THE θ OPERATOR

by

Nick Ramsey

Contents

1. deRham Cohomology

Let $f : X \longrightarrow Y$ be an arbitrary map of schemes. We denote by

$$\Omega^{\bullet}_{X/Y} = (\mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \to \cdots)$$

the associated $f^{-1}\mathcal{O}_Y$-linear relative deRham complex (see EGA IV_4, §16.6 for the construction and functoriality properties). The i^{th} *relative deRham cohomology of* X *over* Y is defined to be the $\mathcal{O}_Y$-module given by the i^{th} relative hypercohomology:

$$\mathbb{H}^i_{\mathrm{dR}}(X/Y) := \mathbb{R}^i f_*(\Omega^{\bullet}_{X/Y}).$$

The deRham cohomology is equipped with a natural filtration (the Hodge filtration) arising from the filtration of the relative deRham complex

$$F^j\Omega^{\bullet}_{X/Y} = (0 \to \cdots \to 0 \to \Omega^j_{X/Y} \to \Omega^{j+1}_{X/Y} \to \cdots)$$

namely

$$F^j\mathbb{H}^i_{\mathrm{dR}}(X/Y) = \mathrm{image}(\mathbb{R}^i f_* F^j\Omega^{\bullet}_{X/Y} \longrightarrow \mathbb{R}^i f_*\Omega^{\bullet}_{X/Y}). \tag{1}$$

The Hodge filtration is descending, exhaustive, and separated.

The main computational tool for deRham cohomology is the Hodge to deRham spectral sequence, whose construction we briefly recall. Let $C^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of the complex $\Omega^\bullet_{X/Y}$. This double complex has a filtration

$$F^j C^{\bullet,\bullet} = C^{\geq j,\bullet}$$

where we replace the terms $C^{n,m}$ with $n < j$ by zeros. This filtration is compatible with the above filtration on $\Omega^\bullet_{X/Y}$ in the sense that $F^j C^{\bullet,\bullet}$ is a Cartan-Eilenberg resolution of $F^j\Omega^\bullet_{X/Y}$ for each $j \geq 0$. This filtration on $C^{\bullet,\bullet}$ induces the filtration

$$F^j \mathrm{Tot}(C^{\bullet,\bullet}) = \mathrm{image}(\mathrm{Tot}(F^j C^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}(C^{\bullet,\bullet}))$$

on the associated total complex, the associated graded pieces of which are $\mathrm{gr}^p(\mathrm{Tot}(C^{\bullet,\bullet})) = C^{p,\bullet}$. Applying the left-exact functor f_* to the above filtered complex, we arrive at the filtered complex $f_*\mathrm{Tot}(C^{\bullet,\bullet})$ whose graded pieces are

$$\mathrm{gr}^p(f_*\mathrm{Tot}(C^{\bullet,\bullet})) = f_* C^{p,\bullet}$$

since $C^{\bullet,\bullet}$ is a double complex of injective objects. By definition of a Cartan-Eilenberg resolution, the complex $C^{p,\bullet}$ is an injective resolution of $\Omega^p_{X/Y}$, so that

$$H^q(\mathrm{gr}^p(f_*\mathrm{Tot}(C^{\bullet,\bullet}))) = R^q f_* \Omega^p_{X/Y}.$$

The *Hodge to deRham spectral sequence* is the spectral sequence associated to the filtered complex $f_*\mathrm{Tot}(C^{\bullet,\bullet})$. By what we have shown the Hodge to deRham spectral sequence is of the form

$$E_1^{p,q} = R^q f_* \Omega^p_{X/Y} \implies \mathbb{H}^{p+q}_{\mathrm{dR}}(X/Y)$$

where the filtration on the right is given by (1) since the filtration on the Cartan-Eilenberg resolution is compatible with filtration on $\Omega^\bullet_{X/Y}$ in the manner stated above. The Hodge to deRham spectral sequence has $\mathcal{O}_Y$-linear differentials and its formation is compatible with the natural pullback maps arising from base change on Y. In particular, the sheaves $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$ are quasi-coherent when f is quasi-compact and separated and even coherent if f is proper and Y is locally noetherian.

Theorem 1.1. — *$f : X \longrightarrow Y$ be a proper, surjective, and smooth morphism of relative dimension* 1 *with geometrically connected fibers. Then the Hodge to deRham spectral sequence degenerates at E_1.*

Proof. The assertion is local on Y, so we may assume that Y is affine. Since cohomology commutes with direct limits, we may further assume that the base is noetherian. By the theorem on formal functions and vanishing of cohomology in degrees > 1 on fibers, we have $R^q f_* = 0$ on quasi-coherent sheaves for $q > 1$. Also, our hypotheses imply that $\Omega^p_{X/Y} = 0$ if $p > 1$. Thus the entire spectral sequence lives in the region $0 \leq p, q \leq 1$, and the only maps are

$$d_1^{0,q} : R^q f_* \mathcal{O}_X \longrightarrow R^q f_* \Omega^1_{X/Y}, \quad q = 0, 1$$

and one checks that these maps are the ones induced from the differential

$$d : \mathcal{O}_X \longrightarrow \Omega^1_{X/Y}$$

Thus to prove degeneration at E_1 it suffices to see that both of these maps are zero. The theorem on formal functions reduces us the case of an Artin local ring (namely, $\mathcal{O}_{Y,y}/m_y^n$ for $y \in Y$ and $n \geq 1$ some integer).

Let A be an Artin local ring and let B be a complete regular local ring of which A is a quotient. By the deformation results of SGA 1, Exposé III, §7, we can lift our curve to a smooth proper curve X over B. By the openness of loci results of EGA IV$_3$, §12.2, $f : X \longrightarrow \mathrm{Spec}(B)$ has geometrically connected fibers. We will prove the result for $f : X \longrightarrow \mathrm{Spec}(B)$, and along the way prove that cohomology and base change holds for $R^q f_* \Omega^p_{X/B}$ for $0 \leq p, q \leq 1$, from which the result for our curve over A will follow.

We claim that cohomology and base change holds for the sheaves $R^q f_* \Omega^p_{X/B}$. Since B is reduced it suffices by Grauert's Theorem (EGA III$_2$ 7.8.4) to prove that

$$\dim_{k(y)} H^q(X_y, \Omega^p_{X_y/k(y)})$$

is constant in $y \in \mathrm{Spec}(B)$. First, since the geometric fibers are reduced and connected, we have $f_* \mathcal{O}_X = \mathcal{O}_Y$, which takes care of the case $p = q = 0$. Since f is flat, the Euler characteristic

$$\chi(\mathcal{O}_{X_y}) = \dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) - \dim_{k(y)} H^1(X_y, \mathcal{O}_{X_y})$$

is constant in $y \in Y$. Since the first term is identically equal to 1, the second term must also be constant in y. That $R^1 f_* \Omega^1_{X/Y}$ is locally free and has formation that is compatible with base change under our hypotheses is a standard consequence of Serre-Grothendieck duality. The same Euler characteristic argument as above now shows that $f_* \Omega^1_{X/Y}$ is compatible with base change. We note also that it follows from the cohomology and base change machinery (see in particular EGA III$_2$, 7.8.4) that all four of these sheaves are in fact locally free (of finite rank).

Since the arrow $f_* \mathcal{O}_X \longrightarrow f_* \Omega^1_{X/Y}$ is induced form the $\mathcal{O}_Y$-linear derivation d, this arrow must be zero since $f_* \mathcal{O}_X = \mathcal{O}_Y$. To check that the arrow $R^1 f_* \mathcal{O}_X \longrightarrow R^1 f_* \Omega^1_{X/Y}$ vanishes, we may (by local freeness of everything in sight) use Serre-Grothendieck duality and check after dualizing. We claim that the induced map on duals

$$f_* \mathcal{O}_X \longrightarrow f_* \Omega^1_{X/Y}$$

is again the one induced by the differential d, and is therefore zero, as we have seen. This doesn't really have anything to do with the duality machinery, and boils down to the commutativity of the diagram

$$\begin{array}{ccc} f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} R^1 f_* \mathcal{O}_X & \xrightarrow{f_* d \otimes 1} & f_* \Omega^1_{X/B} \otimes_{\mathcal{O}_Y} R^1 f_* \mathcal{O}_X \\ \Big\downarrow {\scriptstyle 1 \otimes R^1 f_* d} & & \Big\downarrow {\scriptstyle \cup} \\ f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} R^1 f_* \Omega^1_{X/B} & \xrightarrow{\ \cup\ } & R^1 f_* \Omega^1_{X/B} \end{array}$$

To see that this commutativity holds, we first observe that the cup product map may be factored as

$$R^i f_* \mathcal{F} \otimes_{\mathcal{O}_Y} R^j f_* \mathcal{G} \longrightarrow R^{i+j} f_* (\mathcal{F} \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{G}) \longrightarrow R^{i+j} f_* (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

and that the first map here is functorial in $f^{-1}\mathcal{O}_Y$-linear maps of $\mathcal{O}_X$-modules (such as d!). The above diagram now boils down to trivially commutative diagram of $\mathcal{O}_X$-modules and $f^{-1}\mathcal{O}_Y$-linear maps, which we leave to the reader to write out. ■

Corollary 1.1.1. — *Let $f : X \longrightarrow Y$ be as in the Theorem. Then there is a canonical exact sequence*

$$0 \longrightarrow f_*\Omega^1_{X/Y} \longrightarrow \mathbb{H}^1_{\mathrm{dR}}(X/Y) \longrightarrow R^1 f_*\mathcal{O}_X \longrightarrow 0$$

whose formation commutes with base change on Y and which is functorial in Y-maps $X \to X'$.

Next, we give two very simple calculations with deRham cohomology and its Hodge filtration.

1. We compute $\mathbb{H}^0_{\mathrm{dR}}(X/Y)$. By definition we have

$$\mathbb{H}^0_{\mathrm{dR}}(X/Y) = H^0(f_*(\Omega^\bullet_{X/Y})) = \ker(f_*\mathcal{O}_X \xrightarrow{f_*d} f_*\Omega^1_{X/Y}).$$

   Of course, f_*d is the map $d_1^{0,0}$ in the Hodge to deRham spectral sequence, so if this sequence degenerates at E_1, we have $\mathbb{H}^0_{\mathrm{dR}}(X/Y) = f_*\mathcal{O}_X$.
2. Suppose that f is finitely presented and proper with all fibers of dimension $d \geq 0$. We will compute $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$ for $i \geq 2d$. Not surprisingly, this sheaf will turn out to be 0 for $i > 2d$. Under our hypotheses, $\Omega^p_{X/Y}$ vanishes for $p > d$. Also, by passage to the noetherian case and using the theorem of formal functions (as in the proof of the previous theorem), we see that $R^q f_* = 0$ for $q > d$ on quasi-coherent sheaves. Thus the entire spectral sequence resides in the box $0 \leq p, q, \leq d$. It follows immediately that $\mathbb{H}^i_{\mathrm{dR}}(X/Y) = 0$ for $i > 2d$. It also follows that

$$\mathbb{H}^{2d}_{\mathrm{dR}}(X/Y) = E^{d,d}_\infty = \mathrm{coker}(R^d f_*\Omega^{d-1}_{X/Y} \longrightarrow R^d f_*\Omega^d_{X/Y})$$

   In particular, if the spectral sequence degenerates at E_1, we have $\mathbb{H}^{2d}_{\mathrm{dR}}(X/Y) = R^d f_*\Omega^d_{X/Y}$. If, further, f is smooth, then the trace map of Serre-Grothendieck duality furnishes an isomorphism $\mathbb{H}^{2d}_{\mathrm{dR}}(X/Y) \cong \mathcal{O}_Y$ whose formation commutes with arbitrary base change on Y.

The sheaves $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$ are equipped with functorial bilinear pairings

$$\mathbb{H}^i_{\mathrm{dR}}(X/Y) \otimes_{\mathcal{O}_Y} \mathbb{H}^i_{\mathrm{dR}}(X/Y) \longrightarrow \mathbb{H}^{i+j}_{\mathrm{dR}}(X/Y)$$

which are symmetric (resp. alternating) in the indicated degrees if $(-1)^{ij} = 1$ (resp. $(-1)^{ij} = -1$). This pairing is compatible with the natural pairings

$$R^p f_*\Omega^p_{X/Y} \otimes_{\mathcal{O}_Y} R^{q'} f_*\Omega^{p'}_{X/Y} \longrightarrow R^{q+q'} f_*\Omega^{p+p'}_{X/Y}$$

on the E_1 term of the Hodge to deRham spectral sequence in a sense which we now detail. The parings on the E_1 term induce pairings on the E_r term for each r, and hence on the E_∞ term. The claimed compatibility is that the following diagram commutes.

$$\begin{array}{ccc} F^n\mathbb{H}^i_{\mathrm{dR}}(X/Y)\otimes_{\mathcal{O}_Y} F^m\mathbb{H}^j_{\mathrm{dR}}(X/Y) & \longrightarrow & F^{n+m}\mathbb{H}^{i+j}_{\mathrm{dR}}(X/Y) \\ \downarrow & & \downarrow \\ E_\infty^{n,i-n}\otimes_{\mathcal{O}_Y} E_\infty^{m,j-m} & \longrightarrow & E_\infty^{n+m,n+m-(i+j)} \end{array}$$

We flesh out one special case that will be used in the sequel. Suppose that $f : X \longrightarrow Y$ satisfies the hypotheses of Theorem 1.1. Then we have an alternating pairing on $\mathbb{H}^1_{\mathrm{dR}}(X/Y)$ with values in $\mathbb{H}^2_{\mathrm{dR}}(X/Y)$. Let

$$\langle , \rangle_{\mathrm{dR}} : \mathbb{H}^1_{\mathrm{dR}}(X/Y)\otimes_{\mathcal{O}_Y}\mathbb{H}^1_{\mathrm{dR}}(X/Y)\longrightarrow \mathcal{O}_Y$$

denote the pairing given by post composing the cup product with the isomorphism $\mathbb{H}^2_{\mathrm{dR}}\cong\mathcal{O}_Y$ given by the trace map, as in example 2 above. Let ω be a section of $f_*\Omega^1_{X/Y}$ and let η be any section of $\mathbb{H}^1_{\mathrm{dR}}(X/Y)$. Denote the image of η in $R^1f_*\mathcal{O}_X$ be η'. Then we have

$$\langle\omega,\eta\rangle_{\mathrm{dR}} = \langle\omega,\eta'\rangle_{SG} \tag{2}$$

where

$$\langle , \rangle_{SG} : f_*\Omega^1_{X/Y}\otimes_{\mathcal{O}_Y} R^1f_*\mathcal{O}_X\longrightarrow\mathcal{O}_Y$$

is the canonical pairing followed by the trace map.

2. The Gauss-Manin Connection

Let $f : X \to Y$ be map of schemes and let $\mathcal{E}$ be an $\mathcal{O}_X$-module on X. A *connection on* $\mathcal{E}$ *over* Y is a map of abelian sheaves

$$\nabla : \mathcal{E}\longrightarrow\Omega^1_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{E}$$

with the property that, if $U\subset X$ is an open set, $s\in\mathcal{E}(U)$, and $h\in\mathcal{O}_X(U)$, then

$$\nabla(hs) = h\nabla(s) + dh\otimes s.$$

We note for future use that a connection ∇ on $\mathcal{E}$ induces a connection $\nabla^{\otimes k}$ on the tensor powers $\mathcal{E}^{\otimes k}$ via the Leibnitz formula on sections, namely

$$\nabla^{\otimes k}(s_1\otimes\cdots\otimes s_k) = \nabla(s_1)\otimes s_2\otimes\cdots\otimes s_k + s_1\otimes\nabla(s_2)\otimes\cdots\otimes s_k + \cdots s_1\otimes\cdots\otimes\nabla(s_k)$$

where we commute the $\Omega^1_{X/Y}$ contribution from each term to the front to get a section of

$$\Omega^1_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{E}^{\otimes k}.$$

In addition, one can associate a connection to direct sums, symmetric powers, and exterior powers of any $\mathcal{O}_X$-module with a connections. We leave the definitions to the reader.

Let $\mathcal{E}$ be an $\mathcal{O}_X$-module equipped with a connection ∇. For each $i\geq 0$, ∇ induces a map (which by abuse of notation we also denote ∇)

$$\nabla : \Omega^i_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{E}\longrightarrow\Omega^{i+1}_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{E}$$

defined on sections by

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \otimes \nabla(s)$$

where the second term is interpreted as its image under the canonical map

$$\Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}) \longrightarrow \Omega^{i+1}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

given by contracting (via the wedge product) the first two factors together. Note that these maps are not $\mathcal{O}_X$-linear, though they are $f^{-1}\mathcal{O}_Y$-linear. Thus associated to the pair $(\mathcal{E}, \nabla)$ we have a canonical sequence of $\mathcal{O}_X$-modules and $f^{-1}\mathcal{O}_Y$-linear maps

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\nabla} \Omega^2_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \cdots \tag{3}$$

If this sequence is a complex then ∇ is said to be *integrable* (for reasons arising from the complex-analytic and differential geometry).

Now let S be a base scheme and let $f : X \to Y$ be a smooth map of smooth S-schemes. We will define a canonical connection on the quasi-coherent sheaf $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$ over S called the *Gauss-Manin connection*. The Gauss-Manin connection will arise as a differential in a spectral sequence coming from a certain filtration on the complex $\Omega^\bullet_{X/S}$. Namely, we define

$$F^i\Omega^\bullet_{X/S} = \mathrm{image}(f^*\Omega^i_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet-i}_{X/S} \longrightarrow \Omega^\bullet_{X/S}).$$

It follows from our smoothness hypotheses that

$$\mathrm{gr}^p\Omega^\bullet_{X/S} = f^*\Omega^p_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/Y}$$

Note that

$$\mathbb{R}^n f_*(f^*\Omega^p_{Y/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/Y}) = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{R}^{n-p} f_*\Omega^\bullet_{X/Y} = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^{n-p}_{\mathrm{dR}}(X/Y)$$

and therefore the spectral sequence of a filtered object (EGA 0_{III}, 13.6) becomes

$$E_1^{p,q} = \Omega^p_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^q_{\mathrm{dR}}(X/Y) \implies \mathbb{R}^{p+q} f_*\Omega^\bullet_{X/S}$$

In particular, we have a complex

$$\mathbb{H}^q_{\mathrm{dR}}(X/Y) \xrightarrow{d_1^{0,q}} \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^q_{\mathrm{dR}}(X/Y) \xrightarrow{d_1^{1,q}} \Omega^2_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^q_{\mathrm{dR}}(X/Y) \to \cdots$$

We define the Gauss-Manin connection $\nabla_{X/Y}$ to be $d_1^{0,q}$.

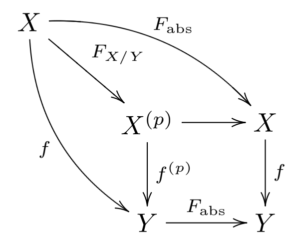
In [**1**] Katz and Oda exploit the cup product structure on this spectral sequence to show that $d_1^{1,q}$ is *induced* from $d_1^{0,q}$ in the manner described above in (3). Thus the associated deRham construction

$$(\Omega^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathbb{H}^q_{\mathrm{dR}}(X/Y), \nabla_{X/Y})$$

is a *complex* and $\nabla_{X/Y}$ is integrable.

3. Frobenius and the Hodge Filtration

Let p be a prime number. Let $f : X \longrightarrow Y$ be a map of $\mathbb{F}_p$-schemes. Recall the usual diagram



where the square is Cartesian (and *defines* $X^{(p)}$), F_{abs} denotes absolute Frobenius morphism on Y, and $F_{X/Y}$ denotes the relative Frobenius morphism. The pullback maps induced by base change on Y and functoriality in Y-maps together give an $\mathcal{O}_Y$-linear map

$$\mathbb{H}^i_{\mathrm{dR}}(X/Y)^{(p)} \longrightarrow \mathbb{H}^i_{\mathrm{dR}}(X^{(p)}/Y) \xrightarrow{F^*_{X/Y}} \mathbb{H}^i_{\mathrm{dR}}(X/Y)$$

where for an $\mathcal{O}_Y$-module $\mathcal{F}$, $\mathcal{F}^{(p)}$ denotes the base change by the absolute Frobenius of Y. By abuse of notation, we also refer to this composite as $F^*_{X/Y}$, and often omit X and Y from the notation. We observe that one can view $F^*_{X/Y}$ as an F_{abs} semilinear *endomorphism* of $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$. If η is any section of $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$, we denote by $\eta^{(p)}$ the section of $\mathbb{H}^i_{\mathrm{dR}}(X^{(p)}/Y)$ given by the image of $1 \otimes \eta$ under natural map

$$\mathbb{H}^i_{\mathrm{dR}}(X/Y)^{(p)} \longrightarrow \mathbb{H}^i_{\mathrm{dR}}(X^{(p)}/Y)$$

above.

Assume that we are in the situation of Theorem 1.1. By Corollary 1.1.1, we have a commutative diagram of $\mathcal{O}_Y$-linear maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{(p)}_* \Omega^1_{X^{(p)}/Y} & \longrightarrow & \mathbb{H}^1_{\mathrm{dR}}(X^{(p)}/Y) & \longrightarrow & R^1 f^{(p)}_* \mathcal{O}_{X^{(p)}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_* \Omega^1_{X/Y} & \longrightarrow & \mathbb{H}^1_{\mathrm{dR}}(X/Y) & \longrightarrow & R^1 f_* \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

with vertical arrows induced by $F^*_{X/Y}$.We claim that the leftmost vertical arrow is 0. By functoriality, this map is induced by pushforward through $f^{(p)}$ from the canonical map

$$\Omega^1_{X^{(p)}/Y} \longrightarrow F_{X/Y*} \Omega^1_{X/Y}. \tag{4}$$

To see that this map vanishes, we work locally on X. Let $\mathrm{Spec}(B)$ and $\mathrm{Spec}(A)$ be affine opens in X and Y, respectively, such that f restricts to $f : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$. In what follows the operation $\otimes_A A$ is understood to be taken with respect to the map $F_{\mathrm{abs}} : A \to A$. The map (4) in this setting is the $B \otimes_A A$-linear map

$$\Omega^1_{(B \otimes_A A)/A} \longrightarrow \Omega^1_{B/A}$$

corresponding by the universal property of Ω^1 to the A-linear derivation D defined the diagram

$$\begin{array}{ccc} B \otimes_A A & \xrightarrow{\quad D \quad} & \Omega^1_{B/A} \\ & {\scriptstyle F}\searrow \quad \nearrow{\scriptstyle d} & \\ & B & \end{array}$$

where F is the A-linear map characterized by $F(b \otimes a) = ab^p$. But then

$$D(b \otimes a) = d(F(b \otimes a)) = d(ab^p) = ad(b^p) = 0$$

so $D = 0$, and the result follows.

The result is that $F^*_{X/Y}$ factors through the first graded piece of the Hodge filtration, and we get an $\mathcal{O}_Y$-linear map (which by abuse of notation we also call $F^*_{X/Y}$)

$$F^*_{X/Y} : R^1 f^{(p)}_* \mathcal{O}_{X^{(p)}} \longrightarrow \mathbb{H}^1_{\mathrm{dR}}(X/Y).$$

If the image of this map is a locally free subsheaf of $\mathbb{H}^1_{\mathrm{dR}}(X/Y)$ which maps isomorphically to $R^1 f_* \mathcal{O}_X$ under the Hodge filtration, then we say that $F^*_{X/Y}$ *splits the Hodge filtration* or simply that *Frobenius splits the Hodge filtration* if we do not wish to name the map $F^*_{X/Y}$. The reason for this name is that, if $\mathcal{F}$ is the aforementioned locally free subsheaf, then one see easily that we have an isomorphism

$$f_* \Omega^1_{X/Y} \oplus \mathcal{F} \xrightarrow{\ \sim\ } \mathbb{H}^1_{\mathrm{dR}}(X/Y),$$

the map being induced by the inclusions. The property that Frobenius splits the Hodge filtration is local on the base in the obvious sense.

4. Applications to Elliptic Curves

Let $N \geq 4$ be an integer and $Y_1(N)$ be the usual fine moduli scheme over $\mathbb{F}_p$. We denote the universal family of elliptic curves over $Y_1(N)$ by $f : E_{\mathrm{univ}} \longrightarrow Y_1(N)$. Note that $Y_1(N)$ and the map f satisfy the hypotheses of Theorem 1.1. Recall that in this situation it is standard to denote $f_* \Omega^1_{E_{\mathrm{univ}}/Y_1(N)}$ by $\underline{\omega}$.

Since the property that Frobenius splits the Hodge filtration is local on the base, it makes sense to ask for the largest open subset of $Y_1(N)$ over which this is the case. Let $Y_1(N)^h$ (the "Hasse domain") denote the ordinary locus of $Y_1(N)$. This is the open subscheme defined as the complement of the zero locus of the Hasse invariant, which is a canonical section of $\underline{\omega}^{\otimes(p-1)}$ defined in previous lectures.

Theorem 4.1. — *$Y_1(N)^h$ is the largest open subset of $Y_1(N)$ over which Frobenius splits the Hodge filtration.*

Proof. We work locally on a sufficiently small open set $U \subset Y_1(N)$. Thus we may assume that $\underline{\omega}|_U$ and $(R^1 f_* \mathcal{O}_{E_{\mathrm{univ}}})|_U$ are trivial. Let ω be a basis of $\underline{\omega}|_U$ and let η be a basis of $(R^1 f_* \mathcal{O}_{E_{\mathrm{univ}}})|_U$ that is Serre-Grothendieck dual to ω. Let $\tilde{\eta}$ be a lift of

η to $\mathbb{H}^1_{\mathrm{dR}} := \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}|_U/U)$. Then $\{\omega, \tilde{\eta}\}$ is a basis of $\mathbb{H}^1_{\mathrm{dR}}$ with $\langle \omega, \tilde{\eta} \rangle_{\mathrm{dR}} = 1$ by equation (2). Let F be the relative Frobenius of $E_{\mathrm{univ}}|_U$ over U. Then we may write

$$F^*(\eta^{(p)}) = B\omega + A\tilde{\eta}$$

for some functions A and B on U. Moreover, by functoriality of the Hodge filtration, A is equal to the value of the Hasse invariant at the pair $(E_{\mathrm{univ}}|_U, \omega)$. But clearly the this element maps isomorphically to a generator of $(R^1 f_* \mathcal{O}_{E_{\mathrm{univ}}})|_U$ if and only if A is invertible on U. ■

Let $\mathcal{F}$ denote the image of

$$F^*_{E_{\mathrm{univ}}/Y_1(N)^h} : R^1 f^{(p)}_* \mathcal{O}_{E^{(p)}_{\mathrm{univ}}} \longrightarrow \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)^h)$$

once and for all. By Theorem 4.1 we have a canonical decomposition

$$\mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)^h) \cong \underline{\omega} \oplus \mathcal{F}$$

where for brevity here and in what follows we also denote $\underline{\omega}|_{Y_1(N)^h}$ by $\underline{\omega}$.

5. The θ operator

We begin by recalling the Kodaira-Spencer isomorphism. Let

$$\nabla : \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)) \longrightarrow \Omega^1_{Y_1(N)/\mathbb{F}_p} \otimes_{\mathcal{O}_{Y_1(N)}} \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N))$$

denote the Gauss-Manin connection. There is a unique $\mathcal{O}_{Y_1(N)}$-linear isomorphism

$$\underline{\mathrm{KS}} : \underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{Y_1(N)/\mathbb{F}_p}$$

that is characterized locally by the property that, if ω is a local basis for $\underline{\omega}$ over an open set U, then $\omega^{\otimes 2}$ maps to $\langle \omega, \nabla\omega \rangle_{\mathrm{dR}} \in \Omega^1_{U/\mathbb{F}_p}$.

Recall from the previous section that, over $Y_1(N)^h$, we have a canonical decomposition

$$\mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)^h) \cong \underline{\omega} \oplus \mathcal{F}.$$

Define a map $\tilde{\theta}$ as the composite

$$
\begin{array}{ccccc}
\underline{\omega}^{\otimes k} & \longrightarrow & \mathrm{Sym}^k \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)^h) & \xrightarrow{\mathrm{Sym}^k \nabla} & \Omega^1_{Y_1(N)^h/R} \otimes \mathrm{Sym}^k \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)^h) \\
 & & & & \downarrow {\scriptstyle \underline{\mathrm{KS}}^{-1}} \\
 & & & & \omega^{\otimes 2} \otimes \mathrm{Sym}^k \mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}/Y_1(N)^h) \\
 & & & & \downarrow {\scriptstyle \sim} \\
 & & & & \underline{\omega}^{\otimes 2} \otimes \underline{\omega}^{\otimes k} \oplus (\underline{\omega}^{\otimes (k-1)} \otimes \mathcal{F}) \oplus \cdots \oplus \mathcal{F}^{\otimes k} \\
 & & & & \downarrow {\scriptstyle \sim} \\
 & & & & \underline{\omega}^{\otimes k+2} \oplus (\underline{\omega}^{\otimes (k+1)} \otimes \mathcal{F}) \oplus \cdots \oplus (\underline{\omega}^{\otimes 2} \otimes \mathcal{F}^{\otimes k}) \\
 & & & & \downarrow \\
 & & \searrow^{\tilde{\theta}} & & \underline{\omega}^{\otimes (k+2)}
\end{array}
$$

We define

$$\theta : H^0(Y_1(N)^h, \underline{\omega}^{\otimes k}) \longrightarrow H^0(Y_1(N)^h, \underline{\omega}^{\otimes (k+p+1)})$$

by

$$\theta(f) = A \cdot \tilde{\theta}(f)$$

where A is the Hasse invariant.

Theorem 5.1. — *The map θ restricts to a map*

$$H^0(Y_1(N), \underline{\omega}^{\otimes k}) \longrightarrow H^0(Y_1(N), \underline{\omega}^{\otimes (k+p+1)}).$$

Proof. We start by finding a nice local expression for $\tilde{\theta}(f)$ on $Y_1(N)^h$. Working locally over some open $U \subseteq Y_1(N)^h$, we assume that $\underline{\omega}|_U$ is free and generated by ω. It follows by duality that $R^1 f_* \mathcal{O}$ is free over U, and hence $\mathcal{F}$ is free over U. Let ϕ be the unique generator of $\mathcal{F}$ which maps to the generator of $R^1 f_* \mathcal{O}$ which is Serre-Grothendieck dual to ω, so $\langle \omega, \phi \rangle_{\mathrm{dR}} = 1$.

Let $\xi \in \Omega^1_{U/\mathbb{F}_p}$ be defined by $\xi = \underline{\mathrm{KS}}(\omega^{\otimes 2})$. By the universal property of $\Omega^1_{U/\mathbb{F}_p}$, we have

$$\mathcal{H}om_{\mathcal{O}_U}(\Omega^1_{U/\mathbb{F}_p}, \mathcal{O}_U) = \mathcal{D}er_{\mathbb{F}_p}(\mathcal{O}_U, \mathcal{O}_U).$$

Let D be the derivation dual to ξ in the sense that it corresponds to the map $\xi \mapsto 1$. By the local characterization of the Kodaira-Spencer isomorphism in terms of the Gauss-Manin connection above, we have $\langle \omega, \nabla_D \omega \rangle_{\mathrm{dR}} = 1$. In particular, $\{\omega, \nabla_D \omega\}$ is a basis of $\mathbb{H}^1_{\mathrm{dR}}(E_{\mathrm{univ}}|_U/U)$.

Define A and B by

$$F^*((\nabla_D \omega)^{(p)}) = B\omega + A \nabla_D \omega.$$

Note that $A = A(E_{\mathrm{univ}}|_U, \omega)$ is the Hasse invariant as usual. By definition, this lies in $\mathcal{F}$, so there exists a unique $C \in \mathcal{O}(U)$ such that $B\omega + A\nabla_D \omega = C\phi$. Pairing with

ω, we see that $C = A$, and since A is invertible on $Y_1(N)^h$, we get

$$\nabla_D\omega = \phi - \frac{B}{A}\omega.$$

Let $f_0\omega^{\otimes k}$ be a section of $\underline{\omega}^{\otimes k}$ over U. Note that

$$\begin{aligned}
\nabla(f_0\omega^{\otimes k}) &= \nabla_D(f_0\omega^{\otimes k})\cdot\xi \\
&= (D(f_0)\omega^{\otimes k} + f_0\nabla_D(\omega^{\otimes k}))\cdot\xi \\
&= (D(f_0)\omega^{\otimes k} + kf_0\omega^{\otimes(k-1)}\nabla_D\omega)\cdot\xi \\
&= \left(D(f_0)\omega^{\otimes k} + kf_0\omega^{\otimes(k-1)}\left(\phi - \frac{B}{A}\omega\right)\right)\cdot\xi \\
&= \left(\left(D(f_0) - kf_0\frac{B}{A}\right)\omega^{\otimes k} + kf_0\omega^{\otimes(k-1)}\phi\right)\cdot\xi
\end{aligned}$$

Under the (inverse of the) Kodaira-Spencer isomorphism ξ maps to $\omega^{\otimes 2}$, so the above maps to

$$\left(D(f_0) - kf_0\frac{B}{A}\right)\omega^{\otimes(k+2)} + kf_0\omega^{\otimes(k+1)}\phi.$$

Since $\phi \in \mathcal{F}$, we conclude that

$$\tilde{\theta}(f\omega^{\otimes k}) = \left(D(f_0) - kf_0\frac{B}{A}\right)\omega^{\otimes(k+2)} \tag{5}$$

Now let $f \in H^0(Y_1(N), \underline{\omega}^{\otimes k})$. Let $y \in Y_1(N)$ be a supersingular point and let U be a neighborhood of y over which $\underline{\omega}$ is trivial and generated by ω. Let $f = f_0\omega^{\otimes k}$ be the local expression of f in this base. The above local calculation applies over the open set $U - \{y\}$ and shows that $A\cdot\tilde{\theta}(f)$ extends to all of U. Thus $\theta(f)$ actually extends to all of $Y_1(N)$. ■

In the proof of the next theorem we will need a slightly more precise version of the above local expression for θ. In particular, we need to determine B.

Lemma 5.1.1. — *In the setting of the local calculation above, we have $B = -D(A)$.*

Proof. Recall that A and B are defined by

$$F^*((\nabla_D\omega)^{(p)}) = B\omega + A\nabla_D\omega.$$

By the local characterization of the Kodaira-Spencer isomorphism in terms of the Gauss-Manin connection and the definition of D, we have

$$B = \langle F^*((\nabla_D\omega)^{(p)}), \nabla_D\omega\rangle_{\mathrm{dR}}.$$

The compatibility of the Gauss-Manin connection with the cup product implies that

$$\langle F^*((\nabla_D\omega)^{(p)}), \nabla_D\omega\rangle_{\mathrm{dR}} + \langle\nabla_D(F^*((\nabla_D\omega)^{(p)})), \omega\rangle_{\mathrm{dR}} = D(\langle F^*((\nabla_D\omega)^{(p)}), \omega\rangle_{\mathrm{dR}}).$$

Since the first term on the left is B and the term of the right is $-D(A)$, it suffices to prove that the second term on the left vanishes. By the compatibility of the Gauss-Manin connection with the relative Frobenius we have

$$\nabla_D(F^*((\nabla_D\omega)^{(p)})) = (1\otimes F^*)\nabla_D^{(p)}((\nabla_D\omega)^{(p)})$$

where $\nabla^{(p)}$ denotes the Gauss-Manin connection on $\mathbb{H}^1_{\mathrm{dR}}(E^{(p)}_{\mathrm{univ}}|_U/U)$. The result will now follow from the following claim.

Claim 5.1.1. — *Let $X \longrightarrow Y$ be a smooth map of smooth $\mathbb{F}_p$-schemes. Then for any section h of $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$ (over any open set) we have $\nabla_{X^{(p)}/Y}(h^{(p)}) = 0$.*

Proof. By the compatibility of the Gauss-Manin connection with base change on Y, there is a commutative diagram

$$\begin{array}{ccc} F^*_{\mathrm{abs}}\mathbb{H}^i_{\mathrm{dR}}(X/Y) & \longrightarrow & \mathbb{H}^i_{\mathrm{dR}}(X^{(p)}/Y) \\ \Big\downarrow{\scriptstyle F^*_{\mathrm{abs}}\nabla_{X/Y}} & & \Big\downarrow{\scriptstyle \nabla_{X^{(p)}/Y}} \\ \Omega^1_{Y/\mathbb{F}_p} \otimes_{\mathcal{O}_Y} F^*_{\mathrm{abs}}\mathbb{H}^i_{\mathrm{dR}}(X/Y) & \longrightarrow & \Omega^1_{Y/\mathbb{F}_p} \otimes_{\mathcal{O}_Y} \mathbb{H}^i_{\mathrm{dR}}(X^{(p)}/Y) \end{array}$$

where the horizontal arrows are the maps arising from cohomology and base-change and $F^*_{\mathrm{abs}}\nabla_{X/Y}$ is the connection on $F^*_{\mathrm{abs}}\mathbb{H}^i_{\mathrm{dR}}(X/Y)$ induced from the Gauss-Manin connection on $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$. Explicitly, $F^*_{\mathrm{abs}}\nabla_{X/Y}$ is given locally on Y by the formula

$$F^*_{\mathrm{abs}}\nabla_{X/Y}(f \otimes h) = df \otimes h + f \otimes \nabla_{X/Y}h$$

where f is a local section of $\mathcal{O}_Y$, h is a local section of $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$, and the tensor product is $\mathcal{O}_Y \otimes_{\mathcal{O}_Y}$ taken with respect to F_{abs}. Notice that the second term on the right side as written lies in

$$\mathcal{O}_Y \otimes_{\mathcal{O}_Y} (\Omega^1_{Y/\mathbb{F}_p} \otimes \mathbb{H}^i_{\mathrm{dR}}(X/Y)).$$

We understand that we mean its image in $\Omega^1_{Y/\mathbb{F}_p} \otimes \mathbb{H}^i_{\mathrm{dR}}(X/Y)$ using the canonical map

$$\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \Omega^1_{Y/\mathbb{F}_p} \longrightarrow \Omega^1_{Y/\mathbb{F}_p}$$

coming from the extension of scalars F_{abs}. But this map is in fact *zero* by the argument in Section 3.

Let h be a local section of $\mathbb{H}^i_{\mathrm{dR}}(X/Y)$. Then $h^{(p)}$ is by definition the image of $1 \otimes h$ under the (top) base-changing arrow in the above commutative square. By the commutativity of the diagram, $\nabla_{X^{(p)}/Y}(h^{(p)})$ is the image under the (bottom) base-changing arrow of

$$F^*_{\mathrm{abs}}\nabla_{X/Y}(1 \otimes h) = d(1) \otimes h = 0.$$

This proves the Claim and the Lemma. ■

Next we wish to determine the effect of θ on q-expansions at cusps. Since we do not wish to introduce the compactified moduli schemes $X_1(N)$ (indeed, the generalized $\Gamma_1(N)$ problem isn't even representable if $N = 4$) we take an "indirect" approach to the issue of q-expansions at cusps. For any point of order N on the Tate curve $\underline{\mathrm{Tate}}(q^N)$ over $K((q))$ for some field K of characteristic p there is associated a unique classifying map

$$g : \mathrm{Spec}(K((q))) \longrightarrow Y_1(N)$$

by the definition of $Y_1(N)$ as a moduli space for elliptic curves with a point of order N. Given a modular form $f \in H^0(Y_1(N), \underline{\omega}^{\otimes k})$ we may pull it back via g map to arrive at a section of $\underline{\omega}^{\otimes k}$ on $\mathrm{Spec}(K((q)))$ (by which we mean the push forward of $\Omega^1_{\underline{\mathrm{Tate}}(q^N)/K((q))}$). We may write $g^*(f) = f(q)\omega_{\mathrm{can}}^{\otimes k}$ for an unique $f(q) \in K((q))$ which we call the q-expansion of f associated to the point of order N on $\underline{\mathrm{Tate}}(q^N)$ with which we began. This is the q-expansion whose behavior with respect to θ we wish to investigate.

The Gauss-Manin connection and the Kodaira-Spencer isomorphism (which is characterized locally in terms of the Gauss-Manin connection) may be pulled back through the classifying map g to arrive at a map

$$\widehat{\nabla} : \mathbb{H}^i_{\mathrm{dR}}(\underline{\mathrm{Tate}}(q^N)/K((q))) \longrightarrow \widehat{\Omega}^1_{K((q))/K} \otimes_{K((q))} \mathbb{H}^i_{\mathrm{dR}}(\underline{\mathrm{Tate}}(q^N)/K((q)))$$

and an isomorphism

$$\widehat{\underline{\mathrm{KS}}} : \underline{\omega}^{\otimes 2} \xrightarrow{\sim} \widehat{\Omega}^1_{K((q))}$$

with the property that if ω is a basis of the free module $\underline{\omega}$, then

$$\widehat{\underline{\mathrm{KS}}}(\omega^{\otimes 2}) = \langle \omega, \widehat{\nabla}\omega \rangle_{\mathrm{dR}}.$$

By reduction from characteristic zero and analytifying over $\mathbb{C}$, one can show by direct computation that

$$\widehat{\underline{\mathrm{KS}}}(\omega_{\mathrm{can}}^{\otimes 2}) = \frac{dq}{q}. \tag{6}$$

Theorem 5.2. — *The map θ has the effect*

$$q\frac{d}{dq} : \sum a_n q^n \longmapsto \sum n a_n q^n$$

on q-expansions at all cusps.

Proof. Let $f \in H^0(Y_1(N), \underline{\omega}^{\otimes k})$. Let

$$g : \mathrm{Spec}(K((q))) \longrightarrow Y_1(N)$$

be the classifying map corresponding to some point of order N on the Tate curve $\underline{\mathrm{Tate}}(q^N)$ over $K((q))$ for a field K of characteristic p, and let $g^*(f) = f(q)\omega_{\mathrm{can}}^{\otimes k}$. Let $U \subseteq Y_1(N)$ be a Zariski open set on which $\underline{\omega}$ is trivial. Let ω be a generator of $\underline{\omega}$ over U and let D be the derivation dual to the image of ω under the Kodaira-Spencer isomorphism. Let $f = f_0\omega^{\otimes k}$ be the local expression for f over U in terms of the basis ω. The local calculation done in the proof of Theorem 5.1 and Lemma 5.1.1 show that over U we have

$$\theta(f) = (AD(f_0) + kf_0D(A))\omega^{\otimes(k+p+1)}$$

Let α be the unique element of $K((q))$ such that $g^*(\omega) = \alpha\omega_{\mathrm{can}}$. It follows from (6) that $g^*(D) = \alpha^{-2}qd/dq$. Since $g^*(f) = f(q)\omega_{\mathrm{can}}^{\otimes k}$, we have $g^*(f_0) = \alpha^{-k}f(q)$. Note also that

$$g^*(A) = A(\underline{\mathrm{Tate}}(q^N), \alpha\omega_{\mathrm{can}}) = \alpha^{1-p}A(\underline{\mathrm{Tate}}(q^N), \omega_{\mathrm{can}}) = \alpha^{1-p}.$$

Thus, pulling the above formula for $\theta(f)$ back through g, we arrive at

$$\begin{aligned}
&\left(\alpha^{1-p}\left(\alpha^{-2}q\frac{d}{dq}\right)(\alpha^{-k}f(q)) + k\alpha^{-k}f(q)\alpha^{-2}q\frac{d}{dq}\left(\alpha^{1-p}\right)\right)\alpha^{k+p+1}\omega_{\mathrm{can}}^{\otimes(k+p+1)} \\
&= \left(-k\alpha^{-k-p-2}q\frac{d\alpha}{dq}f(q) + \alpha^{-k-p-1}q\frac{d}{dq}(f(q)) + k\alpha^{-k-p-2}f(q)q\frac{d\alpha}{dq}\right)\alpha^{k+p+1}\omega_{\mathrm{can}}^{\otimes(k+p+1)} \\
&= q\frac{d}{dq}(f(q))\omega_{\mathrm{can}}^{\otimes(k+p+1)}
\end{aligned}$$

which says exactly that θ has the effect we have claimed on q-expansions. ■

References

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Nick Ramsey • *E-mail :* naramsey@umich.edu