ALTERNATIVE PROOF OF PROPOSITION 2.1 IN MAZUR'S "RATIONAL ISOGENIES OF PRIME DEGREE" PAPER

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We are going to take the same starting point as Ribet in [R1]:

Lemma 0.1. *Let* $\mathbf{T}_0$ *denote the subring of* $\mathrm{End}(J_0(N)_{\mathbb{Q}})$ *generated by the Hecke operators* T_ℓ *for* $\ell \nmid N$. *Then* $\mathbf{T}_0 \otimes \mathbb{Q}$ *lies in the center of* $\mathrm{End}(J_0(N)_{\mathbb{Q}}) \otimes \mathbb{Q}$.

Proof. See first paragraph of proof to Proposition 2.1 in [M] or the method in Silverman's book. For details see my talk. □

We want to show that the map $\mathbf{T}_0 \otimes \mathbb{Q} \to \mathrm{End}^0(J_0(N))$ is surjective (and hence an isomorphism) for prime N. Recall that by Shimura's construction we have $J_0(N)_{\mathbb{Q}} \sim \prod_{[f]} A_f$ with $K_f \hookrightarrow \mathrm{End}^0(A_f)$, where the product is over a set of $G_{\mathbb{Q}}$-representatives of newforms of weight 2 and prime level N, and the totally real K_f's are obtained by adjoining the Hecke eigenvalues of f to $\mathbb{Q}$. We claim that it suffices to show

$$K_f \twoheadrightarrow \mathrm{End}^0(A_f)\,(*).$$

This follows either from proving that $\mathrm{Hom}_{\mathbb{Q}}(A_f, A_g) = 0$ for non-Galois-conjugate f and g (which can be shown also by an argument using the Hom-version of Proposition 0.2 below and Theorem 0.3, see for example [R3] Thm 6.2), or directly from Lemma 0.1 if one uses the construction of A_f as subvarieties, as described in my talk. From the Hecke-equivariant isomorphism of the classical $S_2(\Gamma_0(N))$ with the cotangent space of $J_0(N)_{\mathbb{C}}$, and the corresponding identification of the cotangent space of $A_f \otimes \mathbb{C}$ with the $G_{\mathbb{Q}}$-orbit of the cuspform f, one gets $\mathbf{T}_0 \otimes \mathbb{Q} \simeq \prod_{[f]} K_f$. By the lemma now, any $\varphi \in \mathrm{End}^0(J_0(N))$ commutes with the projectors $\pi_f : \mathbf{T}_0 \otimes \mathbb{Q} \twoheadrightarrow K_f$, which were also used to define the A_f as $\pi_f(J_0(N))$. But this forces φ to be in "block-diagonal" form, i.e. $\varphi = \prod_{[f]} \varphi_f$, with $\varphi_f \in \mathrm{End}^0(A_f)$.

In fact we will show the surjectivity in (*) after extending scalars from $\mathbb{Q}$ to $\mathbb{Q}_\ell$. We also make use of the following

Proposition 0.2. *Let* A *be an abelian variety over* $\mathbb{Q}$, $T_\ell(A)$ *its* ℓ*-adic Tate module. We have an injection*

$$\mathbb{Z}_\ell \otimes \mathrm{End}_{\mathbb{Q}}(A) \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell[G_{\mathbb{Q}}]}(T_\ell(A)).$$

Proof. For a proof, see [M] Theorem 12.5. This is in fact an isomorphism (Tate's conjecture, proved for number fields by Faltings). □

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We obtain

$$\prod_{\lambda|\ell} K_{f,\lambda} \simeq \mathbb{Q}_\ell \otimes K_f \stackrel{\phi}{\hookrightarrow} \mathbb{Q}_\ell \otimes \mathrm{End}^0(A_f) \stackrel{\psi}{\hookrightarrow} \mathrm{End}_{\mathbb{Q}_\ell[G_\mathbb{Q}]}(V_\ell(A_f)),$$

where $V_\ell(A_f) = \mathbb{Q}_\ell \otimes T_\ell(A_f)$. From the lemma we deduce again that ψ actually lands inside $\prod_\lambda \mathrm{End}_{K_{f,\lambda}[G_\mathbb{Q}]}(V_\lambda(A_f))$, where $V_\lambda(A_f) = V_\ell(A_f) \otimes_{K_f \otimes \mathbb{Q}_\ell} K_{f,\lambda}$ (any endomorphism in the image has to commute with the primitive idempotents defining the factors $K_{f,\lambda}$).

The main ingredient is now the following theorem by Ribet (note that the λ-adic Tate module $V_\lambda(A_f)$ is free of rank two over $K_{f,\lambda}$ and gives rise to the $G_\mathbb{Q}$-representation $\rho_f : G_\mathbb{Q} \to \mathrm{End}_{K_{f,\lambda}}(V_\lambda(A_f))$ associated to f).

Theorem 0.3. *$V_\lambda(A_f)$ is an absolutely irreducible $K_{f,\lambda}[G_\mathbb{Q}]$-module.*

Proof. See Proposition 4.1 in [R2]. The idea is the following: Suppose ρ_f is absolutely reducible. By oddness it is reducible (or replace $K_{f,\lambda}$ with a finite extension to get reducibility). Then the semi-simplification of ρ_f would be described by two characters ρ_1, $\rho_2 : G_\mathbb{Q} \to K_{f,\lambda}$. Results of Serre in [S] show that it is locally algebraic so that the ρ_i are of the form $\rho_i = \chi_\ell^{n_i}\epsilon_i$ with χ_ℓ the ℓ-adic cyclotomic character, $n_i \in \mathbb{Z}$ and ϵ_i finite-order characters unramified away from ℓN. The local algebraicity follows in at least two ways from [S], either using a transcendency result of Lang and Siegel ([S], ch. III, pp. 20-30) or by using a result of Tate that local algebraicity is equivalent to the existence of a Hodge-Tate decomposition for semisimple representations ([S], ch. III, p.7, proof on pp. 30-53). That our representation (for $\ell \nmid N$) is of Hodge-Tate type follows from the fact that abelian varieties with good reduction over ℓ-adic fields have Hodge-Tate ℓ-adic Tate-module representations (see Cor. 2 to Thm. 3 in [T]).

From the Eichler-Shimura relations one deduces for $p \nmid \ell N$ that $\mathrm{trace}(\rho_f(\mathrm{Frob}_p)) = a_p$ and $\det(\rho_f(\mathrm{Frob}_p)) = p$, where $\mathrm{Frob}_p \in G_\mathbb{Q}$ is a Frobenius element. For our characters ρ_i we get $n_1 + n_2 = 1$ and therefore a contradiction to the Weil-Ramanujan bound $|a_p(f)| \leq 2\sqrt{p}$ for the Hecke eigenvalues a_p of f for $p \neq \ell$, $p \geq 7$. □

After performing another extension of scalars to the algebraic closure, we may apply Schur's Lemma to see that $\mathrm{End}_{\overline{K}_{f,\lambda}[G_\mathbb{Q}]}(V_\lambda(A_f)) \simeq \overline{K}_{f,\lambda}$. We deduce that the non-zero $\psi \circ \phi$ surjects onto its image and so, since ψ is injective, ϕ must be surjective. We now have proven (*) and therefore

Proposition 0.4 (Mazur Proposition 2.1)**.** $\mathbf{T}_0 \otimes \mathbb{Q} = \mathrm{End}(J_0(N)_\mathbb{Q}) \otimes \mathbb{Q}$.

Remark 0.5. This alternative proof was also used by Ribet in [R3] Cor. 4.2 and Thm. 6.2.

References

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