Classification of quasi-finite étale separated schemes

As we saw in lecture, Zariski's Main Theorem provides a very visual picture of quasi-finite étale separated schemes X over a henselian local ring R. We would like to explain how this can be used to give a similar result over any Dedekind base scheme S. Recall that a *Dedekind scheme* is a regular noetherian scheme whose connected components are 1-dimensional; for purely technical reasons related to descent theory we do not require S to be connected. We let K denote the ring of rational functions (*i.e.*, the product of the function fields at the finitely many generic points). We also let $K_{\rm sep}$ denote the product of separable closures of each of the factor fields of K.

The basic principle is to infer results by descent from the henselization and strict henselization. Although we apply these results for $S = \operatorname{Spec} \mathbf{Z}$, in some steps it will be technically convenient to allow ourselves to replace S with an auxiliary étale (or ind-étale) cover. Hence, it is better to work throughout with an arbitrary Dedekind base S rather than a specific one. Once we have obtained a classification for general X with no extra structure, we will see what happens for commutative S-groups; it is easier to first focus on what happens without the interference of group laws, just as we saw in our earlier study of Zariski's Main Theorem (where the picture was made clearer by initially ignoring group structures).

It should be noted at the outset that what we do in this handout is much more readily understood by thinking in terms of the language of constructible étale sheaves of sets. Thus, the reader should interpret everything that we do as simply a means of making concrete what is more naturally carried out in a more abstract setting. The mathematical input is essentially the same from both points of view, and the main ideas are not fundamentally different either, but it seems that in the language of étale sheaves of sets one gains a certain degree of elegance that is sometimes missing when working throughout with geometric objects as we do here.

1. Some preliminaries

Let X be quasi-finite étale and separated over a Dedekind scheme S, so X_K is a finite étale K-scheme. By the "smearing out" principle, there is a dense open $U \subseteq S$ over which the restriction X_U is *finite.* Now the finite étale map $X_U \to U$ has target U that is normal, so X_U is normal. It follows that X_U must be the normalization of U in the finite étale cover X_K of $\operatorname{Spec} K$. Moreover, if we fix a choice of $K_{\rm sep}$ then we *abuse notation* and write $X(K_{\rm sep})$ to denote $\coprod X(K_{i,{\rm sep}})$ with $K = \prod K_i$ a product of fields (and we like write $\operatorname{Gal}(K_{\rm sep}/K)$ to denote the product of the $\operatorname{Gal}(K_{i,{\rm sep}}/K_i)$'s, with the ith factor only acting on the geometric points of X_{K_i}). This set is unramified at each place of K associated to a closed point $u \in U$ since the finite étale $X_U \to U$ must become a split covering over a strict henselization of U at u (as the only finite étale algebras over $\mathscr{O}_{U,u}^{\rm sh}$ are finite products of copies of $\mathscr{O}_{U,u}^{\rm sh}$).

Conversely, if $Y_K \to \operatorname{Spec} K$ is finite étale and $Y_K(K_{\rm sep})$ is unramified at all places of u, then the normalization Y of U in Y_K is a finite étale U-scheme with K-fiber Y_K. Indeed, this normalization is U-finite since forming integral closure of a normal noetherian ring in a finite étale extension of its total ring of fractions always yields a finite ring extension, and it clearly has K-fiber Y_K. Moreover, this U-finite normalization is U-flat since U is Dedekind (flatness over a Dedekind domain is the same as torsion-freeness). It follows that to check whether or not Y is U-étale it suffices to check that the fibers Y_u over closed points $u \in U$ are étale $k(u)$-schemes. But this is immediate from the hypothesis that $Y(K_{\rm sep}) = Y_K(K_{\rm sep})$ is unramified at u.

We may summarize these conclusions as follows:

Lemma 1.1. *Let S be a Dedekind scheme with ring of rational functions K, and fix a choice of $K_{\rm sep}/K$. Let $U \subseteq S$ be a dense open subscheme.*

The functor $Y \rightsquigarrow Y_K(K_{\rm sep})$ sets up an equivalence of categories between the category of finite étale U-schemes and the category of finite discrete $\operatorname{Gal}(K_{\rm sep}/K)$*-sets that are unramified at all closed points of U (or, as we shall say, are* unramified along U*). The quasi-inverse functor is given by normalization of U in the associated finite étale K-algebra.*

This equivalence of categories respects formation of products, and so it carries group objects to group objects and commutative group objects to commutative group objects.

Remark 1.2. Since $K = \prod K_i$ is a finite product of fields, by a $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-*set* we really mean a finite disjoint union of discrete sets indexed by the i's such that the ith set is given a continuous action by $\mathrm{Gal}(K_{i,\mathrm{sep}}/K_i)$.

The purpose of this handout is to explain how to generalize the lemma to work over the entire base S, and not only over U.

We shall now *fix* U, and we want to describe those quasi-finite étale separated S-schemes $X \to S$ that are finite over U. The idea is to view X as obtained from X_U with some auiliary data to specify fibers over the finite set $S - U$, and since X_U is "the same" as the Galois set $X(K_{\mathrm{sep}})$ that is unramified over U, our real goal is to give a purely Galois-theoretic description of X in terms of the Galois-set $X(K_{\mathrm{sep}})$ and some auxiliary data.

Let us now *fix* a quasi-finite étale separated map $X \to S$ such that X_U is U-finite. Also, for each $s \in S - U$ we fix a choice of separable closure $k(s)_{\mathrm{sep}}$ of $k(s)$ and let $\mathscr{O}_{S,s}^{\mathrm{sh}}$ denote the associated strict henselization of the discrete valuation ring $\mathscr{O}_{S,s}$.

We also fix an embedding of $\mathscr{O}_{S,s}^{\mathrm{sh}}$ into K_{sep}, or what comes to the same thing, we choose a place $\overline{s}$ of K_{sep} over the place of s on K and we take $\mathscr{O}_{S,s}^{\mathrm{sh}}$ to be the union of the local rings under $\overline{s}$ on finite subextensions $K' \subseteq K_{\mathrm{sep}}$ over K such that these local rings are unramified over $\mathscr{O}_{S,s}$ (in the sense of valuation theory). Let us define the *inertia group*

$$I_s = I_{\overline{s}/s} = \mathrm{Gal}(K_{\mathrm{sep}}/K_s^{\mathrm{sh}}),$$

where K_s^{sh} denotes the fraction field of $\mathscr{O}_{S,s}^{\mathrm{sh}}$.

Lemma 1.3. *Assume S is connected. For each $s \in S - U$, there is a canonical injection $X(k(s)_{\mathrm{sep}}) \hookrightarrow X(K_{\mathrm{sep}})^{I_s}$, and this is a bijection with $\#X(k(s)_{\mathrm{sep}}) = \mathrm{rank}(X_K)$ if and only if $X \to S$ is finite over an open neighborhood of s in S.*

Proof. The property of finiteness over an open neighborhood of s is equivalent (by the "smearing out" principle) to finiteness of $X_{/\mathscr{O}_{S,s}} \to \mathrm{Spec}\,\mathscr{O}_{S,s}$. Such finiteness in turn may be checked after any *fpqc* base change, and so since $\mathscr{O}_{S,s}^{\mathrm{sh}}$ is faithfully flat over $\mathscr{O}_{S,s}$ and the desired conclusions only depend on the base change of X over $\mathscr{O}_{S,s}^{\mathrm{sh}}$. We may therefore assume that S is local and strictly henselian. Note that by passing to this strictly henselian local case, K becomes a field and the inertia group I_s becomes the Galois group for K_{sep} over K.

By the structure theorem for quasi-finite separated schemes over a henselian local base, we have

$$X = X_{\mathrm{f}} \coprod X_\eta$$

with X_{f} finite over S and X_η having empty closed fiber. Since X is étale over S, we see that X_{f} is finite étale over S. However, S is local and strictly henselian, so X_{f} is a finite disjoint union of copies of S. Thus, there is a natural bijection

$$X_{\mathrm{f}}(K) \to X_{\mathrm{f}}(k) = X(k)$$

defined by reduction, and so its inverse defines an injection

$$X(k) \hookrightarrow X(K) = X(K_{\mathrm{sep}})^I \tag{1}$$

that is bijective if and only if $X(K) = X_{\mathrm{f}}(K)$.

Hence, this injection is bijective with $X(k)$ of size $\mathrm{rank}(X_K)$ if and only if $X_{\mathrm{f}}(K)$ has size equal to the K-rank of X_K. But this equality happens if and only if X_K is K-split and equal to the K-fiber of X_{f}, which is to say that X_K is K-split and X_η is empty. In other words, the injection (1) is bijective with $X(k)$ of size $\mathrm{rank}(X_K)$ if and only if $X = X_{\mathrm{f}}$, which is to say that X is S-finite.

The construction (1) is visibly functorial in X. ∎

Now observe (still assuming S to be connected) that the injection of sets

$$X(k(s)_{\mathrm{sep}}) \hookrightarrow X(K_{\mathrm{sep}})^{I_s}$$

is compatible with the natural action of $\mathrm{Gal}(k(s)_{\mathrm{sep}}/k(s))$ on both sides. Indeed, this is an immediate consequence of the construction.

We have seen in Lemma 1.1 that the category of finite étale U-schemes may be identified with the category of finite discrete $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-sets that are unramified along U. By Lemma 1.3, we thereby obtain a functor from the category $(S,U)_{\text{ét}}$ of quasi-finite étale separated S-schemes to the category $C_{S,U}$ whose objects are tuples

$$(\Sigma, \{F_s\}_{s\in S-U})$$

where Σ is a finite discrete $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-set unramified along U and F_s is a finite $\mathrm{Gal}(k(s)_{\mathrm{sep}}/k(s))$-subset of $\Sigma_i^{I_s}$, with Σ_i the part of Σ on which $\mathrm{Gal}(K_{i,\mathrm{sep}}/K_i)$ acts (with $K = \prod K_i$ the decomposition into a finite product of fields); morphisms in $C_{S,U}$ are defined in the evident manner. Strictly speaking, $C_{S,U}$ depends on the choices of K_{sep} and $k(s)_{\mathrm{sep}}$ for all $s \in S-U$, but these dependencies are unimportant for our purposes (and could be avoided if we had developed the language of étale sheaves of sets on S).

By construction, the functor

$$F : X \rightsquigarrow (X(K_{\mathrm{sep}}), \{X(k(s)_{\mathrm{sep}})\}_{s\in S-U})$$

from $(S,U)_{\text{ét}}$ to $C_{S,U}$ is obviously faithful and respects formation of fiber products (ultimately because the formation of the "finite part" over a henselian local base is compatible with formation of fiber products).

The main purpose of this handout is to prove:

Theorem 1.4. *The above functor $F : (S,U)_{\text{ét}} \to C_{S,U}$ is an equivalence of categories.*

The equivalence in this theorem carries over to group objects, as well as commutative group objects, in the two categories. One immediate consequence of this theorem and Lemma 1.3 is:

Corollary 1.5. *Let X_U be a finite étale U-group. There exist quasi-finite étale separated S-groups $X^\flat$ and $X^\sharp$ with U-fiber X_U such that every quasi-finite étale separated S-group X with U-fiber X_U contains $X^\flat$ as an open S-subgroup and is contained in $X^\sharp$ as an open S-subgroup. That is, there are maximal and minimal quasi-finite étale separated S-models for X_U.*

The formation of $X^\flat$ and $X^\sharp$ is functorial in X and compatible with fiber products and with ind-étale base change on S. Moreover, if S is connected then $X^\sharp$ is finite over S if and only if $X(K_{\mathrm{sep}})$ is unramified at all $s \in S$.

To prove this corollary, we make the maximal and minimal models by choosing $X^\flat(k(s)_{\mathrm{sep}})$ and $X^\sharp(k(s)_{\mathrm{sep}})$ to respectively be the trivial subgroup and entirety of the finite group $X(K_{\mathrm{sep}})^{I_s}$. The asserted functorial properties follow immediately from this construction. Proposition 1.3 in Ch I of Mazur's IHES paper is a special case of Corollary 1.5.

2. Proof of Theorem 1.4: full faithfulness

Let us first prove the faithfulness of the functor in Theorem 1.4. This is quite easy, since we claim more generally that if $f, g : X \rightrightarrows X'$ are S-maps between S-schemes with X' separated over S and X flat over S, then $f = g$ if $f_K = g_K$. The map $h = f \times g : X \to X' \times_S X'$ that factors through the diagonal on K-fibers, and we want it to factor through the diagonal over S. Since X' is separated, the diagonal

$$\Delta_{X'/S} : X' \to X' \times_S X'$$

is a closed immersion, and its pullback along h is therefore a closed subscheme $i : Z \hookrightarrow X$ that we want to coincide with X. By the hypothesis on K-fibers, $Z_K \hookrightarrow X_K$ is an isomorphism. Hence, we are reduced to checking that a quasi-coherent ideal sheaf on X that vanishes on the K-fiber must globally vanish. This is a local problem, and so reduces to the claim that if A is a flat algebra over a domain R (with fraction field K) and I is an ideal with $I_K = 0$ in $A_K = K \otimes_R A$ then $I = 0$. Since R-flatness of A implies that $A \to A_K$ is injective, we get the desired result.

With faithfulness of Theorem 1.4 settled, we next wish to prove full faithfulness. To this end, let X and X' be quasi-finite separated étale S-schemes that are finite over U, and let $\phi : F(X) \to F(X')$ be a morphism in $C_{S,U}$. We want f to be induced by a (necessarily unique) S-map $f : X \to X'$. The underlying map of Galois sets $X(K_{\mathrm{sep}}) \to X'(K_{\mathrm{sep}})$ arising from ϕ already provides us with a map $f_U : X_U \to X'_U$ of finite étale U-schemes, and the map we seek to construct *must* coincide with f_U over U. Hence, it is necessary and sufficient to prove that f_U extends to an S-map $X \to X'$.

Let us temporarily assume such an $f : X \to X'$ exists, and we will see how to create it from f_U. The task will then be to prove that this creation process yields f without knowing f to exist *a priori*. We will use the important "closure of a graph" technique; this goes as follows. Consider the graph

$$\Gamma_f : X \to X \times_S X'$$

of the hypothetical morphism f. Being the graph of a map to a separated S-scheme, it is a closed immersion whose composite with the projection $X \times_S X' \to X$ is an isomorphism. Moreover, over U this closed subscheme is exactly the graph of f_U. Since X_U is dense open in X, we conclude that the closed subscheme defined by the graph of the hypothetical f has no choice but to be the schematic closure in $X \times_S X'$ of the closed subscheme

$$\Gamma_{f_U} : X_U \hookrightarrow X_U \times_S X'_U = (X \times_S X')_U.$$

Now we turn the situation around and, starting with the given closed subscheme in $(X \times_S X')_U$ provided by the graph Γ_{f_U}, we form its schematic closure Γ in $X \times_S X'$ (recall that schematic-image makes sense for any quasi-compact and quasi-separated map). This meets the open set $(X \times_S X')_U$ in Γ_{f_U}, and we can consider the composite map

$$\Gamma \hookrightarrow X \times_S X' \to X$$

over S. Over U this is an isomorphism, and the preceding considerations show that it is not only necessary that this be an isomorphism, but also that if $\Gamma \to X$ is an isomorphism then the composite of its inverse with

$$\Gamma \hookrightarrow X \times_S X' \to X'$$

defines an S-map $f : X \to X'$ that restricts to f_U over U and hence solves the full-faithfulness problem.

To summarize, it is necessary and sufficient to show that the closure $\Gamma \hookrightarrow X \times_S X'$ of the graph of f_U projects isomorphically onto X. The advantage of this global construction of the desired graph is that it is clearly compatible with flat base change on S (recall that the formation of scheme-theoretic image commutes with flat base change). Hence, not only is it sufficient to check the isomorphism condition for $\Gamma \to X$ by working locally on S, but if we apply a flat base change $S' \to S$ to this map then it is identified with the map constructed from $f_{U'}$ (where $U' = U \times_S S'$). Since the situation is trivial over U (where we have f_U), the problem is local at each of the finitely many points $s \in S - U$. Hence, we first may replace S with $\operatorname{Spec} \mathscr{O}_{S,s}$ (so S is local with closed point s, and U is the generic point), and then it suffices to check the isomorphism property after applying the faithfully flat base change to the strict henselization at s. That is, we are reduced to the case when S is local and strictly henselian.

In the case of a local and strictly henselian base, we have decompositions

$$X = X_{\mathrm{f}} \coprod X_\eta, \;\; X' = X'_{\mathrm{f}} \coprod X'_\eta,$$

and the given map ϕ tells us that $f_K : X_K \to X'_K$ carries the subset $X(k) = X_{\mathrm{f}}(K) \subseteq X(K)$ into the subset $X'(k) = X'_{\mathrm{f}}(K) \subseteq X'(K)$. Since each of X_{f} and X'_{f} is a disjoint union of copies of S, by considering K-points of these copies of S we see that f_K carries $X_{\mathrm{f}/K}$ into $X'_{\mathrm{f}/K}$. But X_{f} and X'_{f} are finite étale (even split) over S, so such a map of their K-fibers uniquely extends to a map $X_{\mathrm{f}} \to X'_{\mathrm{f}}$ over S.

We have now constructed an S-map $X_{\mathrm{f}} \to X'_{\mathrm{f}} \to X'$ and f_K provides an S-map $X_\eta \subseteq X_K \to X'_K$, so these trivially glue to give an S-map

$$f : X = X_{\mathrm{f}} \coprod X_\eta \to X'$$

whose K-fiber is our original f_K. This concludes the proof of full faithfulness in Theorem 1.4.

3. Proof of Theorem 1.4: essential surjectivity

There remains the task of proving that every object in the category $C_{S,U}$ arises from some quasi-finite étale separated S-scheme X. Put another way, by Lemma 1.1, what we have to prove is that if X_U is a finite étale U-scheme and for each $s \in S - U$ we are given a subset $\Sigma_s \subseteq X_U(K_{\mathrm{sep}})^{I_s}$ that is stable under the action of $\mathrm{Gal}(k(s)_{\mathrm{sep}}/k(s))$, then there exists a quasi-finite étale separated S-scheme X restricting to X_U over U such that for all $s \in S - U$ the natural injection

$$X(k(s)_{\mathrm{sep}}) \hookrightarrow X(K_{\mathrm{sep}})^{I_s} = X_U(K_{\mathrm{sep}})^{I_s}$$

is a bijection onto the subset Σ_s.

Let us first show that the problem is local at each $s \in S$. Certainly the problem is Zariski local, so it suffices to replace S with the complement of all but one point of $S-U$ and to solve it for this point. That is, we may suppose $S-U$ consists of a single point s. If we have a solution $X(s)$ over $\operatorname{Spec} \mathscr{O}_{S,s}$, then it "smears out" to a quasi-finite étale and separated scheme $\widetilde{X(s)}$ over an open neighborhood V of s in S such that its K-fiber is $(X_U)_K$. Since $\widetilde{X(s)}_{V-\{s\}} \to V - \{s\}$ therefore becomes finite upon replacing V with $\operatorname{Spec} \mathscr{O}_{S,s}$, it becomes finite upon replacing V with some Zariski-open neighborhood of s. Thus, after shrinking V we may suppose that $\widetilde{X(s)}$ is finite étale over V; since its K-fiber is $(X_U)_K$, we conclude by Lemma 1.1 that there is an isomorphism $\widetilde{X(s)}|_{U\cap V} \simeq X_U|_{U\cap V}$ respecting K-fiber identifications. Hence, gluing $\widetilde{X}(s)|_V$ and X_U over $U \cap V$ gives a global solution to our problem.

Having reduced to the case of a local base S, we see more generally that the problem over any semilocal base is reduced to the local case. Let us first consider the problem over a strictly henselian local base. In this case $X_K(K_{\mathrm{sep}})^{I_s} = X_K(K)$, and upon specifying a subset $\Sigma_s \subseteq X_K(K)$ we see that a disjoint union of copies of S indexed by Σ_s and the K-scheme complement in X_K of the K-point components corresponding to Σ_s gives a solution to our problem. Hence, the problem is solved over the strict henselization of our local base S. Let $\widetilde{X} \to S^{\mathrm{sh}}$ be the solution to our problem over the strict henselization; by expressing $\mathscr{O}_{S,s}^{\mathrm{sh}}$ as a direct limit of local-étale extensions of $\mathscr{O}_{S,s}$ we can find a local-étale cover $S' \to S$ dominated by S^{sh} and a quasi-finite separated étale S'-scheme X' whose S^{sh}-fiber solves the problem over the strict henselization. However, the problem to be solved is expressed in terms $S^{\mathrm{sh}} = {S'}^{\mathrm{sh}}$, and so we see that X' is automatically a solution to our problem posed over the local base S'.

The problem is now one of étale descent. Consider the two pullbacks $p_j^*(X')$ over $S' \times_S S'$. Since $S' \to S$ is local-étale, it is surjective and the base $S' \times_S S'$ is Dedekind (but nearly always disconnected!). The formation of the functor $F : (S, U)_{\text{ét}} \to C_{S,U}$ is compatible with ind-étale base change on S (given compatible extensions on the separable closures of function fields and residue fields at closed points away from U), so $p_1^*(X')$ and $p_2^*(X')$ are respectively solutions to the essential surjectivity problems for the p_j-pullbacks of the object in $C_{S',U'}$ that we obtain by pullback of the initially given object in $C_{S,U}$.

Since the composite maps

$$S' \times_S S' \rightrightarrows S' \to S$$

are *equal*, we conclude that $p_1^*(X')$ and $p_2^*(X')$ are both solutions to the *same* problem whose solution is unique up to unique isomorphism. Hence, we obtain a (unique) isomorphism $\varphi : p_1^*(X') \simeq p_2^*(X')$ over $S'' = S' \times_S S'$ that extends the evident identification (with $U'' \times_U X_U$) over $U'' = U' \times_U S'$. Under pullback to the triple overlap $S''' = S' \times_S S' \times_S S'$, the isomorphisms $p_{12}^*(\varphi)$, $p_{13}^*(\varphi)$ and $p_{23}^*(\varphi)$ are compatible in the evident manner due to the uniqueness of isomorphism among solutions to the essential surjectivity problem (now over the Dedekind base S'''). Thus, we have descent data, and étale descent for quasi-finite separated maps is always effective. That is, we obtain an S-scheme X equipped with an isomorphism $X_{/S'} \simeq X'$ over S' that respects the descent data on both sides (relative to $S' \to S$). In particular, since $X' \to S'$ is quasi-finite étale and separated, the same holds for $X \to S$.

It remains to check that $X \to S$ is a solution to our problem. The only serious issue is to check that the K-fiber of X is identified with $(X_U)_K$, or more specifically that the U-fiber of X is identified with the initially-given X_U. However, this is clear because if we restrict the isomorphism $X_{/S'} \simeq X'$ over U' then we get an isomorphism $X_{/U'} \simeq (X_U)_{/U'}$ that respects the evident descent data on both sides, and so this descends uniquely to a U-isomorphism $X_{/U} \simeq X_U$ by descent theory for morphisms. This completes the proof of essential surjectivity.

4. Group quotients

We conclude by applying the preceding considerations to the case of commutative groups.

Let G be a commutative S-group that is quasi-finite étale and separated, and let H be a closed S-subgroup that is also quasi-finite étale and separated. We would like to establish the existence of a quotient G/H that is quasi-finite étale and separated. To be precise, we seek an S-group with such properties and an

S-group map $G \to G/H$ that is an étale surjection with scheme-theoretic kernel H. Let us first establish the uniqueness of such an object, and then we will address its existence.

Lemma 4.1. *If G' and G'' are quasi-finite étale separated S-groups and $G \to G'$ and $G \to G''$ are two étale surjective S-group maps with scheme-theoretic kernel H, then there exists a unique S-group isomorphism $G' \simeq G''$ that is compatible with the maps from G to each side.*

Proof. There is a dense open $U \subseteq S$ over which all of G, G', G'', and H are finite, and in such cases the quotient G_U/H_U exists as a finite étale U-group equipped with an étale surjection from G_U having scheme-theoretic kernel H_U. Moreover, the theory of quotients for finite flat commutative group schemes ensures that this latter property uniquely characterizes the quotient up to unique isomorphism, and so we see that the desired unique isomorphism between G' and G'' exists over U. The problem is therefore one of extending this G_U-compatible U-group isomorphism $G'_U \simeq G''_U$ over U to a G-compatible S-group isomorphism $G' \simeq G''$.

By using the same "closure of a graph" technique that we employed in §2 (keeping in mind that all schemes in question are flat and separated over the base), it suffices to solve the problem over strict henselizations of the local rings on S at the finitely many points of $S - U$. This reduces us to the case of a local and strictly henselian base S.

Consider the finite-part decompositions

$$G = G_{\mathrm{f}} \coprod G_\eta, \quad H = H_{\mathrm{f}} \coprod H_\eta$$

as S-schemes; recall that the finite parts are (open) S-subgroups. Since G and H are S-étale, we see that G_{f} and H_{f} are S-étale. Functoriality of "finite part" ensures that H_{f} is contained in the open S-subgroup $H \cap G_{\mathrm{f}}$ of H. In fact, we have $H_{\mathrm{f}} = H \cap G_{\mathrm{f}}$ (scheme-theoretic intersection) because $H \cap G_{\mathrm{f}}$ is a closed subscheme of G_{f} (and hence is S-finite, so as a subscheme of H it lies in H_{f}). It follows that inside of $G(K) = G(K_{\mathrm{sep}})^I$ we have

$$H(k) = G(k) \cap H(K) = G(k) \cap H(K_{\mathrm{sep}})^I.$$

Hence, we obtain a natural injection

$$G(k)/H(k) \hookrightarrow G(K)/H(K) \hookrightarrow (G_K/H_K)(K) = (G_K/H_K)(K_{\mathrm{sep}})^I.$$

Now let G/H be the unique quasi-finite étale separated S-group whose K-fiber is G_K/H_K and whose k-fiber corresponds to the subgroup $G(k)/H(k)$ inside of $(G_K/H_K)(K)$. By functoriality of the equivalence in Theorem 1.4, there is a unique S-group map $G \to G/H$ inducing $G_K \to G_K/H_K$ on K-fibers and inducing $G(k) \to G(k)/H(k)$ on k-fibers. In particular, the map $G \to G/H$ between étale S-groups is étale (as is any map between étale S-schemes) and is surjective (as we can see on K-fibers and k-fibers respectively).

It remains to show that the scheme-theoretic kernel of $G \to G/H$ is H. Since the map $G \to G/H$ is étale and G/H is S-separated, the scheme-theoretic kernel H' is an S-étale closed S-subgroup of G and we want $H = H'$ inside of G. Any S-flat closed subscheme of G is the schematic closure of its K-fiber in G_K, so it suffices to show $H_K = H'_K$ inside of G_K, or equivalently that H_K is the kernel of the natural map $G_K \to G_K/H_K$. This follows immediately from the theory of finite flat group quotients (over a field). ■

Now that uniqueness for G/H has been proved, we turn to existence. If we let $U \subseteq S$ be a dense open over which G_U and H_U are finite, the existence of G_U/H_U (with the desired properties) follows from the theory of quotients by finite étale equivalence relations. In view of the uniqueness up to unique isomorphism for the solution to our problem, and in particular its automatic compatibility with respect to étale base change, it suffices to solve the problem étale-locally on S.

If we can solve the problem over a local base ring, then a variation on the "smearing out" argument in §3 allows us to glue the local solutions at each $s \in S - U$ to the solution over U to make a global solution to our quotient problem. Thus, we may assume that the base S is local. If we can solve the problem over a strictly henselian local base then we can chase direct limits to descend this to a solution over some local-étale extension of S, and such a solution then descends to S via descent theory (just as in our earlier considerations with descent theory). Thus, it suffices to treat the case when S is a strictly henselian local base. The existence in this case was settled in the proof of Lemma 4.1.

If we combine the above conclusions with Raynaud's method of scheme-theoretic closure over Dedekind bases, we get an important consequence that recovers §1(c) in Ch. I of Mazur's IHES paper as a special case:

Theorem 4.2. *Let S be Dedekind and let G be a quasi-finite separated and flat commutative S-group. Let $U \subseteq S$ be a dense open such that G_U is U-finite. Assume that G is étale over $S - U$.*

The functor $H \rightsquigarrow H(K_{\rm sep})$ is a bijection between S-flat closed subgroups of G and $\mathrm{Gal}(K_{\rm sep}/K)$-submodules of $G(K_{\rm sep})$, and this functor carries G/H to $G(K_{\rm sep})/H(K_{\rm sep})$.

Note that the only subtlety in this theorem is the consideration of quotients around points of $S - U$, but by hypothesis there is an open V around this locus such that G_V is étale (and so the V-flat closed subscheme H_V is necessarily étale too). It follows that the meaning of G/H over S is unambiguous, as we may work separately over V and over U.