0$. Indeed, suppose that $m > 0$. Divide f by r with a remainder term: $f = ar + s$, where $\deg s < m$. Then $ar = f - s$ and $a(pf + qg) = f - s$, or $(ap - 1)f + qg = -s$. But this means that $s = 0$, because by our assumption m is a minimal degree of a non-zero polynomial which can be presented as a combination of f and g. Hence, $f = ar$, i.e. f is divisible by r. A similar argument shows that g is also divisible by r, but this contradicts to our assumption that f and g are mutually prime. ■

Lemma 3.15. *Let f and g are mutually prime polynomials and $\mathcal{A} : V \to V$ a linear operator. Suppose that $f(\mathcal{A})g(\mathcal{A}) = 0$. Then*

$$V = \operatorname{Ker} f(\mathcal{A}) \oplus \operatorname{Ker} g(\mathcal{A}).$$

Proof. We need to show that every vector $x \in V$ can be uniquely presented as a sum $x = y + z$, where $y \in \operatorname{Ker} g(\mathcal{A})$ and $z \in \operatorname{Ker} f(\mathcal{A})$. According to Lemma 3.14 there exist polynomials p and q such that $pf + qg = 1$. Then

$$p(\mathcal{A}) \circ f(\mathcal{A}) + q(\mathcal{A}) \circ g(\mathcal{A}) = \mathrm{Id}.$$

Hence, for any vector $x \in V$ we have

$$x = \underbrace{p(\mathcal{A}) \circ f(\mathcal{A})(x)}_{y} + \underbrace{q(\mathcal{A}) \circ g(\mathcal{A})(x)}_{z} = y + z.$$

But then

$$g(\mathcal{A})(y) = g(\mathcal{A}) \circ p(\mathcal{A}) \circ f(\mathcal{A})(x) = g(\mathcal{A}) \circ f(\mathcal{A}) \circ p(\mathcal{A})(x) = 0,$$

and

$$f(\mathcal{A})(z) = f(\mathcal{A}) \circ q(\mathcal{A}) \circ g(\mathcal{A})(x) = f(\mathcal{A}) \circ g(\mathcal{A}) \circ q(\mathcal{A})(x) = 0,$$

i.e. $y \in \operatorname{Ker} g(\mathcal{A})$ and $z \in \operatorname{Ker} f(\mathcal{A})$.

Suppose that there exists a different presentation $x = y' + z'$, where $y' \in \operatorname{Ker} g(\mathcal{A})$ and $z' \in \operatorname{Ker} f(\mathcal{A})$. Then

$$\operatorname{Ker} g(\mathcal{A}) \ni y - y' = z - z' \in \operatorname{Ker} f(\mathcal{A}).$$

Then $g(\mathcal{A})(y - y') = 0$ and $f(\mathcal{A})(y - y') = 0$, and hence

$$y - y' = p(\mathcal{A}) \circ f(\mathcal{A})(y - y') + q(\mathcal{A}) \circ g(\mathcal{A})(y - y') = 0.$$

Thus $y = y'$ and $z = z'$, i.e. every vector $x \in V$ can be *uniquely* presented as a sum $x = y + z$, where $y \in \operatorname{Ker} g(\mathcal{A})$ and $z \in \operatorname{Ker} f(\mathcal{A})$. ■

By induction, we can deduce from Lemma 3.15 its generalization for the case of several factors:

Lemma 3.16. *Let $f_1, \dots f_s$ are pairwise mutually prime polynomials and $\mathcal{A} : V \to V$ a linear operator. Suppose that $f_1(\mathcal{A}) \circ \dots \circ f_s(\mathcal{A}) = 0$. Then*

$$V = \bigoplus_{i=1}^{s} \operatorname{Ker} f_i(\mathcal{A}).$$

Proof of Proposition 3.12. Polynomials $f_1(\lambda) := (\lambda - \lambda_1)^{k_1}, \dots, f_s(\lambda) := (\lambda - \lambda_s)^{k_s}$ are pairwise mutually prime. On the other hand, by the Hamilton-Cayley Theorem, see 3.8,

$$f_1(\mathcal{A}) \circ \dots f_s(\mathcal{A}) = \chi_{\mathcal{A}}(\mathcal{A}) = 0.$$

Hence, by Lemma 3.16 we have

$$V = \bigoplus_{i=1}^{s} \operatorname{Ker} f_i(\mathcal{A}) = \bigoplus_{i=1}^{s} \operatorname{Ker} (\mathcal{A} - \lambda_i \mathrm{Id})^{k_i} = \bigoplus_{i=1}^{s} R_{\lambda_i}.$$

■

3.7 Jordan normal formal

A Jordan block of order m is a matrix of of the form

$$J_m + \lambda I = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

One say that a matrix A is in a *Jordan normal form* if it is a block-diagonal matrix with Jordan blocks along the diagonal.

Theorem 3.17 (Jordan Normal Form). *For any linear operator $\mathcal{A} : V \to V$ on a complex vector space V there exists a basis in which its matrix has a Jordan normal form.*

Proof. Let $\chi_{\mathcal{A}}(\lambda) = (-1)^n(\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_s)^{k_s}$ be the characteristic polynomial of the operator $\mathcal{A}$. Then the space V can be decomposed in the direct sum

$$V = \bigoplus_{i=1}^{s} R_{\lambda_i}$$

of its root spaces, which according to Lemma 3.11 are invariant subspaces of the operator $\mathcal{A}$.

Denote $\mathcal{B}_i := (\mathcal{A} - \lambda_i)|_{R_{\lambda_i}}, i = 1, \dots, s$. Then $\mathcal{B}_i^{k_i} = 0$, i.e. $\mathcal{B}_i$ is a nilpotent operator of height k_i. Hence, by Proposition 3.12 there exists a basis $v_1^i, \dots, v_{k_i}^i$ of R_{λ_i} such that in this basis the operator $\mathcal{B}_i$ has a block-diagonal form with matrices $J_{l_1^i}, \dots J_{l_{m_i}^i}$ along the diagonal for some integers $l_1^i, \dots, l_{m_i}^i$ with $l_1^i + \dots + l_{m_i}^i = k_i$. Hence the matrix of $\mathcal{A}|_{R_{\lambda_i}} = \mathcal{B}_i + \lambda_i \mathrm{Id}$ has a a block-diagonal form whose diagonal blocks are Jordan blocks of sizes $l_1^i, \dots, l_{m_i}^i$ with λ_i on the diagonal.

Together the bases $v_1^i, \dots, v_{k_i}^i$ for $i = 1, dots, s$ form a basis of V in which the matrix of the operator $\mathcal{A}$ has a Jordan normal form. ■

3.8 Algorithm

In this section we present a practical algorithm which follows the theory described in the previous section for computing the Jordan normal form of a matrix.

Consider an operator $\mathcal{A} : \mathbb{C}^n \to \mathbb{C}^n$ with the matrix A.

Step 1 Find the characteristic polynomial $\chi(\lambda) = \det(A - \lambda I)$ and decompose as a product of linear factors:

$$\chi(\lambda) = (\lambda_1 - \lambda)^{r_1} \dots (\lambda_k - \lambda)^{r_k}.$$

Step 2 Find the root spaces $V_1, \dots, V_k$ for each eigenvalue:

$$V_j := \mathrm{Ker}\,(\mathcal{A} - \lambda_j \mathrm{Id})^{r_j}.$$

To find a basis of V_j one needs to find a fundamental system of solutions (i.e. a basis of the space of solutions) of the linear system

$$(A-\lambda_j)_j^r = 0. \tag{3.8.1}$$

According to the general theory $\dim V_j = r_j$, so one needs to find r_j linear independent solutions of the system (3.8.1).

Step 3 Now we need to construct the canonical basis of each root space V_j. To simplify the notation we drop the index j and will write denote this root space by V and its basis found in Step 2 by $v_1, \dots, v_r$. We will also write λ instead of λ_j and denote $N := A - \lambda I$.

The algorithm which we describe below attempts to organize a basis as a table of the form

$$\begin{array}{cccccc}
w_1 & Nw_1 & \dots & \dots & \dots & N^{k_1}w_1 \\
w_2 & Nw_2 & \dots & \dots & N^{k_2}w_1 & \\
 & & \dots & \dots & & \\
w_m & Nw_m & \dots & N^{k_m}w_m & &
\end{array}$$

where we have $N^{k_1+1}w_1 = \dots N^{k_m+1}w_m = 0$. If these vectors form a basis of V, i.e. if they are linearly independent and the total number of vectors in this table is equal $\dim V = r$, then if we order these vectors counting first the vectors of the first row from the right to the left, then similarly, the second row, etc. then in this basis the matrix of N will consists of m nilpotent blocks

$$\begin{pmatrix}
0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 \\
 & & \dots & \dots & & \\
0 & 0 & 0 & \dots & 0 & 1 \\
0 & 0 & 0 & \dots & 0 & 0
\end{pmatrix}$$

of sizes $k_1, \dots, k_m$, and hence the matrix of $\mathcal{A}$ will have Jordam blocks of the same sizes with λ on the diagonal.

We will construct the table by an inductive process.

At the beginning we set $w_1 := v_1$ and write the first row of the matrix:

$$w_1 \quad Nw_1 \quad \dots \quad \dots \quad \dots \quad N^{k_1}w_1 \tag{3.8.2}$$

The general theory guarantees that these vectors are linearly independent. Hence, if the total number of these vectors, $k_1 + 1$ is equal to r that this terminates the process, If this is not the case we continue and form in a similar way a row beginning with $w_2 = v_2$:

$$w_2, Nw_2, \dots, N^{k_2}w_2.$$

Next, we form a 2-row table writing the longer row first and aligning the rows to the right. Let, for determinacy, $k_1 \geq k_2$ (otherwise, swap the notation for w_1 and w_2). Thus, we get

$$\begin{array}{ccccc} w_1 & \dots & N^{k_1-k_2}w_1 & \dots & N^{k_1}w_1 \\ & & w_2 & \dots & N^{k_2}w_2 \end{array}$$

If the vectors $N^{k_1}w_1$ and $N^{k_2}w_2$ are linearly dependent, then

$$N^{k_2}w_2 = cN^{k_1}w_1$$

for some $c \in \mathbb{R}$.

We then subtract from the second row the portion of the first row which is above the elements of the second row:

$$\begin{array}{cccccc} w_1 & \dots & N^{k_1-k_2}w_1 & \dots & N^{k_1-1}w_1 & N^{k_1}w_1 \\ & & w_2' = w_2 - cN^{k_1-k_2}w_1 & \dots & N^{k_2-1}w_2' & 0 \end{array}$$

Shift the new second row to the right:

$$\begin{array}{cccccc} w_1 & \dots & N^{k_1-k_2}w_1 & N^{k_1-k_3}w_1 & \dots & N^{k_1}w_1 \\ & & & w_2' & \dots & N^{k_2-1}w_2' \end{array}$$

Now check again whether two vectors in the last column are linear dependent. If they are, i.e. $N^{k_2-1}w_2' = c'N^{k_1}w_1$, then we repeat the process again, i.e. subtract from the second raw the part of the first row which is directly above. We continue till either the second row completely annihilated, or if the vectors in the last column become linearly independent. In this case the general theory guarantees that all the vectors in the table are linearly independent.

At this moment we check again whether the total number of vectors in the table less or equal $\dim V = r$. If we got enough vectors we go to Step 4.

Otherwise, we take a new vector from the original basis, which did not yet use, and again form a row as in (3.8.2).

We add this row to the previously constructed table, ordering the rows in such a way that longer raws take higher positions.

Thus, we get a table with 3 rows (to simplify the notation we renamed vectors on the left side back to w_1, w_2, w_3 and their heights to k_1, k_2, k_3):

$$\begin{array}{ccccccc} w_1 & \dots & N^{k_1-k_2}w_1 & \dots & N^{k_1-k_3+1}w_1 & \dots & N^{k_1}w_1 \\ & & w_2 & \dots & N^{k_2-k_3}w_2 & \dots & N^{k_2}w_2 \\ & & & & w_3 & \dots & N^{k_3}w_3 \end{array}$$

Now we again repeat the procedure beginning from the top two rows and check if the two top vectors in the right column are linearly dependent. If they are then we use the previous algorithm to shorten the second row. If after some iterations the second row becomes shorter than the 3rd one, we reorder the rows again.

After making top two vectors in the right column independent, we check if the 3rd vector of this column is a linear combination of the top two. If it is , i.e.

$$N^{k_3}w_3 = c_1 N^{k_1}w_1 + c_2 N^{k_2}w_2$$

then we subtract from the 3rd row the corresponding portions of the first and the second ones, multiplied by the coefficients c_1 and c_2.

Eventually this process stops when the number of elements in the table reaches r and all vectors in the right column become independent. After that we pass to Step 4.

Step 4 The Jordan basis for the root space is formed by vectors in the table numbered from the right to the left and from the top to the bottom, i.e. we first enumerate the elements of the first row from the right to the left, then similarly the elements of the second row etc. If we terminated with a table of l rows with $m_1, m_2, \dots, m_l$ elements in each row, then the matrix of the operator $\mathcal{A}$ on V in this basis will consists of l Jordan blocks of sizes $m_1, \dots, m_l$.

Chapter 4

Systems of linear differential equations with constant coefficients

4.1 The phase flow of a linear system

In this chapter we study the system

$$\dot{x} = Ax, \ x \in \mathbb{C}^n. \tag{4.1.1}$$

Our main interest is when A is a real matrix, and solutions themselves are real. However, the consistent theory in the real case requires us to look at the complex picture at the same time.

We denote by $\mathcal{A}$ the linear operator $\mathbb{C}^n \to \mathbb{C}^n$ given by the matrix A in the standard basis. When A is a real basis we can consider both, operators $\mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{C}^n \to \mathbb{C}^n$ with this matrix. When we need to distinguish them we will use the notation $\mathcal{A}^{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{A}^{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$.

Proposition 4.1. *The space $\mathcal{S}(A)$ of solutions of the system* (4.1.1) *is an n-dimensional vector space. The map $x(t) \mapsto x(0)$ which associates to a solution $x(t)$ its initial value $x(0)$ is an isomorphism $\mathcal{S}(A) \to \mathbb{C}^n$. The inverse map is given by the formula*

$$x(t) = e^{\mathcal{A}t} x(0),$$

and hence the phase flow of the equation (4.1.1) *consists of the linear operators $e^{\mathcal{A}t}$, $t \in \mathbb{R}$.*

4.2 General form of a solution of a homogeneous linear system with constant coefficients

Let us first consider the case of a diagonalizable matrix A.

Corollary 4.2. *Suppose the operator $\mathcal{A}$ is diagonalizable and let $\lambda_1, \dots, \lambda_n$ and $v_1, \dots, v_n$ be its eigenvalues and the corresponding basis of eigenvectors (some of eigenvalues may coincide). Then a general solution of* (4.1.1) *has the form*

$$x(t) = \sum c_i e^{\lambda_i t} v_i. \tag{4.2.1}$$

Equivalently the solution which correspond to the initial data $x(0) = x_0$ can be written as

$$x(t) = C \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & \dots & \dots & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix} C^{-1} x_0 \,, \tag{4.2.2}$$

where C is the matrix of transition to the basis $v_1, \dots, v_n$, i.e. the matrix whose columns are coordinates of the vectors $v_1, \dots, v_n$ in the standard basis of $\mathbb{C}^n$.

When the matrix A is real then its spectrum (i.e. the set of eigenvalues) has the form $\lambda_1, \bar{\lambda}_1, \dots, \lambda_k, \bar{\lambda}_k$, $\mu_1, \dots, \mu_l$, where $2k + l = n$, $\mu_j \in \mathbb{R}$, $j = 1, \dots, k$, and $\mathrm{Im}\lambda_m \neq 0, m = 1, \dots, l$. The eigenvectors corresponding to conjugated eigenvalues can be chosen themselves conjugated, i.e. $v_1, \overline{v}_1, \dots, v_k, \overline{v}_k$, $w_1, \dots, w_l$, where $w_m \in \mathbb{R}^n$, $m = 1, \dots, l$, and $v_j = X_j + iY_j$, $X_j, Y_j \in \mathbb{R}^n, Y_j \neq 0$, $j = 1, \dots, k$.

Next, we consider the case of a real diagonalizable matrix (with possibly complex eigenvalues)

Corollary 4.3. *Suppose the matrix A is diagonalizable and real. Let $\lambda_1 = \alpha_1 + i\omega_1, \bar{\lambda}_1 \alpha_1 - i\omega_1, \dots, \lambda_k = a_k + i\omega_k, \bar{\lambda}_k = \alpha_k - i\omega_k$, $\mu_1, \dots, \mu_l$, $2k+l = n$ and $v_1 = X_1 + iY_1, \overline{v}_1 = X_1 - iY_1, \dots, v_k = X_k + iY_k, \overline{v}_k = X_k - iY_k$, $w_1, \dots, w_l$, be its eigenvalues and the corresponding basis of eigenvectors, where $X_j, Y_j, w_l \in \mathbb{R}^n$, $j = 1, \dots, k, m = 1, \dots, l$. Then a general* real *solution of* (4.1.1) *has the* *form*

$$x(t) = \mathrm{Re}\left(\sum_1^k r_j e^{i\theta_j} e^{\alpha_j t + i\omega_j t}(X_j + iY_j)\right) + \sum_1^l d_m e^{\mu_m t} w_m$$
$$= \mathrm{Re}\left(\sum_1^k r_j e^{\alpha_j t} e^{i(\theta_j + \omega_j t)}(X_j + iY_j)\right) + \sum_1^l d_m e^{\mu_m t} w_m$$
$$= \sum_1^k r_j e^{\alpha_j t}\left(\cos(\theta_j + \omega_j t)X_j - \sin(\theta_j + \omega_j t)Y_j\right) + \sum_1^l d_m e^{\mu_m t} w_m,$$

where $r_j > 0, \theta_j, d_j \in \mathbb{R}$ are arbitrary constants.

Next, we consider the case of a non-diagonalizable A.

Lemma 4.4. *Suppose that A is a Jordan block*

$$A = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

of order n. Then the phase flow e^{At} has the form

$$e^{At} = \begin{pmatrix} e^{\lambda t} & e^{\lambda t} t & \frac{e^{\lambda t} t^2}{2} & \dots & \frac{e^{\lambda t} t^{n-1}}{(n-1)!} \\ 0 & e^{\lambda t} & e^{\lambda t} t & \dots & \frac{e^{\lambda t} t^{n-2}}{(n-2)!} \\ & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & e^{\lambda t} \end{pmatrix}$$

solution with the initial data $x(0) = x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ . \\ x_n^0 \end{pmatrix}$ has the form

$$x(t) = \begin{pmatrix} e^{\lambda t} \sum_1^n \frac{t^{j-1} x_j^0}{(j-1)!} \\ e^{\lambda t} \sum_2^n \frac{t^{j-2} x_j^0}{(j-2)!} \\ . \\ e^{\lambda t} x_n^0 \end{pmatrix}.$$

.

Corollary 4.5. *Suppose that the matrix A of the system* (4.1.1) *has eigenvalues $\lambda_1, \ldots, \lambda_s$ of mutiplicities $k_1, \ldots, k_s$, respectively. Then each coordinate function $x_l(t)$, $l = 1, \ldots, n$ (in any coordinate system) of any solution has a form*

$$x_l(t) = \sum_{j=1}^{s} p_{l,j}(t) e^{\lambda_j t},$$

where $p_{l,j}(t)$ are polynomials of degree $< k_j$.

It is important to point out that not all vector functions whose coordinates have this form are solutions. The space of solutions is always n-dimensional, while the total number of coefficients of all polynomials $p_{l,j}$ is equal to $n \sum_1^s k_j = n^2 > n$. Thus if $n > 1$ then between the coefficients should be a lot of dependences.

If A is a real matrix and we are interested only in real solutions then Corollary 4.5 takes the following form:

Corollary 4.6. *Suppose that the matrix A of the system* (4.1.1) *has eigenvalues $\lambda_1 = \alpha_1 + i\omega_1, \bar{\lambda}_1 = \alpha_1 - i\omega_1, \ldots, \lambda_p = \alpha_p + i\omega_p, \bar{\lambda}_p = \alpha_p - i\omega_p, \mu_1, \ldots, \mu_q$ of multiplicities $k_1, \ldots, k_p$ and $m_1, \ldots, m_q$, respectively. Then each coordinate function $x_l(t)$, $l = 1, \ldots, n$ (in any coordinate system) of any solution has a form*

$$x_l(t) = \sum_{j=1}^{q} p_{l,j}(t) e^{\mu_j t} + \sum_{j=1}^{q} e^{\alpha_j t} \left(r_{l,j}(t) \cos \omega_j t + s_{l,j}(t) \sin \omega_j \right),$$

where $p_{l,j}(t)$ are polynomials of degree $< k_j$ and $r_{l,j}(t)$ and $s_{l,j}(t)$ are polynomials of degree $< m_j$.

4.3 One linear equation of order n

Let us consider a special case of 1 equation of order n:

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0. \tag{4.3.1}$$

Rewriting (4.3.1) as a system of first order equations we get

$$\begin{aligned} \dot{x} &= x_1 \\ \dot{x}_1 &= \quad x_2 \\ &\dots \\ \dot{x}_{n-1} &= -a_1 x_{n-1} - a_2 x_{n-2} - \dots - a_{n-1} x_1 - a_n x. \end{aligned} \tag{4.3.2}$$

The matrix of this system is equal to

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & \dots & \dots & . \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{pmatrix} \tag{4.3.3}$$

Let us compute the characteristic polynomial

$$\xi_A(\lambda) = \begin{vmatrix} -\lambda & 1 & \dots & 0 \\ 0 & -\lambda & \dots & 0 \\ & \dots & \dots & . \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 - \lambda \end{vmatrix} = (-1)^{n+1}(\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n). \tag{4.3.4}$$

We also note that the rank of the matrix

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & \dots & 0 \\ 0 & -\lambda & \dots & 0 \\ & \dots & \dots & \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 - \lambda \end{pmatrix}$$

for an eigenvalue λ is equal to $n-1$. This means that the eigenspace E_λ is 1-dimensional for each eigenvalue λ. In turn, this implies that the Jordan normal form has blocks of maximal size for each eigenvalue, i.e. if $\xi_A(\lambda) = \pm(\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_s)^{k_s}$ then the Jordan form of A has 1 block of size k_j for each eigenvalue $\lambda_j, j = 1, \dots, s$. According to Corollary 4.5 this implies that

Corollary 4.7. *The general solution of equation* (4.3.1) *has a form*

$$x(t) = \sum_{1}^{s} p_j(t) e^{\lambda_j t}, \tag{4.3.5}$$

where $p_j(t)$ are polynomials of degree $< k_j$. Moreover, every function of this form is a solution of (4.3.1).

We note that there is a short-cut to the fact the characteristic polynomial of (4.3.1) has the form (4.3.4) as well as to the general form (4.3.5) of its solution.

Indeed, one can argue as following. For each eigenvalue λ of the matrix A from (4.3.3) there exists at least one eigenvector $v = (c_1, \ldots, c_n) \neq 0$. Then $x(t) = e^{\lambda t} c_1$ is a solution of (4.3.1) and

$$(x(t), x_1(t) := \dot{x}(t) = \lambda e^{\lambda t} c_1, \ldots, x_{n-1}(t) = x^{(n-1)}(t) = \lambda^{n-1} e^{\lambda t} c_1)$$

is a solution of the corresponding system (4.3.2) of first order equations, and hence $c_2 = \lambda c_1, \ldots c_n = \lambda^{n-1} c_1$. It follows that $c_1 \neq 0$ and we can choose $c_1 = 1$.

Plugging it to (4.3.1) we get

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = (\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n) e^{\lambda t} = 0,$$

and hence

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0,$$

i.e. each eigenvalue is a root of the polynomial (4.3.4). Conversely, for any root λ of (4.3.4) the function$x(t) = e^{\lambda t}$ satisfies (4.3.1), and hence Corollary 4.7 implies that λ is one of eigenvalues of the matrix A. Hence, the characteristic polynomial of A has the form (4.3.4) as claimed.

According to Corollary 4.5 any solution of (4.3.1) has the form (4.3.5). But the space of solutions of equation (4.3.1) is n-dimensional, and the total number of coefficients in the polynomials $p_1(t), p_s(t)$ is equal to $k_1 + \cdots + k_s = n$, and therefore *any expression of the form* (4.3.5) *is a solution of* (4.3.1).

Finally we consider the form of a general *real* solution of equation (4.3.1) with *real* coefficients.

Corollary 4.8. *Suppose that the characteristic polynomial* (4.3.4) *has roots $\lambda_1 = \alpha_1 + i\omega_1, \alpha_1 - i\omega_1, \ldots \lambda_a = \alpha_a + i\omega_a, \alpha_a - i\omega_a$ and $\mu_1, \ldots, \mu_b$ of multiplicities $k_1, \ldots, k_a$ and $l_1, \ldots, l_b$, respectively.*

A general solution of equation (4.3.1) *has a form*

$$x(t) = \sum_{1}^{a} e^{\alpha_j t} \left(p_j(t) \cos \omega_j t + q_i(t) \sin \omega_j t\right) + \sum_{1}^{b} e^{\mu_m t} r_m(t), \tag{4.3.6}$$

where $p_j(t), q_j$ *are polynomials of degree* $< k_j$ *and* $r_m(t)$ *are polynomials of degree* $< l_m$*. Moreover, every function of this form is a solution of* (4.3.1).

4.4 Inhomogeneous linear systems with constant coefficients

Consider a system

$$\dot{x} - Ax = f(t), \; x \in \mathbb{C}^n. \tag{4.4.1}$$

Recall that according to Proposition 4.1 the space $\mathcal{S}(A)$ of solutions of the corresponding homogeneous system is an n-dimensional vector subspace of the space of all smooth vector-functions. Correspondingly, the space of solution $\mathcal{S}(A)$ of the inhomogeneous system (4.4.1) is an *affine* subspace of this space, i.e. if $x(t)$ is any particular solution of (4.4.1) then any other solution has the form $x(t) + y(t)$, where $y(t)$ is a solution of the homogeneous system (4.1.1).

To find a particular solution $x(t)$ which satisfies the initial condition $x(0) = 0$. we use the method of *variation of constants*, i.e. will search for a solution in the form $x(t) = e^{At}c(t)$, $c(t) \in \mathbb{C}^n$. Plugging into equation (4.1.1) we get

$$\begin{aligned} &Ae^{At}c(t) + e^{At}\dot{c}(t) - Ae^{At}c(t) = e^{At}\dot{c}(t) = f(t), \quad \text{or} \\ &\dot{c}(t) = e^{-At} f(t), \end{aligned}$$

and hence

$$c(t) = \int_0^t e^{-As} f(s) ds,$$

and

$$x(t) = e^{At} \int_0^t e^{-As} f(s) ds = \int_0^t e^{A(t-s)} f(s) ds.$$

In practice, there usually are simpler methods for finding a particular solution of an inhomogeneous equation. Let us consider, for example, the case of one inhomogeneous equation of order n

with the right-hand side a quasi-polynomial

$$x^{(n)} + a_1 x^{(n-1}) + \cdots + a_n = f(t), \quad x \in \mathbb{C}. \tag{4.4.2}$$

Lemma 4.9. *Suppose $f(t) = q(t)e^{\mu t}$, where $q(t)$ is a polynomial of degree $\leq k$.*

1. *Suppose that ν is not a root of the characteristic equation*

   $$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{4.4.3}$$

   Then there exists a particular solution of the form $p(t)e^{\nu t}$, where $p(t)$ is a polynomial of degree $\leq k$.

2. *Suppose that ν is a root of (4.4.3) of multiplicity m. Then there exists a particular solution of the form $p(t)e^{\nu t}$, where $p(t)$ is a polynomial of degree $\leq k + m$.*

Chapter 5

Stability

5.1 Asymptotic and Lyapunov stability

Consider a general ODE

$$\dot{x} = F(x), \quad x \in U \subset \mathbb{R}^n. \tag{5.1.1}$$

Let $a \in U$ be an *equilibrium point*, i.e. $F(a) = 0$.

The equilibrium point a is called

- *asymptotically stable* if there exists a neighborhood $\Omega \ni a$, $\Omega \subset U$, such that for any point $x_0 \in \Omega$ the solution $x(t)$ with the initial condition $x(0) = x_0$ exists for all $t \geq 0$ and satisfies the condition $\lim\limits_{t \to +\infty} x(t) = a$;

- *Lyapunov stable* or *stable in the sense of Lyapunov* if for any neighborhood $\Omega \ni a$, $\Omega \subset U$ there exists a smaller neighborhood $\widetilde{\Omega} \ni a$, $\widetilde{\Omega} \subset \Omega$, such that for any point $x_0 \in \widetilde{\Omega}$ the solution $x(t)$ with the initial condition $x(0) = x_0$ exists for all $t \geq 0$ and satisfies the condition $x(t) \in \Omega$ for all $t \in [0, \infty)$.

Asymptotic stability implies Lyapunov stability but not vice versa.

Exercise 5.1. *a) Consider the equation* $\dot{x} = -x$. *Show that* 0 *is asymptotically stable.*

b) Consider the system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1.$$

Show that that the origin is Lyapunov stable but not asymptotically stable.

5.2 Criterion of asymptotic stability

Let $a \in U$ be an equilibrium point of system (5.1.1). Then we have

$$F(x) = A(x-a) + 0(||x-a||).$$

The linear system

$$\dot{y} = Ay, \quad y \in \mathbb{C}^n, \tag{5.2.1}$$

is called the *linearization* of (5.1.1) at the equilibrium point $a \in U$.

Theorem 5.2. *Let $a \in U$ be an equilibrium point of system* (5.1.1) *and $\dot{y} = Ay$ its linearization at the point a. Suppose that* $\mathrm{Spec} A \subset \{z \in \mathbb{C}; \mathrm{Re} z < 0\}$, *i.e. all eigenvalues of the matrix A lie in the half-plane* $\{z \in \mathbb{C}; \mathrm{Re} z < 0\}$. *Then the equilibrium point a is asymptotically stable for the system* (5.1.1).

Remark 5.3. 1. If at least one of the eigenvalues of A has a positive real part, that the equilibrium a is Lyapunov (and asymptotically) unstable.

This fact is more difficult to prove then Theorem 5.2, though we will discuss it later on.

2. If eigenvalues of A satisfy $\mathrm{Re}\lambda \le 0$ then the linearized system (5.2.1) is Lyapunov stable. However, if there is an eigenvalue with $\mathrm{Re}\lambda = 0$ then one cannot draw any conclusion about stability or instability of the original system (5.1.1) (why?).

The proof of Theorem 5.2 will require several lemmas.

Given a vector field X on U a smooth real-valued function $\phi : U \to \mathbb{R}$ is called a *Lyapunov function* for X (and the vector field X is called *gradient-like* for ϕ if $L_X \phi = d\phi(X) \ge c||X||^2$ for some *positive* function $c : U \to \mathbb{R}$.

Exercise 5.4. *Prove that for any function ϕ its gradient vector field is gradient-like.*

Lemma 5.5. *If $\phi : U \to \mathbb{R}$ is a Lyapunov function for a vector field X on U and $f : U \to \widetilde{U}$ a diffeomorphism then the function $\widetilde{\phi} := \phi \circ f^{-1} : \widetilde{U} \to \mathbb{R}$ is a Lyapunov function for the vector field $\widetilde{X} := f_* X$ on $\widetilde{U}$. Moreover, if (ϕ, X) satisfy an inequality $c||X||^2 \le d\phi(X) \le C||X||^2$, then so does $(\widetilde{\phi}, \widetilde{X})$.*

Indeed, by chain rule

$$d\widetilde{\phi}(\widetilde{X}) = d\phi \circ d(f^{-1})(df(X)) = d\phi(X) \ge c||X||^2 \ge \frac{c}{||df||^2}||\widetilde{X}||^2 = \widetilde{c}||\widetilde{X}||^2.$$

We also have

$$||X||^2 \le \frac{1}{||df||^2}||\widetilde{X}||^2,$$

and hence if $d\phi(X) \le C||X||^2$, then

$$d\widetilde{\phi}(\widetilde{X}) = d\phi(X) \le C||X||^2 \le \frac{C}{||df||^2}||\widetilde{X}||^2.$$

We recall that the norm of a linear operator $C : \mathbb{C}^n \to \mathbb{C}^n$ is defined as

$$||C|| = \max_{x \ne 0} \frac{||Ax||}{||x||} = \max_{||x=1||} ||Ax||.$$

A particular case of Lemma 5.5 is

Corollary 5.6. *Consider a linear vector field $Z(x) = Ax$ on $\mathbb{C}^n$. Let ϕ be a Lyapunov function for ϕ. The for any non-singular matrix C the function $\widetilde{\phi}(x) = \phi(C^{-1}x)$ is a Lyapunov function for the vector field $\widetilde{Z}(x) = CAC^{-1}x$.*

Indeed, $\widetilde{Z} = \mathcal{C}_* Z$, where $\mathcal{C} : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map with the matrix C.

Lemma 5.7. *Suppose that* $\mathrm{Spec} A \subset \{z \in \mathbb{C}; \mathrm{Re} z > 0\}$. *Then the vector field $Z(z) = Az, z \in \mathbb{C}$ admits a Lyapunov function.*

We divide the proof of Lemma 5.7 in a few step

Lemma 5.8. *Lemma 5.7 holds when A is a diagonal matrix.*

Proof. Let $\lambda_j = \alpha_j + i\omega_j, j = 1, \ldots, n$ be the eigenvalues of A. Then the vector field Z has in complex coordinates the form $Z(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$.[1] To understand better its geometry let us rewrite it in real coordinates. We identify the space $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ (as usual viewing each complex coordinate $z_j = x_j + iy_j$ as a pair of real coordinates (x_j, y_j). Set $\lambda_j = \alpha_j + i\beta_j$. Then $\lambda_j z_j = \alpha_j x_i - \beta_j y_j + i(\alpha_j y_j + \beta_i x_j)$, i.e. in real notation the vector field Z can be written as

$$Z = \sum_1^n (\alpha_j x_i - \beta_j y_j)\frac{\partial}{\partial x_j} + (\alpha_j y_j + \beta_i x_j)\frac{\partial}{\partial y_j}.$$

Define the function $\phi : \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{R}$ by the formula $\phi(z) = \frac{1}{2}\sum\limits_1^n |z_j|^2 = \frac{1}{2}\sum\limits_1^n (x_j^2 + y_j^2)$. Then ϕ is a Lyapunov function for Z. Indeed,

$$\begin{aligned} L_Z\phi(z) = d\phi(Z) &= \sum_1^n (\alpha_j x_i - \beta_j y_j)\frac{\partial \phi}{\partial x_j} + (\alpha_j y_j + \beta_i x_j)\frac{\partial \phi}{\partial y_j} \\ &= \sum_1^n (\alpha_j x_i - \beta_j y_j)x_j + (\alpha_j y_j + \beta_i x_j)y_j = \sum_1^n \alpha_j (x_j^2 + y_j^2) \\ &= \sum_j^n \alpha_j |z_j|^2. \end{aligned}$$

Denote $c := \min\limits_{j\in\{1,\ldots,n\}} \alpha_j,\ C = \min\limits_{j\in\{1,\ldots,n\}} |\lambda_j|$. By assumption, $c > 0$. Then

$$||Z(z)||^2 = \sum_1^n |\lambda_j z_j|^2 = \sum_1^n ||\lambda_j|^2 |z_j|^2 \leq C||z||^2$$

and therefore

$$L_Z\phi(z) \geq \sum_j^n \alpha_j |z_j|^2 \geq c||z||^2 \geq \frac{c}{C}||Z(z)||^2.$$

■

Lemma 5.9. *Let A be a triangular matrix. Then for any $\epsilon > 0$ there exists a diagonal matrix C such that $C^{-1}AC$ is an upper trangular matrix whose all elements a_{ij} above the diagonal (i.e. with $i < j$) satisfy $|a_{ij}| < \epsilon$.*

[1] We switched the notation for coordinates in $\mathbb{C}^n$ from $x_1, \ldots, x_n$ to $z_1, \ldots, z_n$ and will use the letter x for the real part of z.

Proof. Take

$$C = \begin{pmatrix} N^{n-1} & 0 & \dots & 0 & 0 \\ 0 & N^{n-2} & \dots & 0 & 0 \\ & \dots & \dots & \dots & \\ 0 & 0 & \dots & N & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then for a sufficiently large N the matrix $C^{-1}AC$ has the required properties. ■

Lemma 5.10. *Lemma 5.7 holds when A is an upper triangular matrix.*

Proof. According to Lemma 5.9 we can arrange that for any given $\epsilon > 0$ the elements above the diagonal are $< \epsilon$. Let us denote by $\widetilde{A}$ the diagonal matrix with the same elements on the diagonal, by $\widetilde{Z}$ the vector field $\widetilde{Z}(z) = \widetilde{A}z$ and set $B = A - \widetilde{A}$, $Y = Z - \widetilde{Z}$, so that $Y(z) = Bz$.

Then according to Lemma 5.8 the function $\phi(z) = \frac{1}{2}||z||^2$ is Lyapunov for $\widetilde{Z}$, i.e.

$$C_1||\widetilde{Z}(z)||^2 \geq L_{\widetilde{Z}}\phi(z) \geq c_1||\widetilde{Z}(z)||^2.$$

On the other hand, we have $||Y(z)|| \leq c_2\epsilon||z||$ for some constant $c_2 > 0$ and similarly

$$L_Y(\phi(z)) = \langle Y, \nabla\phi\rangle = \langle Bz, z\rangle \leq c_3\epsilon||z||^2$$

for some constant $c_3 > 0$. Thus,

$$C_1||\widetilde{Z}(z)||^2 + c_3\epsilon||z||^2 \geq L_Z\phi = L_{\widetilde{Z}}\phi + L_Y\phi \geq c_1||\widetilde{Z}(z)||^2 - c_3\epsilon||z||^2.$$

We also have

$$c_4||Z||^2 \leq ||\widetilde{Z}||^2 \leq C_4|Z||^2$$

and

$$c_5||z||^2 \leq ||Z(z)||^2 \leq C_5||z||^2. \tag{5.2.2}$$

Combining all the inequalities we get for a sufficiently small $\epsilon > 0$

$$c||Z||^2 \leq L_Z\phi \leq C||Z||^2.$$

■

Proof of Lemma 5.7. Using Corollary 5.6 we can replace the matrix A by a similar matrix, and thus may assume that it is in the Jordan normal form, and in particular, it is upper triangular. Therefore, it remains to apply Lemma 5.10. ■

Proof of Theorem 5.2. To simplify the notation we assume that the equilibrium point a is the origin $0 \in \mathbb{C}^n$. Thus $-F(z) = Az + Y(z)$, where $Y(z) = o(||z||)$. By assumption, the spectrum of A lies in the right half-plane $\{\mathrm{Re} z > 0\} \subset \mathbb{C}$. Hence, applying Lemma 5.7 we find a Lyapunov function for the vector field $Z(z) = Az$. Let us recall that the Lyapunov function which we constructed for this vector field is equal to $\frac{1}{2}||z||^2$ for some Euclidean structure on $\mathbb{C}^n$.

Let us first check that the same function $\phi(z) = \frac{1}{2}||z||^2$ is also Lyapunov for the vector field $-F$ in a sufficiently small ball $B_\delta(0) = \{||z|| < \delta\}$. In fact, it will be more useful for us to prove the following equivalent (why?) inequality:

$$2c\phi(z) = c||z||^2 \leq -L_F(z)\phi \leq C||z||^2 = 2C\phi(z) \tag{5.2.3}$$

if $||z||$ is small enough. For any $\epsilon > 0$ there exists a sufficiently small $\delta > 0$ such that for any $z \in B_\delta(0)$ we have $||Y(z)|| < \epsilon||z||$.

Hence,

$$|L_Y\phi(z)| = |\langle\nabla\phi(z), Y(z)\rangle| = |\langle z, Y(z)\rangle| \leq ||z||||Y(z)|| \leq \epsilon||z||^2 \tag{5.2.4}$$

On the other hand, according to Lemma 5.7 and taking into account (5.2.2) we have

$$2\widetilde{c}\phi(z) = \widetilde{c}||z||^2 \leq L_Z\phi(z) \leq \widetilde{C}||z||^2 = 2\widetilde{C}\phi(z). \tag{5.2.5}$$

Combining the inequalities (5.2.4) and (5.2.5) we get (5.2.3).

Now we are ready to finish the proof of asymptotic stability.

Take a point $x_0 \in B_\delta(0)$ where δ is chosen so small that the inequality (5.2.3) holds. Let $x(t)$ be the solution of the equation $\dot{x} = F(x)$ with the initial condition $x(0) = x_0$ Denote $h(t) = \phi(x(t)$. Then

$$\frac{dh}{dt}(t) = d_{x(t)}\phi\left(\frac{dx}{dt}(x(t))\right) = d_{x(t)}\phi\left(F(x(t))\right) = L_F\phi(x(t)),$$

and combining with (5.2.3) we get

$$-2Ch(t) = -2c\phi(h(t)) \leq \dot{h}(t) \leq -2c\phi(h(t)) = -2ch(t) \tag{5.2.6}$$

Let us solve the equation $\dot{g}(t) = -2cg(t)$ with the initial condition $g(0) = h(0) = \phi(x_0)$. Then $g(t) = h(0)e^{-2ct}$. On the other hand, the function $\psi(t) := g(t) - h(t)$ satisfies the conditions $\dot{\phi}(t) \leq 0,\ \ \phi(0) = 0$, and hence $\phi(t) > 0$ for $t \geq 0$, i.e. we have

$$0 \leq h(t) \leq g(t) = h(0)e^{-2ct} \underset{t\to\infty}{\to} 0.$$

But $h(t) = \phi(x(t)) = \frac{||x(t)||}{2}$, and therefore $\lim\limits_{t\to\infty} x(t) = 0$. This concludes the proof of the asymptotic stability. ■

5.3 Smooth classification of linear systems

Consider two vector fields X on a domain $U \subset \mathbb{R}^n$ and $\widetilde{X}$ on a domain $\widetilde{X} \subset \mathbb{R}^n$. We say that that the systems $\dot{x} = X(x)$ and $\dot{x} = \widetilde{X}(x)$ are *diffeomorphic, or smoothly equivalent* if there exists a diffeomorphism $f : U \to \widetilde{U}$ such that $f_*X = \widetilde{X}$. If both can be integrated to flows $X^t : U \to U$ and $\widetilde{X}^t : \widetilde{U} \to \widetilde{U}$ then their smooth equivalence can be equivalently defined by the equations

$$X^t = f^{-1} \circ \widetilde{X}^t \circ f,\ \ t \in \mathbb{R}.$$

The two systems are called *linearly equivalent* if they are equivalent via a *linear map* $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^n$. For two linear systems $\dot{x} = Ax$ and $\dot{x} = Bx$ to be linear equivalent via a linear map C just means that $A = C^{-1}BC$ (here we denote by the same letter the linear map $\mathbb{R}^n \to \mathbb{R}^n$ and its matrix in the standard basis). Indeed, the equation $A = C^{-1}BC$ is equivalent the equation $e^{At} = C^{-1}e^{Bt}C, t \in \mathbb{R}$.

Lemma 5.11. *Consider two linear systems, $\dot{x} = Ax$ and $\dot{x} = Bx$. Suppose that they are diffeomorphic. Then they are linearly equivalent.*

Proof. Denote by Y the vector field $Y(x) = Ax$ and by Z the vector field $Z(x) = Bx$. Suppose that $f_*Y = Z$. Both vector fields have unique zeroes at the origin, and hence $f(0) = 0$. Thus $f(x) = df_0(x) + G(x)$, where $G(x) = o(||x||)$. In the computation below we identify the space $\mathbb{R}^n_x$ with $\mathbb{R}^n$ via the parallel transport, and thus think about the differential d_xf as a malp $\mathbb{R}^n \to \mathbb{R}^n$.

We have

$$df_x(Y(x)) = df_x(Ax) = Z(f(x)) = Bf(x).$$

By continuity $df_x = df_0 + H(x)$, where $H(x)$ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ such that $||H(x)|| \underset{||x|| \to 0}{\to} 0$. Hence,

$$df_0(Ax) = Bf(x) - H(x)(Ax) = Bdf_0(x) + B(G(x)) - H(x)Ax.$$

Set $x = \epsilon u$. Then we get

$$df_0 \circ A(u) - B \circ df_0(u) = \frac{1}{\epsilon}(B(G(\epsilon u)) - \epsilon H(\epsilon u) \circ A(u)) \underset{||\epsilon|| \to 0}{\to} 0.$$

Hence, $df_0 \circ A(u) - B \circ df_0(u)$ for all $u \in \mathbb{R}^n$, i.e. $A = df_0^{-1} \circ B \circ df_0$, i.e. A and B are equivalent via the linear map $df_0 : \mathbb{R}^n \to \mathbb{R}^n$. ■

Corollary 5.12. *If the systems $\dot{x} = Ax$ and $\dot{x} = Bx$ are smoothly equivalent then A and B have the same eigenvalues.*

5.4 Topological classification of linear systems: generic case

Corollary 5.12 shows that smooth classification is to rigid: even a small modification of eigenvalues lead to non-equivalent systems. The notion of *topological classification* which we discuss in this section is a coarser notion which still catches some essntial characteristics of the systems.

Consider two vector fields X on a domain $U \subset \mathbb{R}^n$ and $\widetilde{X}$ on a domain $\widetilde{X} \subset \mathbb{R}^n$. Suppose that the phase flows X^t and $\widetilde{X}^t$ for X and $\widetilde{X}$ are well defined for all $t \in \mathbb{R}$. We say that that the systems $\dot{x} = X(x)$ and $\dot{x} = \widetilde{X}(x)$ are *homeomorphic, or topologically equivalent* if there exists a homeomorphism $f : U \to \widetilde{U}$ such that

$$X^t = f^{-1} \circ \widetilde{X}^t \circ f, \ \ t \in \mathbb{R}.$$

Theorem 5.13. *Suppose that the matrix A has no eigenvalues λ with* $\mathrm{Re}\lambda = 0$. *Then the system $\dot{x} = Ax$ is topologically equivalent to the system $\dot{x} = Dx$ for a diagonal matrix $D = (d_{ij})$ such that $d_{jj} = -1$ for $j \leq k$ and $d_{jj} = 1$ for $j > k$, where k is the number of eigenvalues of A with negative real part. Two such systems with different k are not topologically equivalent.*

Lemma 5.14. *The first assertion of Theorem 5.13 holds for $k = 0$.*

Proof. According to Lemma 5.7 there exists a Euclidean structure on $\mathbb{R}^n$ for which the function $||x||^2$ is a Lyapunov function for the vector field $Y(x) = Ax$. The same function is also Lyapunov for the radial vector field $Z = \sum\limits_1^n x_i \frac{\partial}{\partial x_i}$ which defines the system $\dot{x} = x$.

We construct a homeomorphism f which send the trajectory of the second system onto the trajectories of the first one by the formula

$$f(x) = e^{A \log ||x||}(\frac{x}{||x||}), \quad \text{for } x \neq 0.$$

By Theorem 5.2 we have $\lim\limits_{x\to 0} e^{A\log ||x||}(\frac{x}{||x||}) = 0$, and hence we can extend the map f by continuity to 0 by setting $f(0) = 0$. the constructed map is a diffeomorphism in the complement of the origin, but it is *never smooth at* 0 *if* $A \neq I$ (why?). Let us check that the map f is a topological equivalence of the systems $\dot{x} = x$ and $\dot{x} = Ax$. Indeed, for $x \neq 0$ denote $u := \frac{x}{||x||}$, $r := ||x||$. Then we have

$$f(e^t x) = e^{A(t+\log r)}u = e^{At}(e^{\log r A}u) = e^{At} f(x).$$

■

Proof of Theorem 5.13. Lemma 5.14 settles the case when there are no eigenvalues with the negative real part. The same lemma applied to $-A$ implies the result for the case when there are no eigenvalues with the positive real part. Let us now consider the general case. Suppose that the matrix A has exactly k eigenvalues with the negative real part and $(n-k)$ eigenvalues with the positive one (counting their multiplicities). The the phase space $V = \mathbb{R}^n$ of the system splits in the direct sum $V = V_- \oplus V_+$, where $V_\pm$ are both invariant subspaces of the operator A, have dimension k and $n-k$, respectively, and such that all eigenvalues of $A_- := A|_{V_-}$ lie in the left half-plane $\{\mathrm{Re} z < 0\}$ and all eigenvalues of $A+- := A|_{V_+}$ lie in the right half-plane $\{\mathrm{Re} z > 0\}$. Indeed, V_- is the direct sum of all root spaces corresponding to the eigenvalues with the negative real part, and V_+ is the direct sum of all root spaces corresponding to the eigenvalues with the positive real part. Then if one chooses a basis in V which has its first k vectors in V_- and the last $(n-k)$ in V_+ then the matrix of the system will have the block form, and thus the system splits into the direct sum of the systems $\dot{x} = A_- x$ and $\dot{y} = A_+ y$, $x \in \mathbb{R}^k$ and $y \in R^{n-k}$. According to Lemma 5.14 there exists a homeomorphism $f_- : \mathbb{R}^k \to \mathbb{R}^k$ of the system $\dot{x} = -x$ on $\mathbb{R}^k$ and the system $\dot{x} = A_- x$. The same lemma also implies existence of a homeomorphism $h_+ : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ between the systems $\dot{y} = y$ and $\dot{y} = A_+ y$ on $\mathbb{R}^{n-k}$. Then the homeomorphism $h : \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k \oplus \mathbb{R}^{n-k} = \mathbb{R}^n$ given by the formula $h(x, y) = (h_-(x), h_+(y)),\ x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$ is the required topological equivalence.

■

Chapter 6

Solving one first order partial differential equation

6.1 Jet spaces

When studying functions on $\mathbb{R}^n$, or a domain in $\mathbb{R}^n$ it is useful to consider their graphs which live in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, i.e. for $u : \mathbb{R}^n \to \mathbb{R}$ its graph

$$\Gamma_u := \{z = u(x_1, \dots, x_n)\} \subset \mathbb{R}^{n+1}.$$

Similarly, when studying first order partial differential equations with respect to a function on $\mathbb{R}^n$ it is useful to consider a *simultaneous graph of a function and all its derivatives*:

$$\Lambda_u = \{z = u(x), p_1 = \frac{\partial u}{\partial x_1}(x), \dots p_n = \frac{\partial u}{\partial x_n}(x),\ x = (x_1, \dots, x_n) \in \mathbb{R}^n\} \subset \mathbb{R}^{2n+1},$$

where we denoted coordinates in $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by (x, p, z), $x, p \in \mathbb{R}^n$, $z \in \mathbb{R}$. The coordinate z is reserved for graphing the value of a function u and $p_1, \dots, p_n$ for the corresponding first partial derivatives.

The space R^{2n+1} in this context is called the 1*-jet space of functions on* $\mathbb{R}^n$ and usually denoted by $J^1(\mathbb{R}^n)$. We denote by π the projection $J^1(\mathbb{R}^n) \to \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$\pi(x, p, z) = x,\ (x, y, z) \in J^1(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n \times R.$$

A map $s : \mathbb{R}^n \to J^1(\mathbb{R}^n)$ is called a *section* if $\pi \circ s = \mathrm{Id} : \mathbb{R}^n \times \mathbb{R}^n$. In other words, if $s(x) = (x, v(x), u(x)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ for $x \in \mathbb{R}^n$. With every function $u : \mathbb{R}^n \to \mathbb{R}$ one can associate a very special section. Namely,

$$x \mapsto \left(x, \frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x), u(x)\right),\ x \in \mathbb{R}^n,$$

which maps $\mathbb{R}^n$ onto the simultaneous graph of the function u and all its first partial derivatives. Sections of this type are called *holonomic*. We note that most of the sections are not holonomic..

The following lemma gives a necessary and sufficient condition for a section $s : \mathbb{R}^n \to \mathbb{R}^{2n+1}$ to be holonomic. Denote by λ the differential 1-form

$$\lambda := dz - \sum_1^n p_i dx_i,$$

and by ξ the hyperplne field defined by the Pfaffian equation $\lambda = 0$. This hyperplane field is called a *contact structure*.

Lemma 6.1. *A section $s : \mathbb{R}^n \to J^1(\mathbb{R}^n)$ is holonomic if and only if $s^*\lambda = 0$. In other words, s is holonomic if its image is tangent to the contact structure ξ.*

Proof. We have $s(x) = (x, p = v(x), z = u(x))$, and hence the equation

$$0 = s^*\lambda = s^*(dz - pdx) = du - vdx$$

is equivalent to

$$v_1(x) = \frac{\partial u}{\partial x_1}(x), \ldots, v_n(x) = \frac{\partial u}{\partial x_n}(x)$$

which is the definition of a holonomic section.

Submanifolds of dimension n which are tangent to ξ are called *Legendrian*. We note that a general Legendrian submanifold need not be necessarily graphical.

Exercise 6.2. Give an example of a non-graphical Legendrian submanifold $\Lambda \subset J^1(\mathbb{R}^n)$.

6.2 The case $n = 1$

When $n = 1$ then the 1-jet space is 3-dimensional, $J^1(\mathbb{R}) = \mathbb{R}^3$. A holonomic section $s : \mathbb{R} \to J^1(\mathbb{R})$ is a simultaneous graph of a function and its derivative:

$$s(x) = (x, p = f'(x), z = f(x)).$$

The contact structure ξ is the 2-dimensional plane field given by a Pfaffian equation $dz - pdx = 0$.

Let $\Sigma \subset J^1(\mathbb{R})$ be a 2-dimensional submanifold. Suppose that for $a \in \Sigma$ the tangent plane $T_a\Sigma$ is transverse to the contact plane ξ_a. Then the line $\ell_a = T_a\Sigma \cap \xi_a$ is called the *characteristic line*. If Σ is transverse to ξ everywhere, then $\ell = \{\ell_a\}_{a\in\Sigma}$ is a tangent line field to Σ (which is called the *characteristic line field). The integral curves of this line field are called* characteristics.

Lemma 6.3. *Characteristics are Legendrian submanifolds. In particular, if a characteristic $\Lambda \subset \mathbb{R}$ is graphiical with respect to the projection $J^1(\mathbb{R}) \to \mathbb{R}$ then it is a holonomic, i.e. there exists a function $h : (a, b) \to \mathbb{R}$ such that $s(x) = (c, h'(x), h(x)), x \in (a, b)$.*

6.3 Characteristics in the n-dimensional case

Let $\Sigma \subset J^1(\mathbb{R}^n)$ be a hypersurface. A point $a \in \Sigma$ is called *singular* if $T_a\Sigma = \xi_a$. Otherwise, i.e. if $T_a\Sigma$ is transverse to ξ_a, it is called *regular*. At a regular point $a \in \Sigma$ the intersection $\Pi_a = T_a\Sigma \cap \xi_a$ is an $(2n-1)$-dimensional subspace. Here are some conditions which guarantees transversality of $\Sigma \subset J^1(\mathbb{R}^n)$ and $\xi = \{\lambda = 0\}$, i.e. regularity of all points of Σ.

Example 6.4.

1. Suppose a $\Sigma = \{F = 0\}$ where for every point $a \in \Sigma$ there exists $i = 1, \dots, n$ such that $\frac{\partial F}{\partial p_i}(a) \neq 0$. Then Σ is transverse to ξ.

2. Suppose the hypersurface Σ is tangent to the z-directions (e.g. the defining it function F is independent of z. Then Σ is transverse to ξ.

Lemma 6.5. *Suppose Σ is transverse to ξ. Then for any point $a \in \Sigma$ there exists a unique line $\ell_a \subset \Pi_a = \xi_a \cap T_a$ which is characterized by the following condition. Given any vectors $v \in \ell_a$ and $w \in \Pi_a$ we have*

$$d\lambda(v, w) = 0.$$

In other words, ℓ_a is the kernel of the form $d\lambda|_{\Pi_a}$.

Proof. The contact hyperplane field $\xi - \{dz - pdx = 0\}$ is transverse to the z-axis, and hence the form $d\lambda = dp \wedge dx|_\xi$ has the maximal rank $2n$. Therefore the restriction of this form to the codimension 1 subspace $\Pi_a \subset \xi_a$ has rank $2n-1$, because the rank cannot drop more than by 1, but on the other hand the rank of a skew-symmetric form is always even. Hence, there exists a 1-dimensional kernel $\ell_a \subset \Pi_a$ of the form $d\lambda|_{\Pi_a}$, i.e. $d\lambda(v, w) = 0$ for any vectors $v \in \ell_a, w \in \Pi_a$.

The line field $\ell = \{\ell_a\}_{a\in\Sigma}$ which is tangent to Σ is called the *characteristic line field*, and its integral curves are called *characteristics.*

The next lemma gives an explicit expression for a vector field directing the line field ℓ.

Lemma 6.6. *Suppose $\Sigma = \{F(x,p,z) = 0\}$ and $a = (x,p,z) \in \Sigma$ a regular point. Then the line ℓ_a is generated by the vector*

$$v = \sum_1^n F_{p_i}\frac{\partial}{\partial x_i} - \sum_1^n (F_{x_i} - p_i F_z)\frac{\partial}{\partial p_i} + \sum_1^n p_i F_{p_i}\frac{\partial}{\partial z}. \tag{6.3.1}$$

Proof. Given any vector $w = (X,Y,Z) \in \Pi_a = \xi_a \cap T_a\Sigma$ its coordinates should satisfies the following conditions. The equation $dF_a(w) = 0$ takes the form

$$F_x X + F_p P + F_z Z = 0. \tag{6.3.2}$$

The equation $\lambda(w) = 0$ takes the form

$$Z - pX = 0. \tag{6.3.3}$$

Hence, vectors in ξ_a have the form (X,P,pX), and the necessary and sufficient condition for a vector w to be $\xi_a \cap T_a\Sigma$ is that it satisfies the equation

$$(F_x + pF_z)X + F_p P = 0.$$

Let $v = (\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ be a non-zero vector given by (6.3.1). We have $\widetilde{Z} = p\widetilde{X} = \sum\limits_1^n p_x \widetilde{X}_i$ and $(F_x + pF_z)\widetilde{X} + F_p\widetilde{P} = (F_x + pF_z)F_p - F_p(F_x + pF_z) = 0$, and hence $v \in \Pi_a$. We also have

$$v \lrcorner d\lambda = v \lrcorner dp \wedge dx = \widetilde{P}dx - \widetilde{X}dp,$$

and for any vector $w = (X,P,pX) \in \Pi_a$, we have $(F_x + pF_z)X + F_p P = 0$ we have

$$P\widetilde{X} - \widetilde{P}X = (F_x + pF_z)X + F_p P = 0. \tag{6.3.4}$$

Lemma 6.7. *Let $\Sigma \subset J^1(\mathbb{R}^n)$ be a hypersurface transverse to ξ, and ℓ the characteristic line field. Let $L \subset \Sigma$ be a submanifold such that $\lambda|_L = 0$ and L is transverse to ℓ. Let $\widehat{L}$ denote the union of all trajectories of the characteristic foliation intersecting L. Then $\lambda|_{\widehat{L}} = 0$.*

In other words, if we flow a k-dimensional submanifold of Σ tangent to ξ along the characteristics, then it swaps a $(k+1)$-dimensional submanifold of Σ tangent to ξ. **Proof.** Choose a non-vanishing vector field $v \in \ell$. At a point $a \in L$ the tangent $T_a\widehat{L} \subset \Pi_a$ is spanned by T_aL and the vector $v(a)$. Note that $d\lambda_{T_a\widehat{L}} = 0$ because $d\lambda|_{T_aL} = 0$ by assumption, and $d\lambda(v(a), w) = 0$ for all $w \in T_a\widehat{L}$ because $v(a) \in \ell_a = \operatorname{Ker} d\lambda|_{\Pi_a}$. We also note that the flow of the vector field v on $\widehat{L}$ preserves the form $\mu := \lambda|_{\widehat{L}}$. Indeed, the Lie derivative $L_v(\lambda|_{\widehat{L}}) = d(\lambda(v)) + v \lrcorner d\lambda) = 0$. Here the first term vanishes because $v \in \ell \subset \xi$, and the second one vanishes because $v \in \ell = \operatorname{Ker}(d\lambda|_{\Pi})$. Therefore, if λ vanishes in one point of a trajectory of v, then it vanishes at every point of this trajectory. But by definition any trajectory of v on $\widehat{L}$ intersects L, and as we had seen above λ vanishes on $\widehat{L}$ at the points of L. Hence, it vanishes, everywhere.

Lemma 6.8. *Let $\Sigma \subset J^1(\mathbb{R}^n)$ be a hypersurface transverse to ξ, and ℓ the characteristic line field. Then any Legendrian submanifold $L \subset \Sigma$ is tangent to ℓ.*

Proof. Recall that a Legendrian submanifold is an n-dimensional submanifold tangent to Σ. Suppose that for a point $a \in \Sigma$ the characteristic line ℓ_a is transverse to T_aL. Consider the $(n+1)$-dimensional space $S := \operatorname{Span}(T_aL, v)$. We have $S \subset \Pi_a \subset \xi_a$. On the other hand, $d\lambda|_S = 0$. Indeed, $d\lambda|_{T_aL} = 0$ by assumption, and $d\lambda(v, w) = 0$ for all $w \in S$ and $v \in \ell_a$ because $\ell_a = \operatorname{Ker} d\lambda|_{\Pi_a}$. But $d\lambda$ is a non-degenerate form on a $2n$-dimensional space ξ_a. Hence, it cannot vanish on a subspace of dimension $> n$.

Theorem 6.9. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n-1} = \{x_n = 0\}$ be two bounded open domains such that $\overline{\Omega}_1 \subset \Omega_2$, and $\phi : \overline{\Omega}_2 \to \mathbb{R}$ a smooth function. Consider a Cauchy problem*

$$\begin{aligned} &\frac{\partial u}{\partial x_n} = f\left(x_1, \dots, x_{n-1}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{n-1}}, u\right) \\ &u(x_1, \dots, x_{n-1}, 0) = \phi(x_1, \dots, x_{n-1}). \end{aligned} \tag{6.3.5}$$

with respect to a function $u : \mathbb{R}^n \to \mathbb{R}$. Then for a sufficiently small $\epsilon > 0$ the Cauchy problem (6.3.5) *has a unique solution for $(x_1, \dots, x_{n-1} \in \Omega_1,\ |x_n| \le \epsilon$. This solution can be found using the* *following procedure. Consider a system of ordinary differential equations*

$$\begin{aligned}
&\dot{x}_i = -\frac{\partial f}{\partial p_i}(x, p_1, \ldots, p_{n-1}, z), \quad i = 1, \ldots, n-1, \\
&\dot{x}_n = 1, \\
&\dot{p}_i = \frac{\partial f}{\partial x_i}(x, p_1, \ldots, p_{n-1}, z) - p_i \frac{\partial f}{\partial z}(x, p_1, \ldots, p_{n-1}, z), \\
&\dot{z} = f(x, p_1, \ldots, p_{n-1}, z) - \sum_1^{n-1} p_i \frac{\partial f}{\partial p_i}(x, p_1, \ldots, p_{n-1}, z),
\end{aligned} \tag{6.3.6}$$

Let

$$x_j = \alpha_j(c_1, \ldots, c_{n-1}, t), \; j = 1, \ldots, n-1, \; x_n = t, \tag{6.3.7}$$

$$p_j = \beta_j(c_1, \ldots, c_{n-1}, t), \; j = 1, \ldots, n-1, \tag{6.3.8}$$

$$z = \gamma(c_1, \ldots, c_{n-1}, t), \tag{6.3.9}$$

be the solution of system (6.3.6) *with initial data*

$$\begin{aligned}
&x_j(0) = c_j, \; j = 1, \ldots, n-1, \; (c_1, \ldots, c_{n-1}) \in \Omega_2, \\
&x_n(0) = 0., \\
&p_j(0) = \frac{\partial \phi}{\partial x_j}(c_1, \ldots, c_{n-1}), \; j = 1, \ldots, n-1, \\
&z(0) = \phi(c_1, \ldots, c_{n-1}).
\end{aligned}$$

The system of algebraic equations (6.3.7) *can be resolved with respect to* c_i*,* $i = 1, \ldots, n-1$*:*

$$c_j = \delta_j(x_1, \ldots, x_n), \; j = 1, \ldots, n-1,$$

for sufficiently small values of x_n*. Then the function*

$$u(x_1, \ldots, x_n) := \gamma(\delta_1(x_1, \ldots, x_n), \ldots, \delta_{n-1}(x_1, \ldots, x_n), t)$$

is the solution of the Cauchy problem for (6.3.5)*.*

Chapter 7

Proof of basic theorems

7.1 Existence and uniqueness theorem

Theorem 7.1. *Consider a system*

$$\dot{x} = f(x,t), (x,t) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}, \tag{7.1.1}$$

where f is a C^1-map. Then there exists $\epsilon > 0$ such that for any point $(x,t) \in \Omega$ with $||x - x_0|| \leq \epsilon$ there exists a unique solution $g_x(t)$, $|t - t_0| < \epsilon$ which satisfies $g_x(t_0) = x$. Moreover, $g_x(t)$ is a continuous function of x and t.

Proof. To simplify the notation we assume $t_0 = 0, x_0 = 0$. First, choose some $a > 0$ such that

$$C_a = \{||x||, |t| \leq a\} \subset \Omega.$$

Denote

$$C := \max_{(x,t)\in C_a} ||f(x,t)||, \ L = \max_{(x,t)\in C_a} ||d_{x,t}f||.$$

Choose $\epsilon \in (0,a)$ such that $L\epsilon < C$ and $(C+1)\epsilon < a$. Denote $C_\epsilon := \{||x||, |t| \leq \epsilon\}$

Let us denote by M the subset of the space of continuous maps $h : C_\epsilon \to \mathbb{R}^n$ such that $h(x,0) = 0$ and $||h(x,t)|| \leq Ct$. The space M is a closed subset of the space $C^0(C_\epsilon, \mathbb{R}^n)$ endowed with the norm $||h|| = \max_{(x,t)\in C_\epsilon} ||h(x,t)||$. The space $C^0(C_\epsilon, \mathbb{R}^n)$ of continuous maps $C_\epsilon \to \mathbb{R}^n$ with this norm is a complete space (i.e. any Cauchy sequence converges) and hence M is a complete space as well.

Define an integral operator $K : M \to C^0(C_\epsilon, \mathbb{R}^n)$ by the formula

$$K(h)(x,t) = \int_0^t f(x+h(x,s),s)ds,\ (x,t) \in C_\epsilon.$$

The operator K has the following properties:

- $K(h) \in M$, i.e. K maps M to M;
- there exists a positive constant $\mu < 1$ such that

  $$||K(h) - K(\widehat{h})|| \le \mu||h - \widehat{h}||$$

  for all $h, \widehat{h} \in M$; in other words, K is a *contracting operator*.

Indeed, first we note that

$$||(x + h(x,t))|| \le ||x - x_0|| + ||h(x,t)|| \le \epsilon + C\epsilon = (C+1)\epsilon < a,$$

i.e. $(x + h(x,t), t) \in C_a$ if $(x,t) \in C_\epsilon$. Note that $K(h)(x,0) = 0$. Hence, for $|t| < \epsilon$ we have

$$|K(h)(x,t)|| \le \max_{||x|| \le \epsilon} ||K(h)(x,t)|| \le \int_0^t ||f(x+h(x,s),s)||ds \le \int_0^t Cds \le Ct.$$

Thus, $K(h) \in M$. Denote $\mu := L\epsilon$ (recall that $L = \max_{(x,t)\in C_a} ||d_{x,t}f||$)). Then by assumption we have $\mu < 1$. We have

$$||K(h) - K(\widehat{h})|| \le \int_0^t ||f(x+h(x,s),s) - f(x+\widehat{h}(x,s),s)||ds$$

$$\int_0^t L||h - \widehat{h}||ds \le L\epsilon||h - \widehat{h}|| = \mu||h - \widehat{h}||.$$

According to the fixed point theorem for contracting operators there exists a unique $h \in M$ such that $K(h) = h$.

We claim that then $g_x(t) := x + h(x,t)$ is the required solution of the system $\dot{x} = f(x,t)$. Indeed, we have $g_x(0) = x$ and

$$\frac{d}{dt}g_x(t) = \frac{d}{dt}\left(\int_0^t f(g_x(s),s)ds\right) = f(g_x(t),t).$$

Note, that conversely, *any solution of the system is a fixed point of the operator* K, and hence, the solution is unique. ■

7.2 Equation in variations

Consider again the system (7.1.1) and assume here that $f(x,t)$ is at least C^1-smooth.

Let $\phi : (-a,a) \to \Omega$ be a solution of (7.1.1) which corresponds to the initial condition $\phi(0) = x_0$. Let us look for solutions $x(t)$ close to $\phi(t)$ in the form $x(t) := \phi(t) + \epsilon u(t),\ t \in (-a,a)$, we get

$$f(x,t) = f(\phi(t) + u(t), t) = f(\phi(t), t) + \epsilon C(t)u(t) + o(\epsilon).$$

Here $C(t) = d_{x=\phi(t)} f(x,t)$ is the differential (with respect to x of $f(x,t)$ at the point $\phi(t)$. In what follows we will not distinguish between the differential and the corresponding Jacobi matrix.

Thus plugging $x(t) := \phi(t) + \epsilon u(t)$ into equation (7.1.1) we get

$$\dot{\phi} + \epsilon \dot{u} = f(\phi(t), t) + \epsilon C(t)u(t) + o(\epsilon).$$

Taking into account that ϕ is a solution of (7.1.1) we get that $\dot{\phi} = f(\phi(t), t)$. Dividing the remaining terms of the equation by ϵ and passing to the limit when $\epsilon \to 0$ we get the equation

$$\dot{u} = C(t)u. \tag{7.2.1}$$

which is called an *equation in variations* along a solution $x(t)$ of (7.1.1).

If we denote by $\phi_\epsilon(t)$ the solution of (7.1.1) corresponding to the initial condition $\phi_\epsilon(0) = \epsilon u_0$ for some vector $u_0 \in \mathbb{R}^n$, then, *assuming that the solution depends smoothly on the parameter* ϵ we get $\left.\frac{d\phi_{\epsilon(t)}}{d\epsilon}\right|_{\epsilon=0} = u(t)$, where $u(t)$ is the solution of (7.2.1) which satisfies the initial condition $u(0) = u_0$. We will establish the smooth dependence in the next section.

7.3 Smooth dependence on the initial data

Theorem 7.2. *Let $g_x(t)$ be a family of solutions of the system* (7.1.1) *constructed in Theorem 7.1. Suppose that the right-hand side $f(x,t)$ is C^2-smooth. Then $g_x(t)$ depends smoothly on (x,t).*

Consider equations in variations for a solution $x(t)$ of (7.1.1)

$$\dot{u} = C_{x(t)}u,\ u \in \mathbb{R}^n, \tag{7.3.1}$$

where $C_{x(t)}$ is the Jacobi matrix of the map $f(x,t)$ with respect to x at the point $(x(t),t)$. We will not distinguish below between the differential, which is a linear map and its Jacobi matrix. Let us also consider the corresponding matrix equation:

$$\dot{U} = C_{x(t)}U, \;\; U \in M_n, \tag{7.3.2}$$

where M_n is the space of $n \times n$-matrix. If $U(t)$ is the solution of this matrix equation satisfying $U(0) = I$, then $u(t) = U(t)u_0$ is the solution of equation (7.3.1) with the initial condition $u(0) = u_0$. In other words, $U(t)$ is the matrix of the phase flow of equation (7.3.1).

The right-hand side of equation (7.3.3) depends on a solution of (7.1.1), so it is more natural to consider the two equations as a system of equations:

$$\begin{aligned} &\dot{x} = f(x,t), \\ &\dot{U} = C_x(t)U, x \in \Omega \subset \mathbb{R}^n, \; U \in M_n. \end{aligned} \tag{7.3.3}$$

Let us recall from the proof of Theorem 7.1 that the solution $g_x(t)$ of equation (7.1.1) can be obtained as the limit of successive approximations $g_x^{(0)}(t), \ldots, g_x^{(0)}(t), \ldots,$ where $g_x^0(t) = x$ and

$$g_x^{k+1}(t) = x + \int_0^t f(g_x^k(s), s)ds \tag{7.3.4}$$

for $k \geq 0$. Let us apply the same iteration scheme to the system (7.3.3):

Namely, we set $g_x^0(t) = x, \; G_x^0 = \mathrm{Id}$ and define

$$\begin{aligned} g_x^{k+1}(t) &= x + \int_0^t f(g_x^k(s), s)ds, \\ G_x^{k+1}(t) &= I + \int_0^t C_{g_x^k(s)} G_x^k(s)ds \end{aligned} \tag{7.3.5}$$

Note that all maps in both sequences are smooth, $G_x^0 = d_x g_x^0$, and differentiating the first equation with respect to x we inductively get that

$$d_x g_x^k(t) = G_x^k(t) \quad \text{ for all } \; k \geq 0. \tag{7.3.6}$$

On the other hand, according to Theorem 7.1 applied to equation (7.3.3) we conclude that the sequence $(g_x^k(t), G_x^k(t))$ uniformly converges to the solution $(g_x(t), G_x(t))$ of (7.3.3). Then, using

(7.3.6) we conclude that $d_x g_x(t) = G_x(t)$ which implies, in particular, that $g_x(t)$ is continuously differentiable with respect to x on some domain $C_\epsilon \subset \Omega$ where ϵ may need to be further decreased depending on the upper bound for the second derivatives of $f(x,t)$.

We leave it to the reader as an exercise to show that the differentiability with respect to the pair (x,t) of variables follows from what was already proven by the standard trick of passing to the extended phase space of the system.