Math 177: Homework N3

Due on Wednesday, May 27

1. A particle of mass m is moving in $\mathbb{R}^3$ in a central field with potential energy $U(r)$. Write its Hamiltonian function and the equation of motion in the canonical coordinates $(r, \phi, \theta, p_r, p_\phi, p_\theta)$ associated with the spherical coordinates coordinates (r, ϕ, θ).

2. Find an area preserving transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(P, Q) = f(p, q)$, if its graph is given by the generating function $F(q, P) = (q + q^3)P$. In other words, the graph of the area preserving map f in $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq - dP \wedge dQ)$ given by the generating function F with respect to the polarization of $\mathbb{R}^4$ by the coordinate plane (q, P) and (p, Q).

3. Poisson bracket $\{f, g\}$ of two functions $f, g : M \to \mathbb{R}$ on a symplectic manifold (M, ω) is a function $M \to \mathbb{R}$ defined by the formula $\{f, g\} = dg(X_f)$, where X_f is the Hamiltonian vector field of the function f, i.e. $X_f \lrcorner \omega = -df$. Verify the following properties of the Poisson bracket:

Skew-symmetricity: $\{f, g\} = -\{g, f\}$;

Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$;

Leibniz rule: $\{f, gh\} = \{f, g\}h + \{f, h\}g$.

4. Suppose that $\mathbb{R}^2$ is endowed with an area form $\omega = dp \wedge dq$. Let $H_t : \mathbb{R}^2 \to \mathbb{R}$, $t \in [0, 1]$, be a family of smooth functions equal to 0 outside of the unit disc D. Let $X_t := X_{H_t}$ be the

Hamiltonian vector field generated by H_t, i.e. $X_t \lrcorner \omega = -dH_t$. Let $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow of area preserving transformations generated by X_t, i.e.

$$\frac{df_t}{dt}(x) = X_t(f_t(x)).$$

Let $z_0 \in \mathrm{Int}D$ be a fixed point of f_1, i.e. $f_1(z_0) = z_0$. Denote by γ the loop $\gamma : [0,1] \to \mathbb{R}^2$ defined by the formula $\gamma(t) = f_t(z_0),\ t \in [0,1]$. Then the integral $S(z_0) := \int\limits_\gamma pdq - H_t dt$ is called *action* of the fixed point z_0.

Prove that for any path $\delta : [0,1] \to \mathbb{R}^2$ such $\delta(0) \in \mathbb{R}^2 \setminus D$ and $\delta(1) = z_0$ one has

$$\int\limits_\delta pdq - \int\limits_{f_1(\delta)} pdq = S(z_0).$$

In particular, the integral in the left hand side of the equation is independent of the choice of the path δ, so that the action depends only on f_1 and not on a choice of the Hamiltonian H_t which generates it.

Each problem is 10 points.