# Plan of the "Mazur seminar", term 2

This semester will focus on studying the minimal part of Mazur's IHES paper that is necessary to establish that the Eisenstein quotient of $J_0(N)_{/\mathbf{Q}}$ has finite Mordell–Weil group (for prime $N$). The core topics we must discuss are §1 in Ch. I, §7–§8 in Ch. II, and the geometry of $X_0(N)_{/\mathbf{Z}}$ and its relationship with the Néron model of its $\mathbf{Q}$-fiber. These ends are tied together by Prop. 14.1 in Ch. II and §3 in Ch. III, as I explain in my first 2 lectures. We will also have a special lecture on applying Mazur's methods in some concrete examples, and will have additional lectures on Ribet's proof of the converse to Herbrand's theorem (a marvelous application of Mazuresque techniques) and on Parent's refinement of Merel's proof of the torsion boundedness conjecture over number fields.

The schedule below leaves approximately 3 weeks at the end of the term, so that gives some extra room in case some talks required extra time, etc. We may need to insert an extra talk in the middle to deal with the issue of proving that the Eisenstein primes do not have residue characteristic $N$ (a result that follows from Prop. 9.7 in Ch. II, but whose proof there is the culmination of a long series of arguments that we can hopefully avoid for our purposes).

I have assigned a speaker or pair of speakers to each topic below, but some of these topics may require 1 1/2 or 2 meetings to address. If you are unhappy with the topic to which you have been assigned, please let me know asap.

**Talk 1**. (Conrad) The first couple of talks will be an overview of how the main ingredients of the IHES paper fit together (for the purpose of proving Ogg's conjecture, ignoring the voluminous refined information that Mazur obtains and that was historically important for Wiles, etc., but is not so essential anymore). Also included in this is a discussion of some generalities on quasi-finite flat separated (commutative) group schemes (of finite presentation), and §1($a$)–($c$) in Ch. I of the big paper. The main goal here is to become comfortable with various examples of such groups and ways of working with them Galois-theoretically.

**Talk 2**. (Aoki and Liu) The most important quasi-finite groups in Mazur's work are what he calls *admissible groups*: these are a certain special class of quasi-finite flat $\mathbf{Z}$-groups whose generic fiber have a Jordan-Hölder filtration with successive quotients that are constant or dual-to-constant. This material is covered in §1($d$)–($g$) of Ch. I.

The precise definition should be given, including the proof of the important equivalence between the Galois-theoretic formulation of the Jordan-Hölder condition on the $\mathbf{Q}$-fiber and the scheme-theoretic formulation of this as a filtration condition over $\mathbf{Z}$. A key result in this direction is what Mazur calls "Fontaine's theorem" (Theorem 1.4). When presenting the proof of this latter result, use Raynaud's theorem (in Mazur's reference [55]). Also, make sure to establish the results in the table in part ($g$).

**Talk 3**. (Parson) Mazur's methods resting on §1 in Ch. I can be applied in many concrete examples. This talk will illustrate the utility of these ideas by revisiting some low-level genus-1 modular curves that were considered last semester and establishing that they have rank 0 by pure-thought methods and not requiring the use of computers (other than the human brain). This talk will be partly based on the discussion in the final section of Mazur's paper in Inventiones **18** on abelian varieties over towers of numbers fields.

**Talk 4**. (Shastry) Last semester we were able to avoid studying the integral structure of modular curves at primes dividing the level. This time we must work over all of $\operatorname{Spec}\mathbf{Z}$, and so we need to understand the geometry of $X_0(N)_{\mathbf{Z}}$ (especially its behavior modulo $N$).

This talk will explain how one gets a handle on such geometry, particularly that $X_0(N)_{\mathbf{Z}}$ has semistable reduction modulo $N$, and that in general one can use semistable integral proper models of curves to compute the special fibers of the Néron model of the Jacobian of the generic fiber. For example, the Néron model of $J_0(N)$ at $N$ has reduction that is a split torus. These results will be used in §8 of Ch. II.

Beware that the Appendix with Rapoport is riddled with errors, so do not trust too much in what the tables there say unless you have checked things for yourself. A good reference for some background on this talk is §9 (esp. §9.2) in *Néron Models*.

**Talk 5**. (Klosin, Cais) This talk presents the arguments in §7–§8 in Ch. II (replace the reference to the fishy Appendix with a reference to Example 8 in §9.2 of *Néron Models*, upon noting that $\mathrm{Pic}^0_{\mathbf{P}^1/\mathbf{Z}}$ vanishes) and explains how these results are applied in conjunction with the Grothendieck orthogonality theorem and the results in §1 of Ch. I to settle items (*a*) and (*b*) on p. 148 to wrap up the proof that the Eisenstein quotient has rank 0.

In particular, it should be explained how the results in Ch. I ensure that for an Eisenstein prime $\mathfrak{P}$, the optimal quotient $J_{\mathfrak{P}}$ has $p$-adic Tate module equal (up to isogeny) to the $\mathfrak{P}$-adic factor of the $p$-adic Tate module of $J$ and that this quotient has *ordinary* reduction.

Feel free to speak with either me or Parson (or look in SGA7) to find out about the meaning of the Grothendieck orthogonality theorem for abelian varieties with semiabelian reduction, such as $J_0(N)_{\mathbf{Z}_{(N)}}$.

**Talk 6**. (Berger, Xue). As a further application of the methods of Mazur (especially the creative use of finite group schemes), Ribet proved the converse to Herbrand's theorem concerning cyclotomic class groups and Bernoulli numbers. This talk will present the contents of Ribet's paper, but with one crucial technical improvement: the irritating proof of Theorem 3.3 (which is given as a consequence of some numerological facts about class numbers) should be replaced with a conceptual proof resting on the results of Deligne–Rapoport.

**Talk 7**. (Arnold) Following Mazur's work, several authors gradually refined his ideas until Merel solved the general torsion boundedness conjecture for elliptic curves over number fields. The work of Parent gives a more detailed explanation of Merel's refinements and provides some better upper bounds than Merel's original paper. This talk will be an exposition of the highlights of a paper of Parent's in Crelle **506**.