TALK 9: FORMAL IMMERSIONS AND QUOTIENTS OF MODULAR JACOBIANS

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We establish some notation. Let N be prime[1] and set $S = \operatorname{Spec} \mathbb{Z}[1/2N]$. Denote by $X_{\mathbb{Q}} = X_0(N)_{/\mathbb{Q}}$ and $J_{\mathbb{Q}} = J_0(N)$ the usual modular curve and its Jacobian over $\mathbb{Q}$, where we normalize the embedding $X_{\mathbb{Q}} \hookrightarrow J_{\mathbb{Q}}$ by requiring $\infty \mapsto 0$. $X_{\mathbb{Q}}$ is the generic fiber of the (smooth) S-curve $X = X_0(N)_{/S}$, and $J_{\mathbb{Q}}$ has Néron model $J = \text{Néron}_S(J_0(N))$, which is in fact an abelian scheme over S. The Néron mapping property gives us a map $X \to J$ extending the map over $\mathbb{Q}$. $\mathbb{T}$ will denote the usual Hecke algebra, viewed (when convenient) as a subalgebra of $\operatorname{End}_{\mathbb{Q}}(J_{\mathbb{Q}})$, and we put $\mathbb{T}_A = \mathbb{T} \otimes_{\mathbb{Z}} A$ for an abelian group A. Note that $\mathbb{T}$ also acts on J (over S) by the Néron mapping property.

1. Optimal Quotients and Ideals of $\mathbb{T}$

Recall from Tobias's talk that an *optimal quotient* of $J_{\mathbb{Q}}$ is defined to be a quotient $J_{\mathbb{Q}} \to A$ with connected kernel. It is easier to work with quotients of the form $J_{\mathbb{Q}}/IJ_{\mathbb{Q}}$, where $I \subseteq \mathbb{T}$ is an ideal. The purpose of this section is to show that any optimal quotient arises this way.

We first examine how tangent spaces behave under taking quotients. These results will also be useful later.

Proposition 1.1 *Let K be a field of characteristic 0, $f : A \to B$ a morphism of abelian varieties over K, and $\tilde{f} : \operatorname{Tan}_0 A \to \operatorname{Tan}_0 B$ the tangent map at the identity. Then $\operatorname{Tan}_0(f(A)) = \tilde{f}(\operatorname{Tan}_0(A))$.*

Proof. The category of commutative proper K-groups is abelian, and all such groups are smooth, since $\operatorname{char} K = 0$. Note that the additive functor (comm. proper K-gps.) $\to$ (K-vect. sp.) given by $G \mapsto \operatorname{Tan}_0 G$ is exact: it is clearly left exact (e.g., by the interpretation of the tangent space as $K[\varepsilon]$-points), but also right exact for dimension reasons. We have an exact sequence of proper K-groups

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} f(A) \longrightarrow 0,$$

so this gives us

$$\operatorname{Tan}_0(f(A)) = \operatorname{Tan}_0(A)/\operatorname{Tan}_0(\ker f) = \tilde{f}(\operatorname{Tan}_0(A)). \qquad \square$$

Corollary 1.2 *Suppose a ring R acts by K-endomorphisms ($\operatorname{char} K = 0$) on an abelian variety $A_{/K}$ and $I \subseteq R$ is a finitely generated ideal. Then $\operatorname{Tan}_0(IA) = I \operatorname{Tan}_0(A)$. (Note that R also acts on $\operatorname{Tan}_0(A)$.)*

[1] This assumption is largely superfluous.

Proof. If $\{i_1, \ldots, i_n\}$ generates I, then IA is, by definition, the image of the map

$$\prod_{j=1}^{n} A \xrightarrow{\Sigma} A$$

sending $\vec{a} = (a_1, \ldots, a_n)$ to $\Sigma(\vec{a}) = \sum_{j=1}^{n} i_j(a_j)$. The tangent map

$$\prod_{j=1}^{n} \operatorname{Tan}_0 A \xrightarrow{\tilde{\Sigma}} \operatorname{Tan}_0 A$$

is then given by $\tilde{\Sigma}(\vec{a}) = \sum_{j=1}^{n} i_j(a_j)$, so we can apply the previous proposition. □

We can now prove the stated goal of this section.

Theorem 1.3 *Let $f : J_{\mathbb{Q}} \to A$ be an optimal quotient. Then $\ker f = IJ$ for some saturated ideal $I \subseteq \mathbb{T}$. (That I is* saturated *means $\mathbb{T}/I$ is torsion-free.)*

Proof. Tobias proved that, in this situation, $\mathbb{T}$ descends to an action on A, i.e., there is an action of $\mathbb{T}$ on A for which f is $\mathbb{T}$-equivariant. Define I to be the kernel of the action $\mathbb{T} \to \operatorname{End}_{\mathbb{Q}}(A)$. (Note I is saturated because $\operatorname{End}_{\mathbb{Q}}(A)$ is torsion-free.) Then clearly $IJ_{\mathbb{Q}} \subseteq \ker f$. Since $\ker f$ is connected, it will suffice to prove that $\dim IJ_{\mathbb{Q}} = \dim \ker f$, or equivalently that $\dim \operatorname{Tan}_0 IJ_{\mathbb{Q}} = \dim \operatorname{Tan}_0 \ker f$.

Tobias showed that $J_{\mathbb{Q}}$ decomposes (up to isogeny) as a product of $\mathbb{Q}$-simple abelian varieties $\prod_{g \in C} A_g$, where the product is over Galois conjugacy classes of normalized eigenforms. Thus we have an exact sequence (cf. Prop. 1.1) of $\mathbb{T}_{\mathbb{Q}}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Tan}_0 \ker f & \longrightarrow & \operatorname{Tan}_0 J_{\mathbb{Q}} & \longrightarrow & \operatorname{Tan}_0 A & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \prod_{g \notin C'} \operatorname{Tan}_0 A_g & \longrightarrow & \prod_{g \in C} \operatorname{Tan}_0 A_g & \longrightarrow & \prod_{g \in C'} \operatorname{Tan}_0 A_g & \longrightarrow & 0 \end{array}$$

for some subset $C' \subseteq C$. As Tobias discussed, $\mathbb{T}_{\mathbb{Q}} = \prod_g K_g$ where $K_g = \operatorname{End}^0(A_g)$. It is then clear by definition that $I_{\mathbb{Q}} = \prod_{g \notin C'} K_g$, and so it follows from Cor. 1.2 that $\dim \operatorname{Tan}_0 IJ_{\mathbb{Q}} = \dim \operatorname{Tan}_0 \ker f$. □

Remark 1.4. It is easy see that this establishes a bijective correspondence between saturated ideals in $\mathbb{T}$ and optimal quotients of $J_{\mathbb{Q}}$. We won't need this here.

2. Tangent Spaces and Modular Forms

In this section, following closely the treatment in [6], we will show that the tangent space of J (working over S now) at the identity is a free $\mathbb{T}_{\mathbb{Z}[1/2N]}$-module of rank 1 with an explicit generator $\frac{d}{dq}\big|_0$. We recall two facts which make this at least believable.

Theorem 2.1 *$S_2(N)_{\mathbb{Q}}$ is a free $\mathbb{T}_{\mathbb{Q}}$-module of rank 1.*

Proof. See Brian's book [1]. □

Theorem 2.2 *For any ring R flat over $\mathbb{Z}[1/2N]$ or over $\mathbb{Z}/m$ for $(m, 2N) = 1$, there are natural Hecke-equivariant isomorphisms*

$$S_2(N)_R \cong H^0(X_{/R}, \Omega^1_{X/R}) \cong H^0(J_{/R}, \Omega^1_{J/R}) \cong \operatorname{Cot}_0(J_{/R}).$$

Proof. See Brian's book [1]. Also cf. Mazur's big paper [4], Ch. II, §4. □

As discussed in Hui's talk, the formal completion of X along the cuspidal section $\infty_{/S}$ is identified with $\operatorname{Spf}\mathbb{Z}[1/2N][\![q]\!]$. Thus the tangent space $\operatorname{Tan}_{\infty_{/S}}(X)$ is free of rank 1 over $\mathbb{Z}[1/2N]$ with generator $\frac{d}{dq}$. Denote by $\frac{d}{dq}\big|_0$ its image in $\operatorname{Tan}_0 J$; we will show it generates $\operatorname{Tan}_0 J$ as a $\mathbb{T}_{\mathbb{Z}[1/2N]}$-module. We need several lemmas.

Lemma 2.3 *The $\mathbb{T}_{\mathbb{Z}[1/2N]}$-module $\operatorname{Tan}_0 J$ is faithful.*

Proof. Since $\mathbb{T}_{\mathbb{Z}[1/2N]}$ and $\operatorname{Tan}_0 J$ are $\mathbb{Z}$-torsion-free[2], we can check this after extending scalars to $\mathbb{Q}$. The functor Tan_0 is faithful on smooth connected algebraic groups in characteristic 0 (this can be seen in our case from Cor. 1.2; better: use Lie theory). □

Lemma 2.4 *For every maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_{\mathbb{Z}[1/2N]}$, $\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J$ is nonzero.*

Proof. Suppose $\mathfrak{m}\operatorname{Tan}_0 J = \operatorname{Tan}_0 J$. Then NAK implies that $(\operatorname{Tan}_0 J)_\mathfrak{m} = 0$, so $(\operatorname{Tan}_0 J)_t = 0$ for some $t \notin \mathfrak{m}$ since $\operatorname{Tan}_0 J$ is $\mathbb{T}_{\mathbb{Z}[1/2N]}$-finite. After replacing t by some power, we may assume that t kills $\operatorname{Tan}_0 J$. Since $t \notin \mathfrak{m}$, in particular $t \neq 0$, contradicting the faithfulness of $\operatorname{Tan}_0 J$ as a $\mathbb{T}_{\mathbb{Z}[1/2N]}$-module (Lemma 2.3). □

Lemma 2.5 *For every maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_{\mathbb{Z}[1/2N]}$, the element $\frac{d}{dq}\big|_0 \in \operatorname{Tan}_0 J$ has nonzero image in the quotient $\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J$.*

Proof. The idea of the proof is to observe that, for suitable ℓ, the dual of the module $(\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J) \otimes \overline{\mathbb{F}}_\ell$ (nonzero by the previous lemma) is naturally a submodule of $S_2(N)_{\overline{\mathbb{F}}_\ell}$ and show that $\frac{d}{dq}\big|_0$ acts nontrivially on this space. To be precise (after [6]):

Since $\mathbb{T}$ is $\mathbb{Z}$-finite, $F = \mathbb{T}_{\mathbb{Z}[1/2N]}/\mathfrak{m}$ is a finite extension of $\mathbb{F}_\ell$ for some prime ℓ. For notational convenience, set $R = \mathbb{T}_{\mathbb{Z}[1/2N]} \otimes_\mathbb{Z} \overline{\mathbb{F}}_\ell$ and $M = \operatorname{Tan}_0 J \otimes_\mathbb{Z} \overline{\mathbb{F}}_\ell$. Thus there is an exact sequence

$$\mathfrak{m} \otimes_\mathbb{Z} \overline{\mathbb{F}}_\ell \longrightarrow R \longrightarrow F \otimes_\mathbb{Z} \overline{\mathbb{F}}_\ell \longrightarrow 0.$$

Choosing an embedding $i : F \to \overline{\mathbb{F}}_\ell$, we get an $\overline{\mathbb{F}}_\ell$-algebra map $i \otimes 1 : F \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \to \overline{\mathbb{F}}_\ell$. Let $\overline{\mathfrak{m}}$ be the kernel of the composite $R \to F \otimes_\mathbb{Z} \overline{\mathbb{F}}_\ell \to \overline{\mathbb{F}}_\ell$, so that

$$0 \longrightarrow \overline{\mathfrak{m}} \longrightarrow R \longrightarrow \overline{\mathbb{F}}_\ell \longrightarrow 0$$

is exact. One checks that $\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J$ injects into $(\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J) \otimes_F \overline{\mathbb{F}}_\ell \cong M/\overline{\mathfrak{m}}M$.

Let $M^\vee[\overline{\mathfrak{m}}]$ denote the R-submodule of $M^\vee$ killed by $\overline{\mathfrak{m}}$, where $M^\vee$ denotes the dual $\overline{\mathbb{F}}_\ell$-vector space of M. In this notation we have,

$$(M/\overline{\mathfrak{m}}M)^\vee \cong M^\vee[\overline{\mathfrak{m}}]$$

as R-modules. Thus, using tangent-cotangent duality and Thm. 2.2, we have

$$M^\vee[\overline{\mathfrak{m}}] = H^0(X_{/\overline{\mathbb{F}}_\ell}, \Omega^1_{X/\overline{\mathbb{F}}_\ell})[\overline{\mathfrak{m}}] \cong S_2(N)_{\overline{\mathbb{F}}_\ell}[\overline{\mathfrak{m}}].$$

We showed in Lemma 2.4 that $\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J$ is nonzero. Thus we can find a nonzero form f in $S_2(N)_{\overline{\mathbb{F}}_\ell}[\overline{\mathfrak{m}}]$. Ch. III, Cor. 12.9 of [2] implies that

$$H^0(X_{/\overline{\mathbb{F}}_\ell}, \Omega^1_{X/\overline{\mathbb{F}}_\ell}) = H^0(X, \Omega^1_{X/\mathbb{Z}[1/2N]}) \otimes \overline{\mathbb{F}}_\ell.$$

[2] Recall that $\mathbb{T}$ is a subring of $\operatorname{End}_\mathbb{Q}(J_\mathbb{Q})$.

Thus we can lift f to characteristic 0, and by working in $S_2(N)_{\mathbb{C}}$ we see that f satisfies $a_1(T_n f) = a_n(f)$ for all n, even if f doesn't lift to a newform (see formula (3.5.12) in [7]).

Hence, for all n, we have $\frac{d}{dq}\big|_0(T_n f) = a_1(T_n f) = a_n(f)$ by Hui's talk. But since $X_0(N)_{/\overline{\mathbb{F}}_\ell}$ is smooth and geometrically connected, its differential forms are determined by their images in the completed stalk at any point, so $\frac{d}{dq}\big|_0$ can't have trivial image. □

Lemma 2.6 *For every maximal ideal* $\mathfrak{m} \subseteq \mathbb{T}_{\mathbb{Z}[1/2N]}$, $\operatorname{Tan}_0 J/\mathfrak{m} \operatorname{Tan}_0 J$ *is a free* $\mathbb{T}_{\mathbb{Z}[1/2N]}/\mathfrak{m}$*-module of rank 1 with basis the image of* $\frac{d}{dq}\big|_0$.

Proof. In the notation of Lemma 2.5, we have

$$\begin{aligned}\dim_F \operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J &= \dim_{\overline{\mathbb{F}}_\ell}(\operatorname{Tan}_0 J/\mathfrak{m}\operatorname{Tan}_0 J)\otimes_F \overline{\mathbb{F}}_\ell\\ &= \dim_{\overline{\mathbb{F}}_\ell} M/\overline{\mathfrak{m}}M\\ &= \dim_{\overline{\mathbb{F}}_\ell} M^\vee[\overline{\mathfrak{m}}].\end{aligned}$$

Let a_n be the image in $R/\overline{\mathfrak{m}} \cong \overline{\mathbb{F}}_\ell$ of the Hecke operator T_n and choose any nonzero $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}[\overline{\mathfrak{m}}] \cong M^\vee[\overline{\mathfrak{m}}]$. Because R acts on this space through the quotient $R/\overline{\mathfrak{m}}$, each T_n acts by multiplication by a_n, i.e., $T_n(f) = a_n f$. As above, we know that $a_1(T_n f) = a_n(f)$, so the Fourier expansion of f at infinity must have the form $q + a_2q^2 + a_3q^3 + \cdots$. Since the differential forms on $X_0(N)_{/\overline{\mathbb{F}}_\ell}$ are determined by their q-expansions at infinity, we see that f generates the $\overline{\mathbb{F}}_\ell$-vector space $M^\vee[\overline{\mathfrak{m}}]$, and so this space has dimension 1. □

We are at last ready to prove this section's theorem and its important corollary:

Theorem 2.7 *The tangent space* $\operatorname{Tan}_0 J$ *to* J *at* 0 *is a free* $\mathbb{T}_{\mathbb{Z}[1/2N]}$*-module of rank* 1*, generated by* $\frac{d}{dq}\big|_0$.

Proof. We show that the module $N = \operatorname{Tan}_0 J/(\mathbb{T}_{\mathbb{Z}[1/2N]}\frac{d}{dq}\big|_0)$ is trivial. According to Lemma 2.6, for every maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_{\mathbb{Z}[1/2N]}$, we have

$$N/\mathfrak{m}N = N_\mathfrak{m}/\mathfrak{m}N_\mathfrak{m} = 0.$$

Therefore, NAK implies that $N_\mathfrak{m} = 0$. This shows that N is zero locally everywhere, hence zero[3]. Finally, $\operatorname{Tan}_0 J$ is a free $\mathbb{T}_{\mathbb{Z}[1/2N]}$-module since it is faithful (Lemma 2.3). □

Corollary 2.8 *Let* $I \subseteq \mathbb{T}$ *be a saturated ideal, and let* J_I *be the Néron model over* $S = \operatorname{Spec}\mathbb{Z}[1/2N]$ *of the quotient* $J_\mathbb{Q}/IJ_\mathbb{Q}$*. Then the tangent space* $\operatorname{Tan}_0 J_I$ *at* 0 *of* J_I *is a free* $\mathbb{T}/I \otimes \mathbb{Z}[1/2N]$*-module of rank* 1 *with basis the image of* $\frac{d}{dq}\big|_0$.

Proof. Consider the exact sequence

$$0 \longrightarrow IJ_\mathbb{Q} \longrightarrow J_\mathbb{Q} \longrightarrow J_\mathbb{Q}/IJ_\mathbb{Q} \longrightarrow 0$$

of abelian varieties over $\mathbb{Q}$. Since $e(\mathbb{Q}_p) < p - 1$ away from 2, we can apply the result at the end of Tong's talk (i.e., Cor. 1.1 in [5]) to get an exact sequence on tangent spaces:

$$(*) \qquad 0 \longrightarrow \operatorname{Tan}_0 \text{Néron}_S(IJ_\mathbb{Q}) \longrightarrow \operatorname{Tan}_0 J \longrightarrow \operatorname{Tan}_0 J_I \longrightarrow 0.$$

[3]Zeroness is a local property.

By the theorem, $\operatorname{Tan}_0 J$ is a free $\mathbb{T}_{\mathbb{Z}[1/2N]}$-module of rank 1 with basis $\frac{d}{dq}\big|_0$, so there is an ideal $I' \subseteq \mathbb{T}_{\mathbb{Z}[1/2N]}$ such that the sequence $(*)$ is isomorphic to the sequence

$$0 \longrightarrow I' \longrightarrow \mathbb{T}_{\mathbb{Z}[1/2N]} \longrightarrow \mathbb{T}_{\mathbb{Z}[1/2N]}/I' \longrightarrow 0.$$

But over $\mathbb{Q}$, Cor. 1.2 shows that $I_{\mathbb{Q}} = I'_{\mathbb{Q}}$, and since both I and I' are saturated (the former by assumption, and the latter because $\mathbb{T}_{\mathbb{Z}[1/2N]}/I' \cong \operatorname{Tan}_0 J_I$ is torsion-free), we have $I' = I_{\mathbb{Z}[1/2N]}$. Hence $\operatorname{Tan}_0 J_I \cong (\mathbb{T}/I)_{\mathbb{Z}[1/2N]}$. □

3. Formal Immersions

In this section, we discuss (for any Noetherian scheme S) what it means for a map of S-schemes $f : Y \to Z$ to be a formal immersion along a section $s \in Y(S)$ and show that any map $X \to \mathscr{A}$ of X to the Néron model of a nontrivial optimal quotient of $J_{\mathbb{Q}}$ is a formal immersion along ∞. This will imply that, for $p \nmid 2N$, $\infty_{/\mathbb{Z}_{(p)}} \in X(\mathbb{Z}_{(p)})$ is the only point which both reduces to $\infty_{/\mathbb{F}_p}$ in $X(\mathbb{F}_p)$ and maps to 0 in $A(\mathbb{Z}_{(p)})$. First we need to define formal immersion:

Definition 3.1. Let $f : Y \to Z$ be a locally finite type morphism of locally Noetherian schemes. Then for a point $y \in Y$, we say that f is a *formal immersion at* y if the induced map $\widehat{\mathcal{O}}_{Z,f(y)} \to \widehat{\mathcal{O}}_{Y,y}$ on completed local rings is surjective.

Suppose further that Y and Z are finite type and separated over a locally Noetherian scheme S and that f is an S-morphism. Then for a section $s \in Y(S)$, we say that f is a *formal immersion along* s if f is a formal immersion for all $y \in s(S)$.

Remark 3.2. It can be shown using NAK that if $s \in Y(S)$ is a section with image $f(s) \in Z(S)$, and if $\widehat{Y}$ and $\widehat{Z}$ denote the formal completions of Y and Z along these sections (which are necessarily closed immersions), then f is a formal immersion along s in the sense of the above definition if and only if $\widehat{Y} \to \widehat{Z}$ is a closed immersion of formal schemes.

The importance of formal immersions is basically the following:

Proposition 3.3 *Suppose that Y is separated and that $f : Y \to Z$ is a formal immersion at $y \in Y$. Suppose that there is an integral Noetherian scheme T and two T-valued points $p_1, p_2 \in Y(T)$ such that for some point $t \in T$ we have $y = p_1(t) = p_2(t)$. If moreover $f \circ p_1 = f \circ p_2$, then $p_1 = p_2$.*

Proof. The subscheme[4] $A = \{x \in T \mid p_1(x) = p_2(x)\} \subseteq T$ is closed because Y is separated. By integrality of T, it will suffice to show that $\operatorname{Spec} \mathcal{O}_{T,t} \to T$ factors through A. Hence we can assume that T is local with closed point t. The maps $p_i : T \to Y$ then factor (uniquely) through $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$, so we may assume that Y is local with closed point y. Because $\mathcal{O}_{T,t}$ is a domain, the rightmost map in the diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{Y,y} & \underset{\hat{p}_2^\sharp}{\overset{\hat{p}_1^\sharp}{\rightrightarrows}} & \widehat{\mathcal{O}}_{T,t} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & \underset{p_2^\sharp}{\overset{p_1^\sharp}{\rightrightarrows}} & \mathcal{O}_{T,t} \end{array}$$

[4]Of course, we really mean that A is the pullback of the diagonal under $(p_1, p_2) : T \to Y \times Y$.

is injective by [3], Thm. 8.10(ii). Hence it will suffice to show that $\hat{p}_1^\sharp = \hat{p}_2^\sharp$. But $\hat{p}_1^\sharp \hat{f}^\sharp = \hat{p}_2^\sharp \hat{f}^\sharp$ and $\hat{f}^\sharp : \widehat{\mathcal{O}}_{Z,f(y)} \to \widehat{\mathcal{O}}_{Y,y}$ is surjective. □

In the situation we're considering, we have:

Theorem 3.4 *Let $J_\mathbb{Q} \to A$ be any nonzero optimal quotient, say $A = J_\mathbb{Q}/IJ_\mathbb{Q}$ (cf. Thm. 1.3), and let $\mathscr{A} = \text{Néron}_S(A)$ (where $S = \operatorname{Spec} \mathbb{Z}[1/2N]$). If $f : X \to \mathscr{A}$ is the map extending $X_\mathbb{Q} \hookrightarrow J_\mathbb{Q} \twoheadrightarrow A$, then f is a formal immersion along $\infty_{/S} \in X(S)$.*

Proof. Let $x \in \infty_{/S}(S)$. The residue fields of x and $f(x)$ are the same, so by [3], Thm. 8.4 it will suffice to show that the tangent map $\operatorname{Tan}_x X \to \operatorname{Tan}_{f(x)} \mathscr{A}$ is injective. But then this follows from Cor. 2.8 and the definition of $\frac{d}{dq}\big|_0$. □

As a result of Prop. 3.3 and Thm. 3.4, we immediately get:

Corollary 3.5 *With notation and assumptions as in Thm. 3.4, $\infty_{/\mathbb{Z}_{(p)}} \in X(\mathbb{Z}_{(p)})$ is the only point reducing to $\infty_{/\mathbb{F}_p}$ in $X(\mathbb{F}_p)$ that also maps to 0 in $\mathscr{A}(\mathbb{Z}_{(p)})$.*

Proof. As f is a formal immersion along $\infty_{/\mathbb{Z}_{(p)}}$, we can apply Prop. 3.3 with $T = \operatorname{Spec} \mathbb{Z}_{(p)}$. □

An application of the Atkin-Lehner involution allows us to prove:

Corollary 3.6 *With notation and assumptions as in Thm. 3.4, $0_{/\mathbb{Z}_{(p)}} \in X(\mathbb{Z}_{(p)})$ is the only point reducing to $0_{/\mathbb{F}_p}$ in $X(\mathbb{F}_p)$that also maps to $f(0_{/\mathbb{Z}_{(p)}})$ in $\mathscr{A}(\mathbb{Z}_{(p)})$.*

Proof. The Atkin-Lehner involution is an automorphism of X over S and switches the cusps at 0 and ∞. Tobias showed that f is equivariant for the Atkin-Lehner involution on X and J and that the involution on J induces an involution on $\mathscr{A}$, so we are done. □

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