FINITENESS OF CLASS NUMBERS FOR ALGEBRAIC GROUPS

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1. Introduction

Let G be an affine group scheme of finite type over a global field F. (We do not assume G to be reductive or smooth or connected.) Let $\mathbf{A}_F$ denote the locally compact adele ring of F, S be a finite non-empty set of places of F that contains the archimedean places, and $\mathbf{A}_F^S$ the factor ring of adeles with vanishing component along S; for $F_S = \prod_{v \in S} F_v$, we have $\mathbf{A}_F = F_S \times \mathbf{A}_F^S$. Recall that if X is a finite type affine scheme over a topological ring R then the set $X(R)$ inherits a natural topology that is functorial in X, and the formation of this topology is compatible with fiber products (in the categories of R-schemes and topological spaces respectively). In particular, $X(R)$ is locally compact when R is, and $X(R)$ is a topological group when X is an R-group scheme. We are interested in the locally compact topological group $G(\mathbf{A}_F)$.

Since F is a discrete subring of $\mathbf{A}_F$, the subgroup $G(F)$ inside of $G(\mathbf{A}_F)$ is discrete (and closed). Let K be a compact open subgroup in $G(\mathbf{A}_F^S)$. Consider the double coset space

$$\Sigma_{G,S,K} = G(F)\backslash G(\mathbf{A}_F)/G(F_S)K = G(F)\backslash G(\mathbf{A}_F^S)/K.$$

Clearly for any two compact open subgroups K and K', $K \cap K'$ is again compact open and hence of finite index in each of K and K'. It follows that the finiteness property of $\Sigma_{G,S,K}$ is independent of the choice of K, and so is an intrinsic property of G (and S). In fact, it is equivalent to the compactness of the (typically non-Hausdorff) coset space $G(F)\backslash G(\mathbf{A}_F)/G(F_S) = G(F)\backslash G(\mathbf{A}_F^S)$.

Definition 1.1. The F-group scheme G *has finite class numbers with respect to* S if $\Sigma_{G,S,K}$ is finite for one (equivalently, every) compact open subgroup $K \subseteq G(\mathbf{A}_F^S)$. If this holds for all choices of S then G *has finite class numbers.*

For $G = \mathbf{G}_m$ over a number field and suitable pairs (S, K), the sets $\Sigma_{G,S,K}$ coincide with classical generalized ideal class groups. In general, the size of the set $\Sigma_{G,S,K}$ (when finite) is therefore called the (S, K) *class number* of G. If G is non-commutative then $\Sigma_{G,S,K}$ is merely a pointed set rather than a group. For each non-archimedean place v of F, the group of local points $G(F_v)$ has an open neighborhood of the identity that is a profinite group. Thus, G has finite class numbers (with respect to a fixed choice of S) if and only if for every compact open subgroup K in $G(\mathbf{A}_{F,f})$ (with $\mathbf{A}_{F,f}$ the factor ring of "finite adeles") the double coset space $G(F)\backslash G(\mathbf{A}_F)/G(F_S)K$ is finite; the distinction here is that K is a compact open subgroup in $G(\mathbf{A}_{F,f})$ rather than in $G(\mathbf{A}_F^S)$. In practice one often sees this latter point of view because it uses compact open subgroups of the group $G(\mathbf{A}_{F,f})$ that has nothing to do with the choice of S, but for our purposes it will be more convenient to use compact open subgroups of $G(\mathbf{A}_F^S)$ rather than compact open subgroups of $G(\mathbf{A}_{F,f})$.

If $G(F)\backslash G(\mathbf{A}_F)$ is compact then obviously G has finite class numbers. This includes the case $G = \mathbf{G}_a$, as well as any solvable smooth G having no nontrivial F-rational characters (by [11, IV, 1.3]). The finiteness of generalized ideal class groups implies that $\mathbf{G}_m$ over number fields has finite class numbers. The same holds over global function fields by using the compactness of the norm-1 subgroup of the idele class group and the discreteness of the idelic norm in the function field case. By using a compactness result of Godement–Oesterlé [11, IV, 1.3] and the cases of $\mathbf{G}_a$ and $\mathbf{G}_m$ over any global field, it can be deduced that solvable smooth groups over global fields have finite class numbers. For another class of examples (with fixed choice of S), if G is a

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smooth connected F-simple semisimple and simply connected F-group and $G(F_v)$ is non-compact for some $v \in S$ then the strong approximation property [13] implies that $G(F)$ has dense image in $G(\mathbf{A}_F^S)$, so $\Sigma_{G,S,K}$ is a single point for any compact open subgroup $K \subseteq G(\mathbf{A}_F^S)$.

It is a general theorem of Borel [1, Thm. 5.1] that any affine group scheme G of finite type over a number field F has finite class numbers. In particular, one does not need to assume connectivity or reductivity of G. The first step in Borel's proof is to reduce the general case to the connected case by means of Lang's trick [2, 16.5(i)] and a compactness argument (this part works over any global field), and to then use structure theorems for connected smooth affine groups over perfect fields to reduce the connected case to the unipotent and reductive cases separately. A similar technique reduces the unipotent case to that of $\mathbf{G}_a$. The real work is in the reductive case, for which Borel uses archimedean places via the theory of Siegel domains as developed in earlier work of Borel and Harish-Chandra. In particular, the argument does not make use of reduction to the semisimple case.

The situation in positive characteristic cannot be treated by Borel's methods from the number field case. For connected adjoint semisimple groups over a global field F the finiteness of class numbers was proved by Borel and Prasad via finiteness theorems for volumes of quotients of semisimple adelic groups modulo certain discrete subgroups; see Theorem 5.2 below. To go beyond smooth solvable and connected adjoint semisimple groups (e.g., reductive groups) it seems necessary to introduce some new ideas. The first difficulty is that if $1 \to G' \to G \to G'' \to 1$ is an exact sequence of smooth connected affine F-groups then the open image of $G(\mathbf{A}_F) \to G''(\mathbf{A}_F)$ can fail to have finite index (even if G' is a torus); the same problem can arise for the map $G(F_S) \to G''(F_S)$ when G' is unipotent and smooth with $\mathrm{char}(F) > 0$. This is why finiteness of class numbers for G is not a trivial consequence of such finiteness for G' and G''. It is also not evident if finiteness of class numbers is invariant under purely inseparable isogenies over global function fields (or even general isogenies), so a serious problem over such fields (which are not perfect) is to overcome the subtle nature of unipotent smooth groups over imperfect fields.

Two basic difficulties over an imperfect field k are that (i) for a smooth affine k-group G the unipotent radical of $G_{\overline{k}}$ may fail to be defined over k as a subgroup of G (with $\overline{k}$ an algebraic closure of k), and (ii) non-trivial unipotent smooth k-groups may fail to admit a composition series over k with successive quotients of the form $\mathbf{G}_a$ (and in fact the cohomology of such groups over arithmetically interesting rings can be infinite).

Example 1.2. By [7, XVII, App. II, 5.1], if k'/k is a nontrivial purely inseparable extension of fields and G' is a connected smooth non-solvable k'-group then for the Weil restriction $G = \mathrm{Res}_{k'/k}(G')$ the radical of $G_{\overline{k}}$ is not defined over k as a subgroup of G. In particular, if G' is semisimple then the unipotent radical of $G_{\overline{k}}$ is not defined over k.

Example 1.3. Consider the additive subgroup $U \subseteq \mathbf{G}_a \times \mathbf{G}_a$ defined by $y^p = x + tx^p$ over $F = k(t)$ with k a field of characteristic $p > 0$. This is unipotent because it is isomorphic to $\mathbf{G}_a$ after base change to $F(t^{1/p})$, but it is not isomorphic to $\mathbf{G}_a$ over F as a curve (let alone as a group) because it has a unique point at infinity and this point has residue field $F(t^{1/p}) \neq F$. It is a pleasant exercise to check that $U(F) = 0$ when $p > 2$ (whereas for $p = 2$ the group $U(F)$ is naturally parameterized since U is a smooth affine conic). One sees via the exact sequence $1 \to U \to \mathbf{G}_a^2 \to \mathbf{G}_a \to 1$ (resting on the map $(x, y) \mapsto y^p - (x + tx^p)$) that $\mathrm{H}^1(k((t)), U)$ is infinite if $p > 2$ and $\mathrm{H}^1(F, U)$ and $\mathrm{H}^1(\mathscr{O}_{F,S}, U)$ are infinite for any non-empty finite set of places of S of F that are trivial on k. (Also see [11, V, 2.2].)

Our main result is the following analogue of Borel's theorem over number fields.

Theorem 1.4. *Any smooth affine group scheme over a global function field has finite class numbers.*

The key to overcoming difficulties with unipotent smooth groups and the field of definition of the radical (and unipotent radical) is to work with possibly non-reduced subgroups. Most of the results we prove along the way toward the proof of Theorem 1.4 are valid (and so are proved) without smoothness hypotheses. There are only two places where smoothness is used: in passing to the identity component (Theorem 2.1, hence Corollary 2.2), and in proving a certain finite-index lemma for smooth connected groups (Lemma 3.8).

This latter lemma is false without smoothness hypotheses, and this can be used to give a counterexample to Theorem 1.4 in the absence of smoothness, as we now show.

Example 1.5. Let $G' \to G$ be a purely inseparable isogeny of connected semisimple groups over a global function field F such that G contains a rank-1 split torus $T_0 = \mathbf{G}_m$ which, up to isogeny, lifts into G' over F_{v_0} but does not do so over F_{v_1} for some places v_0 and v_1 of F. (Such isogenies cannot exist in characteristic $p > 0$ unless $p \le 3$. An example [4, 1.2] in characteristic 2 is the μ_2-covering $G' = \mathrm{SO}(Q) \to \mathrm{Sp}(B_Q) \simeq \mathrm{SL}_2$ where $\{Q = 0\}$ is a smooth plane conic over F with no rational point and B_Q is the associated bilinear form with defect 1, viewed as a non-degenerate alternating binary form.) The centralizer $Z_G(T_0)$ is a connected reductive subgroup of G. Let $H \subseteq Z_G(T_0)$ be any connected reductive F-subgroup containing T_0 such that H has F_{v_1}-rank 1. (We can take $H = T_0$.) By conjugacy of maximal split tori in connected reductive groups, it follows that $(T_0)_{F_{v_1}}$ is the only nontrivial split torus in $H_{F_{v_1}}$. Since T_0 is central in the reductive group $Z_G(T_0)$, there is a character $Z_G(T_0) \to \mathbf{G}_m$ whose restriction to $T_0 = \mathbf{G}_m$ is nontrivial. Hence, there is likewise such a character $\chi : H \to \mathbf{G}_m$. Let $H' = G' \times_G H$ be the scheme-theoretic preimage of H in G', so $H' \subseteq G'$ is a closed subgroup scheme and the finite flat covering $\pi : H' \to H$ has infinitesimal covering group. In particular, H' is connected since H is connected. We will show that H' does not have finite class numbers.

By hypothesis there is a torus $T_{v_0} = \mathbf{G}_m$ in $G'_{F_{v_0}}$ that maps onto $(T_0)_{F_{v_0}}$ in $G_{F_{v_0}}$. This must lie in the fiber product $H'_{F_{v_0}}$, so the canonical projection $\pi : H' \to H$ followed by χ restricts (over F_{v_0}) to a nontrivial character on T_{v_0}. Hence, the image of the $H'(F_{v_0}) \to F_{v_0}^\times$ induced by $\chi \circ \pi$ is not contained in the units of the valuation ring. In contrast, I claim that $H'(F_{v_1})$ is compact, so $\chi \circ \pi$ carries $H'(F_{v_1})$ into $\mathscr{O}_{F_{v_1}}^\times$. Since H is reductive with F_{v_1}-rank 1, by using χ and the projection $H \to H_1 = H/T_0$ we get an F-isogeny $f : H \to \mathbf{G}_m \times H_1$ in which the first factor is defined by χ and the factor H_1 is connected reductive over F with F_{v_1}-rank 0. Hence, $H_1(F_{v_1})$ is compact. The composite map $f \circ \pi : H' \to \mathbf{G}_m \times H_1$ is finite, so it is proper on F_{v_1}-points. Hence, $H'(F_{v_1})$ has closed image and is compact if and only if its image is compact. To verify such compactness it therefore suffices to prove that $H'(F_{v_1}) \to F_{v_1}^\times$ has finite image. Since $F_{v_1}^\times$ has only finitely many elements of finite order, we can assume that some $h' \in H'(F_{v_1})$ has image in $F_{v_1}^\times$ with infinite order. Let $C \subseteq G'_{F_{v_1}}$ be the closure of the subgroup generated by h', so this is a smooth F_{v_1}-subgroup of $H'_{F_{v_1}}$ with positive dimension and the map $C^0 \to \mathbf{G}_m$ induced by $\chi \circ f \circ \pi$ is a non-trivial character since $h' \in C(F_{v_1})$ has image with infinite order under this map. This nontrivial character on C^0 must have nontrivial restriction to any maximal F_{v_1}-torus in C^0, so such a torus contains a $\mathbf{G}_m$ on which $\chi \circ f \circ \pi$ is non-trivial. The image of this latter $\mathbf{G}_m$ under $H'_{F_{v_1}} \to H_{F_{v_1}}$ is a rank-1 split torus, but we have already seen that $(T_0)_{F_{v_1}}$ is the only such F_{v_1}-torus in $H_{F_{v_1}}$. Thus, up to isogeny we have lifted $T_0 \subseteq G$ into G' locally at v_1, contrary to the initial requirements on v_1. This completes the proof that $H'(F_{v_1})$ is compact.

Now consider the double coset space $\Sigma_{H',v_1,K}$ for a compact open subgroup K in $H'(\mathbf{A}_F^{v_1})$. The global points $H'(F)$ and the compact subgroup $H'(F_{v_1})K$ are carried into the subgroup of norm-1 ideles under the map $\chi \circ \pi : H'(\mathbf{A}_F) \to \mathbf{A}_F^\times$, so by composing with the idelic norm we get a well-defined map $\Sigma_{H',v_1,K} \to q^{\mathbf{Z}}$ whose restriction back to $H'(F_{v_0})$ has image containing $q^{n\mathbf{Z}}$ for some $n > 0$. Hence, $\Sigma_{H',v_1,K}$ is infinite. (Although $\chi \circ \pi$ is a nontrivial character of the group scheme H', there is no $\mathbf{G}_m$ contained in H' on which $\chi \circ \pi$ is nontrivial since even locally at v_1 no such $\mathbf{G}_m$ exists. This is why the proof of Lemma 3.8 breaks down for the connected F-group H' with $S = \{v_1\}$, and this is the only step where the proof of Theorem 1.4 does not work for H'.)

In Remark 3.9 we expand on Example 1.5 to address when Theorem 1.4 holds for connected but possibly non-smooth groups G over global function fields. Let us now briefly give an overview of the paper. In §2 we use arguments of Borel over number fields in [1, §1] to show that a smooth affine group over a global field has finite class numbers if its identity component does. Several lemmas concerning properties of adelic coset fibrations and of tori over local fields are recorded in §3. In §4 we prove a technical result on properness of a certain map involving possibly non-reduced group schemes (due to G. Prasad in the reductive case); the intervention of non-smooth groups at this step seems to be unavoidable. Finally, in §5 we use §3 and §4 to prove Theorem 1.4 via passage to the case of connected adjoint semisimple groups that was settled by Borel and Prasad. In Appendix A we discuss some generalities concerning adelic points of separated schemes of finite type over a global field F, comparing the viewpoints of Weil and Grothendieck. (It is recommended that the reader merely skim this appendix on a first reading; it elaborates on [11, I, §3], but we include it to serve as a more detailed reference on this topic.) Throughout our arguments, we aim to treat all global fields on an equal footing as much as possible (so, for example, Borel's original finiteness theorem over number fields is deduced from the connected adjoint semisimple case).

Notation and Terminology. We do not make any connectivity assumptions on group schemes. If G is an affine group scheme of finite type over a field k then $\mathrm{X}_k(G)$ denotes the character group $\mathrm{Hom}_k(G, \mathbf{G}_m)$ over k; this is a finitely generated $\mathbf{Z}$-module (and torsion-free when G is smooth and connected).

If $R \to R'$ is a map of rings and Z is a scheme over R then $Z_{R'}$ denotes the base change of Z to an R'-scheme. If Y is a scheme, then Y_{red} denotes the underlying reduced scheme. A map of schemes $f : Y \to Z$ is *radiciel* if it is injective and induces a purely inseparable extension on residue fields $\kappa(f(y)) \to \kappa(y)$ for all $y \in Y$; a typical example is a map of smooth algebraic groups such that the kernel has only one geometric point. A surjective map of finite type schemes over a field k is radiciel precisely when it induces a bijection on $\overline{k}$-points (with $\overline{k}$ an algebraic closure of k), and in the case of normal varieties over k it is equivalent to say that the extension on function fields is purely inseparable.

If $Y \to Z$ is a map of $\mathbf{F}_p$-schemes (the case $Z = \mathrm{Spec}\, k$ for a field k will be of most interest to us), then $Y^{(p^n)}$ denotes the Z-scheme $Y \times_{Z, F_Z^n} Z$, where $F_Z : Z \to Z$ is the absolute Frobenius map (identity on topological spaces, pth-power map on the structure sheaf); loosely speaking, $Y^{(p^n)}$ is the Z-scheme obtained from Y by raising the coefficients in the defining equations of Y (over Z) to the p^nth power. The *n-fold relative Frobenius map* $\phi_{Y/Z,n} : Y \to Y^{(p^n)}$ over Z is induced by $F_Y^n : Y \to Y$ and the structure map $Y \to Z$; loosely speaking, it corresponds to the p^nth-power map in local coordinates (over Z). The formation of both $Y^{(p^n)}$ and $\phi_{Y/Z,n}$ commute with base change on Z and fiber products over Z and are functorial in the Z-scheme Y. In particular, if Y is a Z-group scheme then $\phi_{Y/Z,n}$ is a homomorphism of group schemes.

We use the scheme-theoretic Weil restriction of scalars (in the affine case) with respect to possibly inseparable finite extensions of the base field (as well as a variant for base rings), and for a development of Weil restriction in the context of schemes we refer the reader to [5, §7.6] and [11, App. 2, 3]; this second reference addresses issues specific to the case of smooth affine groups over fields. In the special case of finite Galois extensions of fields, this scheme-theoretic construction recovers the Galois descent construction as used in [15] and many other works on algebraic groups.

Whenever we speak of a finite set of places S (of a global field F that is usually understood from context) it is always understood that S is non-empty and contains all archimedean places.

2. Connected components

Let G be an affine group scheme of finite type over a global field F, and let G^0 be the connected component of the identity. The subgroup $G^0(R)$ in $G(R)$ is normal for every F-algebra R (such as $R = \mathbf{A}_F$). Indeed, this comes down to the claim that the universal conjugation $c : G \times G^0 \to G$ defined by $c(g, g') = gg'g^{-1}$ factors through G^0. By [8, IV_2, 4.5.13], G^0 is geometrically connected over F, so $G_\alpha \times G^0$ is connected for each connected component G_α of G. Hence, the desired factorization holds for topological reasons (look at $c(G \times \{1\})$).

We now review (in scheme-theoretic language) an argument of Borel [1, 1.9] to show that G has finite class numbers if G is smooth and G^0 has finite class numbers. Since G^0 is a closed subscheme of G, it follows from Appendix A that $G(\mathbf{A}_F)$ is a locally compact and Hausdorff topological group containing $G^0(\mathbf{A}_F)$ as a closed (normal) subgroup. In particular, the quotient space $G(\mathbf{A}_F)/G^0(\mathbf{A}_F)$ is locally compact and Hausdorff, and it is naturally a topological group. Moreover, Theorem A.8 ensures that if G is smooth then for some finite set of places S of F we can find a smooth affine group scheme G_S over $\mathrm{Spec}\, \mathscr{O}_{F,S}$ and another such group scheme G_S^0 equipped with a closed and open immersion $G_S^0 \to G_S$ whose generic fiber is $G^0 \to G$ and whose closed fiber G_x^0 over each closed point $x \in \mathrm{Spec}\, \mathscr{O}_{F,S}$ is a (geometrically) connected smooth group scheme over the finite field $\kappa(x)$. This latter connectivity property for fibers over finite fields is crucial in the proof of:

Theorem 2.1 (Borel). *If G as above is smooth then the quotient $G(\mathbf{A}_F)/G^0(\mathbf{A}_F)$ is compact.*

Before we give the proof, to get a feel for this theorem consider the special case when G is a constant group $\mathbf{Z}/n\mathbf{Z}$. In this case G^0 is trivial, and $G(\mathbf{A}_F)$ is the set of n-tuples of mutually orthogonal idempotents in $\mathbf{A}_F$ with sum adding up to 1. In other words, if V_F denotes the set of places of F (index set for the "factors" of $\mathbf{A}_F$), then $G(\mathbf{A}_F)$ is the set $\mathrm{Hom}_{\mathrm{Set}}(V_F, \mathbf{Z}/n\mathbf{Z})$, or $\prod_{V_F} \mathbf{Z}/n\mathbf{Z}$ (product with index set V_F). The topology induced by $\mathbf{A}_F$ is equal to the product topology, so compactness follows in this case.

Proof. Pick a finite set of places S for which we have $G^0_S \to G_S$ as considered above the statement of the theorem. For S' containing S, $G^0_{S'}(\mathbf{A}_{F,S'})$ is a closed subgroup of $G_{S'}(\mathbf{A}_{F,S'})$. This is the inclusion

$$\prod_{v\in S'} G^0(F_v) \times \prod_{v\notin S'} G^0_{S,v}(\mathscr{O}_v) \to \prod_{v\in S'} G(F_v) \times \prod_{v\notin S'} G_{S,v}(\mathscr{O}_v)$$

using product topologies. By Theorem A.5, the inclusions in each factor are open and closed embeddings. Clearly $G(F_v)/G^0(F_v)$ injects into $(G/G^0)(F_v)$, which is a finite set (since G/G^0 is finite étale over F), and likewise for $v \notin S$ we see that $G_S(\mathscr{O}_v)/G^0_S(\mathscr{O}_v)$ injects into the finite set $G(F_v)/G^0(F_v)$ (since $G^0_S(\mathscr{O}_v) = G^0(F_v) \cap G_S(\mathscr{O}_v)$ inside of $G(F_v)$, due to the fact that G^0_S is closed and open in G_S and $\mathrm{Spec}\,\mathscr{O}_v$ is connected). Thus, the topological quotient $G_S(\mathbf{A}_{F,S})/G^0_S(\mathbf{A}_{F,S})$ is topologically a product of finite discrete sets, so it is compact Hausdorff.

For finite S' containing S, since $\mathbf{A}_{F,S'} = \mathscr{O}_{F,S'} \otimes_{\mathscr{O}_{F,S}} \mathbf{A}_{F,S}$ and G^0_S is closed and open in G_S we see that

$$G^0_S(\mathbf{A}_{F,S}) = G_S(\mathbf{A}_{F,S}) \cap G^0_{S'}(\mathbf{A}_{F,S'}) = G_S(\mathbf{A}_{F,S}) \cap G^0_S(\mathbf{A}_{F,S})$$

inside of $G_{S'}(\mathbf{A}_{F,S'}) = G_S(\mathbf{A}_{F,S'})$. Thus, the continuous map of compact Hausdorff groups

$$G_S(\mathbf{A}_{F,S})/G^0_S(\mathbf{A}_{F,S}) \to G_{S'}(\mathbf{A}_{F,S'})/G^0_{S'}(\mathbf{A}_{F,S'})$$

is injective and hence a closed embedding. However, $G_S(\mathbf{A}_{F,S})$ is open in $G_{S'}(\mathbf{A}_{F,S'})$, so this closed embedding of compact Hausdorff groups is also an open embedding, whence it has finite index. The same holds with the pair (S, S') replaced by (S', S'') for any pair of finite sets of places of F containing S with $S' \subseteq S''$.

Recall that $G(\mathbf{A}_F)$ is the directed union of open subgroups $G_{S'}(\mathbf{A}_{F,S'})$, and similarly for G^0 with the $G^0_{S'}$'s, so it is easy to deduce that $G(\mathbf{A}_F)/G^0(\mathbf{A}_F)$ is the directed union of open subgroups $G_{S'}(\mathbf{A}_{F,S'})/G^0_{S'}(\mathbf{A}_{F,S'})$ with their compact quotient topologies. Thus, our problem is exactly to prove that this directed chain stops. By (7) in Theorem A.9, it is equivalent to show that for all sufficiently large S', $G_{S'}(\mathbf{A}_{F,S'})G^0(F_v)$ contains $G(F_v)$ for every $v \notin S'$. That is, $G_S(\mathscr{O}_v)G^0(F_v) \stackrel{?}{=} G(F_v)$ for all but finitely many v (outside of S). Here is the key: if we consider the short exact sequence

$$1 \to G^0 \stackrel{j}{\to} G \stackrel{\pi}{\to} G/G^0 \to 1 \tag{1}$$

of F-group schemes, the map π is smooth, separated, and faithfully flat (i.e., surjective), with G/G^0 a finite (étale) F-group scheme, so by Theorem A.8 if we enlarge S we may find a finite (étale) $\mathscr{O}_{F,S}$-group scheme E_S with generic fiber G/G^0 and a smooth, separated, surjective group scheme map $\pi_S : G_S \to E_S$ with generic fiber π. The kernel $H_S = \ker(\pi_S)$ is a smooth $\mathscr{O}_{F,S}$-group scheme, and its generic fiber is identified with G^0. Thus, again by Theorem A.8, by increasing S we may find an isomorphism $H_S \simeq G^0_S$ compatible with the closed immersions into G_S. By increasing S we can therefore "smear out" (1) to a diagram

$$1 \to G^0_S \to G_S \to E_S \to 1$$

of finite type separated $\mathscr{O}_{F,S}$-group schemes, with the left map exactly as above and $G_S \to E_S$ a smooth surjective map onto a target E_S that is finite étale over $\mathscr{O}_{F,S}$.

Now it suffices to prove the following general claim. Suppose R is a complete (or just henselian) discrete valuation ring with fraction field K and finite residue field k, and G is a separated smooth finite type group scheme over R. Suppose there is a smooth surjection $G \to E$ onto a finite R-group scheme E for which the (necessarily smooth) kernel G^0 has (necessarily geometrically) connected fibers. Then we claim that

$G(K) = G^0(K)G(R)$. Since $E(K) = E(R)$ by finiteness of E, and $G(K) \to E(K)$ has kernel $G^0(K)$, for $g \in G(K)$ with image $\overline{g} \in E(K) = E(R)$ we just have to show that $\overline{g}$ is in the image of $G(R)$. Consider the cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & G \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow[\overline{g}]{} & E \end{array}$$

We want $X(R)$ to be non-empty.

From the hypotheses, $G \to E$ is a smooth surjection that is a principal homogenous space for the E-group scheme $E \times_{\operatorname{Spec} R} G^0$, so applying pullback along $\overline{g}$ shows that X admits a structure of fiberwise non-empty principal homogenous space for the R-group G^0. The closed fiber X_0 is a non-empty principal homogenous space for a smooth connected group scheme G^0_0 over a finite field k. Lang's trick [2, 16.5(i)] ensures that such homogeneous spaces are always trivial, which is to say that $X(k)$ must be non-empty. Since X is R-smooth (!) and R is henselian local, a rational point on the closed fiber lifts to an R-point. ■

Corollary 2.2 (Borel). *An affine smooth F-group scheme G has finite class numbers if its identity component G^0 does.*

Proof. The inclusion

$$G(F)/G^0(F) \hookrightarrow (G/G^0)(F)$$

implies that $G(F)/G^0(F)$ is finite (since G/G^0 is F-finite). Let K be a compact open subgroup in $G(\mathbf{A}_F^S)$, so $K^0 = K \cap G^0(\mathbf{A}_F^S)$ is a compact open subgroup of $G^0(\mathbf{A}_F^S)$ (since $G^0(\mathbf{A}_F^S)$ is a closed subgroup of $G(\mathbf{A}_F^S)$). By the hypothesis that G^0 has finite class numbers with respect to S, there exists a finite set $\{\gamma_j^0\}$ in $G^0(\mathbf{A}_F^S)$ such that

$$G^0(\mathbf{A}_F^S) = \coprod G^0(F)\gamma_j^0 K^0.$$

By Theorem 2.1, $G(\mathbf{A}_F^S)/G^0(\mathbf{A}_F^S)$ is compact, so there exists a finite subset $\{g_i\}$ in $G(\mathbf{A}_F^S)$ such that

$$G(\mathbf{A}_F^S) = \coprod G^0(\mathbf{A}_F^S) g_i K = \coprod G^0(F)\gamma_j^0 K^0 g_i K.$$

Since $G^0(F) \subseteq G(F)$ and each compact open subset $K^0 g_i K$ in $G(\mathbf{A}_F^S)$ is a finite union of right cosets $g_{i,\alpha}K$, we obtain finiteness of $\Sigma_{G,S,K}$. ■

3. Adelic quotients and tori

This section consists of several lemmas that we will need later. Let us first recall a general result in the theory of topological groups that will often be used in what follows.

Theorem 3.1. *Let G be a locally compact Hausdorff topological group, and X a second-countable locally compact Hausdorff topological space endowed with a continuous G-action, say on the right. Let $x \in X$ be a point and let $G_x \subseteq G$ be its stabilizer for the G-action. If the orbit $x \cdot G$ is locally closed in X then the natural map $G_x \backslash G \to X$ induced by $g \mapsto xg$ is a homeomorphism onto the orbit of x.*

Proof. See [6, Ch. IX, §5] for a proof in a more general setting. The role of second-countability is so that the Baire category theorem may be applied. ■

Definition 3.2. Let H be an affine F-group scheme of finite type. For each $\chi \in \mathrm{X}_F(H) = \mathrm{Hom}_F(H, \mathbf{G}_m)$, let

$$|\chi| : H(\mathbf{A}_F) \to \mathbf{R}^\times_{>0}$$

denote the continuous composite of $\chi : H(\mathbf{A}_F) \to \mathbf{G}_m(\mathbf{A}_F) = \mathbf{A}_F^\times$ and the idelic norm $\|\cdot\|_F : \mathbf{A}_F^\times \to \mathbf{R}^\times_{>0}$. The closed subgroup $H(\mathbf{A}_F)^0 \subseteq H(\mathbf{A}_F)$ is

$$H(\mathbf{A}_F)^0 := \bigcap_{\chi \in \mathrm{X}_F(H)} \ker |\chi|.$$

Example 3.3. If H is a (connected) semisimple smooth F-group, a unipotent F-group, an anisotropic F-torus, or more generally $\mathrm{X}_F(H) = \{1\}$, then $H(\mathbf{A}_F)^0 = H(\mathbf{A}_F)$. In general, $H(\mathbf{A}_F)^0 \subseteq H(\mathbf{A}_F)$ is a normal subgroup with abelian quotient, and it is functorial in H. When F is a global function field this subgroup is open because the idelic norm is discretely valued for such F and $\mathrm{X}_F(H)$ is finitely generated over $\mathbf{Z}$.

Our interest in Definition 3.2 is due to the following lemma (which is well-known in the smooth case):

Lemma 3.4. *Let $H \hookrightarrow H'$ be a closed subgroup of an affine F-group scheme of finite type. The natural map*

$$H(F)\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)^0$$

is a closed embedding. In particular, $H(F)\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)$ is a closed embedding.

Proof. The target is a locally compact Hausdorff space admitting a continuous right action by $H'(\mathbf{A}_F)^0$ and hence by $H(\mathbf{A}_F)^0$, and $H(\mathbf{A}_F)^0$ is a second-countable locally compact Hausdorff group. It follows from Theorem 3.1 that for any $x \in H'(F)\backslash H'(\mathbf{A}_F)^0$ with stabilizer subgroup S_x in $H(\mathbf{A}_F)^0$, the natural orbit map

$$S_x\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)^0$$

is a homeomorphism onto the orbit of x if the orbit is closed. Taking x to be the coset of the identity gives $S_x = H'(F) \cap H(\mathbf{A}_F)^0 = H(F)$, and so the problem is to prove that this orbit is closed.

That is, we have to prove that $H'(F)H(\mathbf{A}_F)^0$ is closed in $H'(\mathbf{A}_F)^0$. A short elegant proof of this is given in [11, IV, §1.1], where it is assumed that H' and H are smooth but this smoothness is not really needed. More precisely, the only role of smoothness is to invoke the standard result that if G is a smooth affine group scheme over a field k and G' is a smooth closed subgroup scheme then there is a closed immersion of k-groups $G \hookrightarrow \mathrm{GL}(V)$ for a finite-dimensional k-vector space V such that G' is the scheme-theoretic stabilizer of a line. The proof of this result in [2, 5.1] works without smoothness by using points valued in artin local rings (not just fields). ■

To reduce Theorem 1.4 to the case of (connected adjoint) semisimple groups that was settled by Borel and Prasad (see Theorem 5.2), we will require the following fibration lemma.

Lemma 3.5. *Let*

$$1 \to H' \to H \to H'' \to 1$$

be a short exact sequence of affine F-group schemes of finite type such that H' admits a composition series of closed subgroup schemes with successive quotients of the form $\mathbf{G}_a$ or $\mathrm{Res}_{F'/F}(\mathbf{G}_m)$ for finite separable extensions F'/F.

For every place v the map $H(F_v) \to H''(F_v)$ is a topological fibration, and the map $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ is a topological fibration. Also,

$$H(F)\backslash H(\mathbf{A}_F) \to H''(F)\backslash H''(\mathbf{A}_F)$$

is a topological fibration whose fibers are all homeomorphic to $H'(F)\backslash H'(\mathbf{A}_F)$.

Proof. By descent theory, H' is smooth and hence $H \to H''$ is a smooth morphism. By Hilbert's Theorem 90 and the hypothesis on H', for any field L/F the map $H(L) \to H''(L)$ is surjective. Thus, $H(F_v) \to H''(F_v)$ is a surjection of topological groups (so it induces the quotient topology, by Lemma 3.1). Since smooth maps are Zariski-locally on the source expressed as étale over an affine space, by the local structure theorem for étale morphisms and the classical theorem on continuity of simple roots of a varying monic polynomial of fixed degree, the map $H(F_v) \to H''(F_v)$ locally admits continuous sections and hence is a topological fibration.

To handle the adelic points, we first "smear out" the given exact sequence to an exact sequence

$$1 \to \mathscr{H}' \to \mathscr{H} \to \mathscr{H}'' \to 1$$

of affine flat group schemes of finite type over $\mathrm{Spec}\, \mathscr{O}_{F,S}$ for a finite set of places S of F. By "short exact sequence" we mean that $\mathscr{H} \to \mathscr{H}''$ is faithfully flat with functorial kernel $\mathscr{H}'$. We can moreover arrange that $\mathscr{H}'$ is $\mathscr{O}_{F,S}$-smooth with geometrically connected fibers. Thus, by descent theory we see that the map $\mathscr{H} \to \mathscr{H}''$ is a smooth morphism and expresses $\mathscr{H}$ as an $\mathscr{H}'$-torsor over $\mathscr{H}''$ for the étale topology. It is a standard consequence of Lang's trick (arguing as near the end of the proof of Theorem 2.1) that the map of topological groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ (for $v \notin S$) is surjective for all $v \notin S$, as we now quickly review. For $h'' \in \mathscr{H}''(\mathscr{O}_v)$, consider the pullback $X_{h''}$ of the smooth $\mathscr{O}_v$-morphism $\mathscr{H}_{\mathscr{O}_v} \to \mathscr{H}''_{\mathscr{O}_v}$ along the map $h : \operatorname{Spec} \mathscr{O}_v \to X_{h''}$. We need to prove that $X_{h''}(\mathscr{O}_v)$ is non-empty, so since the $\mathscr{O}_v$-scheme $X_{h''}$ is an $\mathscr{H}'_{\mathscr{O}_v}$-torsor for the étale topology (due to the torsor property for $\mathscr{H} \to \mathscr{H}''$) it is smooth and hence by Hensel's Lemma it has a section if its closed fiber has a rational point. But its special fiber is a torsor (in the étale topology) for a connected smooth group over a finite field, and hence by Lang's trick it has a rational point.

Since the map of $\mathscr{O}_v$-points $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ is induced by the fibration map $H(F_v) \to H''(F_v)$ via restriction to open subgroups, we can construct local cross-sections for the map of topological groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$, so this map is a fibration for all but finitely many v. Since $\mathscr{H}''(\mathscr{O}_v)$ is compact and totally disconnected for all $v \notin S$, there exists a global cross-section to $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$. It then follows immediately that the map of topological groups $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ admits local cross-sections (for the adelic topology) and so is a fibration.

Finally, consider the map

$$\pi : H(F)\backslash H(\mathbf{A}_F) \to H''(F)\backslash H''(\mathbf{A}_F).$$

Since $H''(F)$ is discrete in $H''(\mathbf{A}_F)$, so

$$H''(\mathbf{A}_F) \to H''(F)\backslash H''(\mathbf{A}_F)$$

admits local cross-sections, we get local cross-sections for π by using local cross-sections for $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$. Since $H(F) \to H''(F)$ and $H(\mathbf{A}_F) \to H''(\mathbf{A}_F)$ are surjective, the right action by $H'(\mathbf{A}_F)$ is transitive on fibers of π and all fibers are homeomorphic. Thus, all fibers of π are homeomorphic to the fiber $\pi^{-1}(1)$.

The left action of $H'(\mathbf{A}_F)$ on fibers is continuous, and the stabilizer in $H'(\mathbf{A}_F)$ for the identity coset in $H(F)\backslash H(\mathbf{A}_F)$ is $H'(\mathbf{A}_F) \cap H(F) = H'(F)$. Since $\pi^{-1}(1)$ is closed, it follows from Theorem 3.1 that the natural map $H'(F)\backslash H'(\mathbf{A}_F) \to \pi^{-1}(1)$ is a homeomorphism. Since $H'(F)$ is discrete in $H'(\mathbf{A}_F)$ and $H(F)$ is discrete in $H(\mathbf{A}_F)$, we can use the local cross-sections and the $H'(\mathbf{A}_F)$-action to verify that π is a topological fibration, since the topological diagram

$$\begin{array}{ccc} H'(F)\backslash H(\mathbf{A}_F) & \longrightarrow & H''(\mathbf{A}_F) \\ \downarrow & & \downarrow \\ H(F)\backslash H(\mathbf{A}_F) & \xrightarrow[\pi]{} & H''(F)\backslash H''(\mathbf{A}_F) \end{array}$$

is cartesian. ∎

We next record some well-known finiteness properties for tori over local fields k. For an arbitrary k-torus T, we define

$$T(k)^0 = \bigcap_{\chi \in \mathrm{X}_k(T)} \ker |\chi|.$$

For example, $T(k)^0 = T(k)$ if T is k-anisotropic. This subgroup of $T(k)$ is functorial in T.

Lemma 3.6. *The subgroup $T(k)^0$ is compact for any k-torus T.*

Proof. By functoriality with respect to the closed immersion of k-tori

$$T \hookrightarrow \mathrm{Res}_{k'/k}(T_{k'})$$

for a finite separable extension k'/k that splits T, it is enough to consider the special case $T = \mathrm{Res}_{k'/k}(\mathbf{G}_m)$. In this case $\mathrm{X}_k(T)$ is infinite cyclic and $\mathrm{N}_{k'/k}$ is a nontrivial element, so

$$T(k)^0 = \ker(\mathrm{N}_{k'/k} : T(k) \to \mathbf{G}_m(k)) = \ker(\mathrm{N}_{k'/k} : {k'}^{\times} \to k^{\times}).$$

This kernel is the compact $\mathscr{O}_{k'}^{\times}$ in the non-archimedean case and it is easily checked to be compact in the archimedean case. ∎

Lemma 3.7. *Let G be a smooth affine group over a local field k. Let $G \twoheadrightarrow T$ be a torus quotient with smooth kernel and let $Z \subseteq G$ be a normal torus with associated quotient $\widetilde{G} = G/Z$. The natural maps*

$$G(k) \to T(k),\ \ G(k) \to \widetilde{G}(k)$$

have open image with finite index.

Proof. Since $G \to T$ and $G \to \widetilde{G}$ are smooth morphisms, the implicit function theorem for k-manifolds ensures that the maps on k-points have open images.

Surjective maps between connected smooth affine groups carry maximal tori to maximal tori [2, 11.14], so for a maximal torus T' in G over k the map $T' \to T$ is a surjection of tori. Such surjections can be split in the isogeny category of k-tori, so $T'(k) \to T(k)$ has image containing $N \cdot T(k)$ for some nonzero integer N. Hence, to show that the image Γ of $G(k) \to T(k)$ has finite index it suffices to prove that $T(k)/N \cdot T(k)$ is compact (as then the open subgroup $\Gamma/N \cdot T(k)$ has finite index); note that $N \cdot T(k)$ is a closed subgroup of $T(k)$ because $N : T(k) \to T(k)$ is proper (this is clear if $T = \mathrm{Res}_{k'/k}(\mathbf{G}_m)$ for a finite separable extension k'/k, and in general we use the closed immersion $T \hookrightarrow \mathrm{Res}_{k'/k}(T_{k'})$ for a finite separable k'/k that splits T).

To prove that the locally compact Hausdorff group $T(k)/N \cdot T(k)$ is compact, let $T_0 \subseteq T$ be the maximal k-split subtorus and $\overline{T} = T/T_0$, so $\overline{T}$ is anisotropic. By Hilbert's Theorem 90 the sequence of groups

$$1 \to T_0(k) \to T(k) \to \overline{T}(k) \to 1$$

is exact, and $T(k) \to \overline{T}(k)$ is open because $T \to \overline{T}$ is smooth. Thus,

$$T_0(k)/N \cdot T_0(k) \to T(k)/N \cdot T(k) \to \overline{T}(k)/N \cdot \overline{T}(k) \to 1$$

is a right exact sequence with the second map open. Since all terms are locally compact and Hausdorff, if the outer terms are compact then the middle term is as well. Hence, we are reduced to the cases of split tori and anisotropic tori. The split case is trivial, and the anisotropic case follows from Lemma 3.6.

Let us now turn to the problem of showing that the open image of $G(k) \to \widetilde{G}(k)$ has finite index. Since we have an exact sequence of groups

$$G(k) \to \widetilde{G}(k) \to \mathrm{H}^1(k, Z),$$

it suffices to prove that $\mathrm{H}^1(k,T)$ is finite for any k-torus T. The closed immersion $T \to \mathrm{Res}_{k'/k}(T_{k'})$ for a finite separable extension k'/k that splits T can be split in the isogeny category of k-tori, so we get the result in characteristic 0 since finite discrete Galois modules for such local fields have finite Galois cohomology. To handle the general non-archimedean case (allowing positive characteristic), consider the duality pairing

$$\mathrm{H}^1(k,T) \times \mathrm{H}^1(k, \mathrm{X}(T)) \to \mathrm{H}^2(k, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z},$$

where $\mathrm{X}(T) = \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$ is the geometric character group (for a separable closure k_s/k). Since $\mathrm{X}(T)$ is a finite free $\mathbf{Z}$-module, it follows from local class field theory (see [10, Ch. I, Thm. 1.8(a)]) that this pairing identifies $\mathrm{H}^1(k,T)$ with the $\mathbf{Q}/\mathbf{Z}$-dual of $\mathrm{H}^1(k, \mathrm{X}(T))$. Thus, we just have to show that $\mathrm{H}^1(k, \mathrm{X}(T))$ is finite, and this follows by using inflation-restriction with respect to a finite Galois extension k'/k that splits the discrete torsion-free Galois module $\mathrm{X}(T)$ over k. ■

We conclude with a crucial finiteness lemma.

Lemma 3.8. *For any connected smooth affine group scheme G over F and any finite non-empty set of places S of F containing the archimedean places, the subgroup $G(\mathbf{A}_F)^0 \cdot G(F_S)$ in $G(\mathbf{A}_F)$ has finite index.*

Proof. We first give an argument that works in characteristic 0, and then we modify it in the case of positive characteristic (using the discreteness of the idelic norm) to circumvent difficulties caused by inseparable homomorphisms. Consider the maximal split torus quotient $G \to T = \mathrm{X}_F(G)^\vee \otimes_{\mathbf{Z}} \mathbf{G}_m$, so $\mathrm{X}_F(T) = \mathrm{X}_F(G)$. Hence, we have a natural injection of groups

$$G(\mathbf{A}_F)/G(\mathbf{A}_F)^0 \to T(\mathbf{A}_F)/T(\mathbf{A}_F)^0.$$

Consider the commutative diagram of groups

$$\begin{array}{ccc} G(\mathbf{A}_F)/G(\mathbf{A}_F)^0 & \longrightarrow & T(\mathbf{A}_F)/T(\mathbf{A}_F)^0 \\ \uparrow & & \uparrow \\ G(F_S) & \longrightarrow & T(F_S) \end{array}$$

We need to prove that the map along the left has image with finite index, so by injectivity of the top row it is enough to prove the maps along the bottom and right sides have images with finite index.

First we check that the cokernel along the right side has finite size. Since T is a split torus, we only have to consider the analogous question for $\mathbf{G}_m$. This case is obvious by separately considering number fields and function fields (using that S contains archimedean places in the number field case and that S is not empty in the function field case).

By Lemma 3.7 applied to the local fields F_v with $v \in S$, the map $G(F_S) \to T(F_S)$ has image with finite index as long as the scheme-theoretic kernel of the quotient map $G \to T$ is smooth. This is automatic in characteristic 0, so the case of number fields is settled.

It remains to consider the case when F has characteristic $p > 0$, and we have to address the possibility that $G \twoheadrightarrow T$ may have non-smooth kernel. If q is the size of the constant field in F then the idelic norm on $\mathbf{A}_F^\times$ has image $q^{\mathbf{Z}}$, so $G(\mathbf{A}_F)/G(\mathbf{A}_F)^0$ is a subgroup of the finite free $\mathbf{Z}$-module $\mathrm{Hom}(\mathrm{X}_F(G), q^{\mathbf{Z}})$. Thus, it is also a finite free $\mathbf{Z}$-module, so the abelian group $G(\mathbf{A}_F)/G(\mathbf{A}_F)^0 G(F_S)$ is finite if it is killed by some nonzero integer. Thus, instead of having to prove that $G(F_v) \to T(F_v)$ has image with finite index for each $v \in S$, we only need to prove that the cokernel is killed by some nonzero integer. Since G is smooth, for a maximal torus $T' \subseteq G$ over F the map $T' \to T$ is a surjection of tori [2, 11.14]. (If G is not smooth then such a T' may not exist.) It suffices to prove that $T'(F_v) \to T(F_v)$ has cokernel killed by a nonzero integer. Since surjections of tori over a field are split in the isogeny category over the same field, we are done with this case. ■

Remark 3.9. For connected affine groups of finite type over a global field, we will see that the smoothness requirement in Lemma 3.8 is the only place where smoothness is required in the proof of Theorem 1.4 (in the connected case). Note that if G is connected and possibly non-smooth, failure of the conclusion of Lemma 3.8 for a particular S implies that G does not have finite class numbers with respect to S, since for any compact open subgroup K in $G(\mathbf{A}_F^S)$ the double coset $\Sigma_{G,S,K}$ maps onto $G(\mathbf{A}_F)/G(\mathbf{A}_F)^0 G(F_S)$. In other words, for a connected affine group G of finite type over F, with q the size of the constant field of F, it will follow from our proof of Theorem 1.4 that such a G has finite class numbers with respect to S if and only if $G(F_S) \to G(\mathbf{A}_F)/G(\mathbf{A}_F)^0$ has image with finite index. Thus, a sufficient condition for finiteness of class numbers with respect to S (for connected G over F) is that the map $G(F_S) \to \mathrm{Hom}(\mathrm{X}_F(G), q^{\mathbf{Z}})$ defined by $g \mapsto (\chi \mapsto \|\chi(g)\|_F)$ has image with finite index. This criterion is trivially satisfied whenever $\mathrm{X}_F(G)$ is a torsion group, which is to say (for F_p the perfect closure of F) that G_{F_p} modulo its unipotent radical has F_p-anisotropic center, and Lemma 3.8 says that this criterion is always satisfied for smooth connected G. In particular, counterexamples to finiteness of class numbers in the non-smooth connected case must admit a surjective homomorphism to $\mathbf{G}_m$. Note that H' in Example 1.5 satisfies this latter property, as it must.

For later reference we wish to record one further result that relates non-smooth groups to smooth groups via a canonical map. This result involves quotients of non-smooth group schemes modulo normal subgroup schemes, so for readers who are uncomfortable with non-reduced group schemes we now make some brief remarks concerning quotients in this setting. For the definition and existence of quotients of finite type group schemes G modulo closed normal subgroup schemes H when working over a field k, see [7, VI, 3.2(*iv*), 5.2]. In particular, the quotient map $G \to G/H$ is faithfully flat, so if G is k-smooth then the quotient G/H is k-smooth (since the coordinate ring of G/H must inject into that of G by faithful flatness, so it is geometrically reduced over k because that of G is). By [7, $\mathrm{VI_B}$, 11.17], G/H is affine when G is affine. In particular, if G is smooth and affine and H is smooth then this recovers the concept of quotient as used in the theory of linear algebraic groups. Indeed, if we temporarily write $(G/H)_{\mathrm{alg}}$ to denote the quotient in the sense of algebraic groups then the natural homomorphism $G \to (G/H)_{\mathrm{alg}}$ has scheme-theoretic kernel H, so by the universal property of G/H we get an induced homomorphism $G/H \to (G/H)_{\mathrm{alg}}$ of smooth affine groups that is surjective with trivial scheme-theoretic kernel, so it must be an isomorphism.

Theorem 3.10. *Let G be a group scheme of finite type over a field k of characteristic $p > 0$, and let $\phi_{G/k,n} : G \to G^{(p^n)}$ be its n-fold relative Frobenius morphism. For sufficiently large n, the quotient $G/\ker \phi_{G/k,n}$ is k-smooth.*

Proof. This is [7, $\mathrm{VII}_{\mathrm{A}}$, 8.3], ■

4. A properness result

If $1 \to G' \to G \to G'' \to 1$ is a short exact sequence of connected smooth affine groups over a global field F, it is natural to wonder if finiteness of class numbers for G'' and G' implies the same for G. An obstruction to carrying out such an argument is that the open map $G(F_S) \to G''(F_S)$ may fail to have image with finite index. The situation of interest to us is when G' is solvable and G'' has finite class numbers. (We noted in §1 that solvable groups have finite class numbers, but this fact will later drop out from our treatment of the general case.) The following result is inspired by these considerations.

Theorem 4.1. *Let G be a connected affine F-group scheme of finite type and $H \subseteq G$ a closed normal subgroup scheme with multiplicative component group such that $(H_{\overline{F}})_{\mathrm{red}}$ is solvable. Let $\overline{G} = G/H$ be the connected affine quotient, and assume that $\overline{G}$ is smooth and semisimple. The natural map*

$$G(F)\backslash G(\mathbf{A}_F)^0 \to \overline{G}(F)\backslash \overline{G}(\mathbf{A}_F)$$

is proper.

We will only require Theorem 4.1 for smooth G, but it is essential to not require smoothness for H. As a first step in the proof of the theorem, we wish to reduce to the case when H is a smooth group. In case F has characteristic 0, H is automatically smooth. Since number fields are perfect, in this case we have that the solvable H with multiplicative component group contains a connected smooth unipotent normal subgroup U such that H/U is an extension of a finite étale multiplicative group by a torus. To get to the same situation in positive characteristic we shall use the following lemma.

Lemma 4.2. *Let $H \to H'$ be a radiciel surjective homomorphism between affine group schemes of finite type over a global function field F. The natural continuous map*

$$H(F)\backslash H(\mathbf{A}_F) \to H'(F)\backslash H'(\mathbf{A}_F)$$

is a closed embedding.

Proof. We let $p = \mathrm{char}(F) > 0$. By Theorem 3.10, for sufficiently large n the quotients $H_n = H/\ker \phi_{H/F,n}$ and $H'_n = H'/\ker \phi_{H'/F,n}$ are F-smooth. Consider the commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & H' \\ \downarrow & & \downarrow \\ H_n & \longrightarrow & H'_n \end{array}$$

in which the two vertical sides are finite flat quotient maps and the bottom side is also finite flat since the source and target are smooth and the map is a surjective homomorphism with finite fibers. To prove the theorem for the top arrow it obviously suffices to prove it for the other three sides. Hence, we can assume that $H \to H'$ is finite flat. In this case $H' = H/K$ for a finite infinitesimal normal closed subgroup scheme K in H. Such a K is killed by its own n-fold relative Frobenius morphism for some $n \geq 0$, so by the universal property of flat quotients we see that the corresponding relative Frobenius $H \to H^{(p^n)}$ for H factors through the map $H \to H'$. In general, if $f : X \to Y$ and $f' : Y \to Z$ are continuous maps between topological spaces with f' separated and $f' \circ f$ a closed embedding, then f is a closed embedding. Thus, it suffices to treat a relative Frobenius morphism $H \to H^{(p^n)}$ (with possibly non-smooth H).

Let $F' = F^{1/p^n}$, a global field of degree p^n over F. We have a natural isomorphism of topological groups $H^{(p^n)}(\mathbf{A}_F) \simeq H(\mathbf{A}_{F'})$, and this carries $H^{(p^n)}(F)$ over to $H(F')$. The composite

$$H(\mathbf{A}_F) \stackrel{\phi_{H/F,n}}{\to} H^{(p^n)}(\mathbf{A}_F) \simeq H(\mathbf{A}_{F'})$$

is induced by functoriality with respect to the inclusion map of F-algebras $\mathbf{A}_F \to \mathbf{A}_{F'}$, and likewise

$$H(F) \to H^{(p^n)}(F) \simeq H(F')$$

is induced by functoriality with respect to the inclusion map of fields $F \to F'$. Hence, our problem is to prove that the natural map

$$H(F)\backslash H(\mathbf{A}_F) \to H(F')\backslash H(\mathbf{A}_{F'})$$

is a closed embedding.

Rather more generally, for any finite extension of global function fields F'/F and any affine finite type F-group H, the natural map

$$H(F)\backslash H(\mathbf{A}_F) \to H(F')\backslash H(\mathbf{A}_{F'})$$

is a closed embedding. To see this, let $H' = \mathrm{Res}_{F'/F}(H_{F'})$, so by Example A.12 this map is the map

$$H(F)\backslash H(\mathbf{A}_F) \to H'(F)\backslash H'(\mathbf{A}_F)$$

induced by the canonical closed immersion $H \to H'$. By applying Lemma 3.4 to the map $H \to H'$, the natural map

$$H(F)\backslash H(\mathbf{A}_F)^0 \to H'(F)\backslash H'(\mathbf{A}_F)^0 \tag{2}$$

is a closed embedding. Due to the topological structure of the idelic norm in the case of global function fields, $H(F)\backslash H(\mathbf{A}_F)$ is topologically a disjoint union of copies of $H(F)\backslash H(\mathbf{A}_F)^0$. More precisely, if we let $\Lambda = \mathrm{Hom}(\mathrm{X}_F(H), q^{\mathbf{Z}})$ (where q is the size of the finite constant field of F) then each $h \in H(\mathbf{A}_F)$ induces an element of Λ via $\chi \mapsto \|\chi(h)\|_F$, so if $\Lambda_H \subseteq \Lambda$ is the subgroup obtained from such h's and for each $\lambda \in \Lambda_H$ we choose $h_\lambda \in H(\mathbf{A}_F)$ giving rise to λ in this way then topologically we have

$$H(F)\backslash H(\mathbf{A}_F) = \coprod_{\lambda \in \Lambda_H} H(F)\backslash H(\mathbf{A}_F)^0 \cdot h_\lambda.$$

An analogous decomposition holds for $H'(F)\backslash H'(\mathbf{A}_F)$ using $\Lambda'_{H'} = \mathrm{Hom}(\mathrm{X}_F(H'), q^{\mathbf{Z}})$ defined in a similar manner.

The natural map $\mathrm{X}_F(H')_{\mathbf{Q}} \to \mathrm{X}_F(H)_{\mathbf{Q}}$ is surjective because if $\chi : H \to \mathbf{G}_m$ is a homomorphism then $\chi^{[F':F]}$ factors as

$$H \longrightarrow H' \xrightarrow{\mathrm{Res}_{F'/F}(\chi_{F'})} \mathrm{Res}_{F'/F}(\mathbf{G}_m) \xrightarrow{\mathrm{N}_{F'/F}} \mathbf{G}_m$$

due to functoriality of Weil restriction and the fact that $\mathbf{G}_m \to \mathrm{Res}_{F'/F}(\mathbf{G}_m) \to \mathbf{G}_m$ is raising to the $[F' : F]$th power. We conclude that Λ_H is naturally identified with a subgroup of $\Lambda'_{H'}$. For each $\lambda \in \Lambda_H$ and associated choice $h_\lambda \in H(\mathbf{A}_F)$ we can use the image of h_λ in $H'(\mathbf{A}_F)$ as the corresponding choice $h'_{\lambda'}$ for the image λ' of λ in $\Lambda'_{H'}$. In this way, the initial map that we want to be a closed embedding is identified with a disjoint union of copies of the closed embedding (2), followed by a further open and closed embedding. ■

We now return to the proof of Theorem 4.1, beginning with a reduction step in positive characteristic.

Step 1. By applying Lemma 4.2 to the n-fold relative Frobenius morphisms $G \to G^{(p^n)}$ and $\overline{G} \to \overline{G}^{(p^n)}$ in case $\mathrm{char}(F) = p > 0$, to prove that $G(F)\backslash G(\mathbf{A}_F)^0 \to \overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$ is proper for such F it suffices to check the analogous assertion for the map $G^{(p^n)} \to \overline{G}^{(p^n)}$ (with kernel $H^{(p^n)}$) induced via base change along the p^n-Frobenius of F for some $n \geq 0$. Using the isomorphism $F^{1/p^n} \simeq F$ induced by the p^nth-power map, we get an isomorphism of schemes $F^{1/p^n} \otimes_F H \simeq H^{(p^n)}$ (even an isomorphism of group schemes over the Frobenius isomorphism $F^{1/p^n} \simeq F$). Over the perfect closure F_p of F, the underlying reduced scheme of a finite type group scheme is smooth and hence a subgroup scheme. By expressing F_p as the direct limit of the extensions F^{1/p^n} of F, we thereby get some $n \geq 0$ such that $F^{1/p^n} \otimes_F H$ has underlying reduced scheme that is a smooth F^{1/p^n}-subgroup scheme, so for such n the underlying reduced scheme of $H^{(p^n)}$ is an F-smooth subgroup scheme. Hence, by passing to $G^{(p^n)}$ and $H^{(p^n)}$ for such n we may assume that $H_{\rm red}$ is a smooth F-subgroup of H.

Since $G/H_{\rm red} \to G/H$ is a radiciel surjective homomorphism, by Lemma 4.2 we can replace H by $H_{\rm red}$ to reduce to the case when H is smooth (and hence solvable). A solvable smooth affine group with multiplicative component group over the perfect closure F_p has a smooth split connected unipotent normal subgroup modulo which it is an extension of a finite étale multiplicative group by a torus. Thus, by repeating this same direct limit and base change argument, we may use further Frobenius base change and descent from the perfect closure to get to the case when H has a smooth split connected unipotent normal subgroup U such that H/U an extension of a finite étale multiplicative group by a torus. This is the same property of H that we have already noted is automatically satisfied in the number field case.

One final reduction step in positive characteristic is to get to the case when G (and thus $\overline{G}$) is smooth. Choose $n \geq 0$ such that $G_n = G/\ker\phi_{G/F,n}$ and $\overline{G}_n = \overline{G}/\ker\phi_{\overline{G}/F,n}$ are F-smooth. The surjective homomorphism $G_n \to \overline{G}_n$ is automatically faithfully flat since the source and target are smooth, so it expresses $\overline{G}_n$ as the quotient of G_n modulo a closed normal subgroup scheme whose $\overline{F}$-fiber has underlying reduced subgroup that is isogenous to $(H_{\overline{F}})_{\rm red}$ and hence is connected and semisimple. By Lemma 4.2 applied to $G \to G_n$ and $\overline{G} \to \overline{G}_n$ it therefore suffices to prove Theorem 4.1 when G is smooth. Now we can go back through the preceding reduction steps concerning H (which did not involve changing G beyond some extension of the base field), so we have that G and H are both smooth, and H contains a smooth split connected unipotent normal subgroup U such that H/U is an extension of a finite étale multiplicative group by a torus.

Step 2. Now working in any characteristic, H is normal in G and U must be the unipotent radical of H, so U is normal in G since G is smooth. Lemma 3.5 gives that $G(F)\backslash G(\mathbf{A}_F)$ is topologically fibered over $(G/U)(F)\backslash(G/U)(\mathbf{A}_F)$ with fibers homeomorphic to $U(F)\backslash U(\mathbf{A}_F)$. Since U is a connected smooth split unipotent group, it admits a composition series with successive quotients equal to $\mathbf{G}_a$. Thus, successive applications of Lemma 3.5 give that $U(F)\backslash U(\mathbf{A}_F)$ is compact since $F\backslash\mathbf{A}_F$ is compact. Hence,

$$G(F)\backslash G(\mathbf{A}_F) \to (G/U)(F)\backslash(G/U)(\mathbf{A}_F)$$

is proper. We may therefore replace G with G/U to reduce to the case when H is an extension of a finite étale multiplicative group by a torus T. For dimension and smoothness reasons, we must have $T = H^0$. Thus, the normality of H in G implies that of T in G, but since G is connected and the automorphism functor of a torus is étale we must have that T is in the center of G. It follows that H is a central extension of an étale multiplicative group H/H^0 by a torus T. Since G/T is connected and semisimple (as it is isogenous to G/H), H must be central in G. The group G must therefore be reductive with T the identity component of its reduced center. Moreover, since H is commutative and H/T is étale and multiplicative, it follows that H is multiplicative.

Let F'/F be a finite separable extension that splits the multiplicative H, so $H_{F'}$ is a closed subgroup of a split torus T' over F'. Consider the central pushout $G \to \widetilde{G}$ of G by the canonical closed immersion $H \to {\rm Res}_{F'/F}(H_{F'}) \to {\rm Res}_{F'/F}(T') = Z$. We have $G/H = \widetilde{G}/Z$ and the map $G \to \widetilde{G}$ is a closed immersion. Lemma 3.4 ensures that $G(F)\backslash G(\mathbf{A}_F)^0 \to \widetilde{G}(F)\backslash\widetilde{G}(\mathbf{A}_F)^0$ is a closed embedding, so we can replace $H \to G$ by $Z \to \widetilde{G}$. That is, we are reduced to the case when H is a power of ${\rm Res}_{F'/F}(\mathbf{G}_m)$ for some finite separable extension F'/F. (In particular, H is now connected.) Hence, by Lemma 3.5 and the multiplicative Hilbert Theorem 90,

$$G(F)\backslash G(\mathbf{A}_F) \to \overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$$

is a fibration whose fibers are orbits for the continuous free right action of $H(F)\backslash H(\mathbf{A}_F)$ on $G(F)\backslash G(\mathbf{A}_F)$. Lemma 3.5 also ensures that the map $H(F)\backslash H(\mathbf{A}_F) \to G(F)\backslash G(\mathbf{A}_F)$ is a closed embedding.

We now separately treat the cases of number fields and function fields, due to the different structure of the idelic norm and idelic topology in the two cases. The case $H = 1$ is trivial, so we can assume $H \neq 1$.

Step 3. Suppose F is a number field. In this case, we will show that the local cross-sections to the central fibration $G(\mathbf{A}_F) \to \overline{G}(\mathbf{A}_F)$ can be chosen to land inside of $G(\mathbf{A}_F)^0$. This will provide local cross-sections to the natural map $\pi : G(F)\backslash G(\mathbf{A}_F)^0 \to \overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$, showing that π is a fibration whose fibers are orbits for the continuous free action of $H(F)\backslash H(\mathbf{A}_F)^0$. But this latter quotient is compact since it is a power of the norm-1 subgroup of $F'^{\times}\backslash\mathbf{A}_{F'}^{\times}$. By Theorem 3.1 the properness of π will then follow. To build local cross-sections landing in $G(\mathbf{A}_F)^0$, it suffices to construct a continuous map of topological spaces $c : G(\mathbf{A}_F) \to H(\mathbf{A}_F)$ such that $c(g)^{-1}g \in G(\mathbf{A}_F)^0$ for all g.

Let $Z \subseteq H$ be the maximal F-split subtorus of H (clearly $Z \neq 1$, due to the description of H in terms of Weil restriction). Since H is the identity component of the reduced center of the reductive G, Z is the maximal F-split central torus in G. By the structure theory of reductive groups, $\mathrm{X}_F(G)$ maps isomorphically onto a finite-index subgroup of $\mathrm{X}_F(Z)$. Thus, we can choose a basis $\chi_1, \dots, \chi_r$ of $\mathrm{X}_F(Z)$ such that $\chi_1^{e_1}, \dots, \chi_r^{e_r}$ is a basis of $\mathrm{X}_F(G)$. We use the χ_j's to define an isomorphism $Z \simeq \mathbf{G}_m^r$, so upon choosing an archimedean place $v \in S$ we get a closed embedding

$$t : (\mathbf{R}_{>0}^{\times})^r \hookrightarrow (F_v^{\times})^r = Z(F_v) \hookrightarrow Z(\mathbf{A}_F) \subseteq H(\mathbf{A}_F)$$

via the canonical inclusion $\mathbf{R}_{>0}^{\times} \hookrightarrow F_v^{\times}$ for the archimedean place v. For any $g \in G(\mathbf{A}_F)$, define

$$c(g) = t(\|\chi_1^{e_1}(g)\|_F^{1/e_1}, \cdots, \|\chi_r^{e_r}(g)\|_F^{1/e_r}).$$

It is clear that c has the desired property, due to unique divisibility of $\mathbf{R}_{>0}^{\times}$ and how the χ_j's were chosen.

Step 4. Next, suppose F is a global function field with constant field of size q, so the idelic norm on $\mathbf{A}_F^{\times}$ has image $Q = q^{\mathbf{Z}}$ in $\mathbf{R}_{>0}^{\times}$. Once again, let $Z \subseteq H$ be the maximal F-split subtorus and let $\{\chi_1, \dots, \chi_r\}$ be a basis of $\mathrm{X}_F(Z)$ such that $\{\chi_1^{e_1}, \dots, \chi_r^{e_r}\}$ is a basis of the finite-index image of $\mathrm{X}_F(G)$. The continuous homomorphism

$$\Phi = (\|\chi_1^{e_1}\|_F, \dots, \|\chi_r^{e_r}\|_F) : G(\mathbf{A}_F) \to Q^{\oplus r} \subseteq (\mathbf{R}_{>0}^{\times})^{\oplus r}$$

has image equal to a subgroup $\Gamma \subseteq Q^{\oplus r}$ (even of finite index, though we do not use this fact), and the restriction of this map to $H(\mathbf{A}_F)$ has image that we denote $\Lambda \subseteq \Gamma$. The maximal split torus quotient of G admits an isogeny from the maximal split central torus Z in G, so there is a map between these tori in the other direction such that their composite is multiplication by some nonzero integer on T. Hence, Γ/Λ is killed by this nonzero integer and so Λ has finite index in Γ.

Since $G(F) \to \overline{G}(F)$ is surjective and $G(F) \subseteq G(\mathbf{A}_F)^0$, for each $\overline{g} \in \overline{G}(\mathbf{A}_F)$ the elements $g \in G(\mathbf{A}_F)$ mapping to $\overline{G}(F) \cdot \overline{g}$ in $\overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$ all give rise to the same left coset $H(\mathbf{A}_F)G(\mathbf{A}_F)^0 g$. Thus, by local constancy of the idelic norm we arrive at a natural decomposition of $\overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$ into open and closed subsets Y_j labelled by the finitely many elements $\overline{\gamma}_j \in \Gamma/\Lambda$. For each such j, let $g_j \in G(\mathbf{A}_F)$ be an element whose image in Γ/Λ is $\overline{\gamma}_j$, so we get a finite disjoint union decomposition

$$G(F)\backslash G(\mathbf{A}_F) = \coprod G(F)\backslash H(\mathbf{A}_F)G(\mathbf{A}_F)^0 g_j$$

into the open and closed preimages of the Y_j's. Define

$$E = \coprod G(\mathbf{A}_F)^0 g_j = \coprod g_j G(\mathbf{A}_F)^0,$$

an open and closed set in $G(\mathbf{A}_F)$ that is stable under left and right translations by the normal subgroup $G(\mathbf{A}_F)^0$.

Since $G(\mathbf{A}_F)$ has a topological base of compact open sets, on the open and closed subgroup

$$\Phi^{-1}(\Lambda) = H(\mathbf{A}_F)G(\mathbf{A}_F)^0 \subseteq G(\mathbf{A}_F)$$

it is trivial to construct a (locally constant) continuous map $c : \Phi^{-1}(\Lambda) \to H(\mathbf{A}_F)$ such that $c(g)^{-1}g \in G(\mathbf{A}_F)^0$ for all $g \in \Phi^{-1}(\Lambda)$. Thus, by using the continuous maps

$$g \mapsto c(gg_j^{-1})^{-1}g \in G(\mathbf{A}_F)^0 g_j$$

on each $H(\mathbf{A}_F)G(\mathbf{A}_F)^0 g_j$ and using local cross sections to $G(\mathbf{A}_F) \to \overline{G}(\mathbf{A}_F)$ we see that the restriction of the $H(F)\backslash H(\mathbf{A}_F)$-equivariant fibration

$$G(F)\backslash G(\mathbf{A}_F) \to \overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$$

to the $H(F)\backslash H(\mathbf{A}_F)^0$-stable open and closed set $G(F)\backslash E$ admits local cross-sections. The resulting map

$$\pi_E : G(F)\backslash E \to \overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$$

must therefore be a $H(F)\backslash H(\mathbf{A}_F)^0$-equivariant fibration (the topology is easy to follow because E is open and closed in $G(\mathbf{A}_F)$ and we are using quotients by discrete subgroups).

The continuous free action of $H(F)\backslash H(\mathbf{A}_F)^0$ on fibers of π_E is transitive. Thus, the fibration π_E has fibers homeomorphic to $H(F)\backslash H(\mathbf{A}_F)^0$. Since $H(F)\backslash H(\mathbf{A}_F)^0$ is compact (argue as in the number field case), the map π_E is therefore a fibration with compact fibers and thus it is proper. The restriction π of π_E to the closed subset $G(F)\backslash G(\mathbf{A}_F)^0$ is therefore also proper, as desired.

5. Proof of Theorem 1.4

Let G be a smooth affine group scheme over a global field F. To prove that G has finite class numbers, by Corollary 2.2 we may assume that G is connected.

Lemma 5.1. *Let G be a connected affine group scheme of finite type over a field k. There exists a closed normal k-subgroup scheme $H \subseteq G$ with multiplicative component group such that $(H_{\overline{k}})_{\mathrm{red}}$ is solvable and G/H is smooth and semisimple with trivial scheme-theoretic center.*

This lemma will only be needed for smooth G, but in positive characteristic it cannot generally be arranged that H is smooth even when G is. (This is the problem of the radical not being defined over an imperfect base field.) Note that the solvability hypothesis is equivalent to the group-theoretic condition that $H(\overline{k})$ is a solvable group. If we assume G is smooth then H usually cannot be assumed to be smooth (since over any imperfect field there are smooth connected affine groups whose radical is not defined over the base field, as noted at the end of Example 1.2). This is why non-reduced group schemes play a crucial role in the proof of Theorem 1.4.

Proof. First we reduce to the case of smooth G, so we may assume k has positive characteristic p. By Theorem 3.10 we can find $n \geq 0$ so that $G_n = G/\ker \phi_{G,n}$ is smooth. Any closed normal k-subgroup scheme of G_n has closed normal pullback in G, and if we can find an H_n as in the lemma for the group G_n then its pullback H in G is what we seek (since $G/H \simeq G_n/H_n$, $H(\overline{k}) \simeq H_n(\overline{k})$ is a solvable abstract group, and $H/H^0 \to H_n/H_n^0$ is a radiciel surjection between étale groups so it is an isomorphism). Hence, we now can assume G is smooth.

If k is perfect then the radical of $G_{\overline{k}}$ descends to a smooth connected k-subgroup $\mathscr{R}(G) \subseteq G$ and we can take H to be the pullback in G of the scheme-theoretic center of the connected semisimple group $G/\mathscr{R}(G)$. (Recall that the scheme-theoretic center of a connected semisimple group is always multiplicative, by the classification theory in terms of root systems.) It remains to consider imperfect k, so $\mathrm{char}(k) = p > 0$. By descent from the perfect closure of k, there exists a finite purely inseparable extension k'/k such that there is a closed smooth normal solvable k'-subgroup $H' \subseteq G_{k'}$ such that $G_{k'}/H'$ is semisimple with trivial scheme-theoretic center. For some $n \geq 0$ we have $k' \subseteq k^{1/p^n}$, so there is a natural k-group isomorphism $k \otimes_{\phi_n,k'} G_{k'} \simeq G^{(p^n)}$ where $\phi_n : k' \to k$ is $c \mapsto c^{p^n}$. Thus, H' gives rise to a closed smooth normal solvable k-subgroup $k \otimes_{\phi_n,k'} H'$ in $k \otimes_{\phi_n,k'} G_{k'} \simeq G^{(p^n)}$ with multiplicative component group and yielding a semisimple quotient having trivial scheme-theoretic center. The pullback of this subgroup under the n-fold Frobenius isogeny $G \to G^{(p^n)}$ is a closed normal k-subgroup scheme $H \subseteq G$ such that $G/H \simeq k \otimes_{\phi_n,k'} (G_{k'}/H')$ is semisimple with trivial scheme-theoretic center, and once again H/H^0 is multiplicative because a radiciel surjection between étale schemes is an isomorphism. The reduced geometric fiber $(H_{\overline{k}})_{\mathrm{red}}$ is solvable if and only if its abstract group $H(\overline{k})$ of geometric points is solvable, but the finite radiciel homomorphism $H \to k \otimes_{\phi_n,k'} H'$ arising from the definition of H induces an isomorphism on $\overline{k}$-point groups and hence we want to show that $(k \otimes_{\phi_n,k'} H')(\overline{k})$ is solvable. This latter group is isomorphic to ${H'}^{(p^n)}(\overline{k})$, and the smooth affine k'-group ${H'}^{(p^n)}$ is solvable since H' is solvable. ■

For our initial connected affine F-group G, by taking H as in Lemma 5.1 we get a connected semisimple F-group $\overline{G} = G/H$ having trivial scheme-theoretic center. That is, $\overline{G}$ is an adjoint group. The adjoint property is crucial in the proof of:

Theorem 5.2 (Borel, Prasad)**.** *A connected adjoint semisimple group over a global field has finite class numbers.*

Proof. In [3, 3.9] finiteness of class numbers is proved for any centrally isogenous quotient of a connected absolutely almost simple semisimple group over a global field. The following argument of Prasad uses this result to handle any connected adjoint semisimple group over a global field F. If we let G be such a group, then it is a finite product $\prod G_i$ with F-simple semisimple groups G_i. (The reason we have a product structure is because the simply connected member of the central k-isogeny class is a product due to [2, 22.9, 22.11, 22.15], and its scheme-theoretic center is then visibly a product too. The adjoint member of the central isogeny class is the quotient by this finite central subgroup scheme.) It suffices to treat the G_i's separately, so we may assume that G is F-simple.

In general, we claim that if k is a field and G is a connected k-simple semisimple adjoint group then $G \simeq \mathrm{Res}_{k'/k}(G')$ for some finite separable extension k'/k and connected absolutely simple semisimple adjoint group G' over k'. (Conversely, it is easy to check that any such Weil restriction is a connected k-simple semisimple algebraic group.) Granting this, in our situation we have $G \simeq \mathrm{Res}_{F'/F}(G')$ for some finite separable extension F'/F and some G' over F' to which [3, 3.9] applies; that is, G' has finite class numbers. If S is a finite non-empty set of places of F containing the archimedean places and S' is its preimage in F' then by Example A.4 we have compatible topological identifications $G(\mathbf{A}_F^S) = G'(\mathbf{A}_{F'}^{S'})$ and $G(F) = G'(F')$, so $G(F)\backslash G(\mathbf{A}_F^S) = G'(F')\backslash G'(\mathbf{A}_{F'}^{S'})$ is compact, as required. That is, G has finite class numbers.

To prove the general claim concerning Weil restrictions (which is well-known, but for which we lack a reference), we can first choose a finite Galois extension L/k such that the adjoint group G_L is a product of connected absolutely simple semisimple factors $G_{L,j}$ that are necessarily adjoint groups. This product decomposition is canonical: the $G_{L,j}$'s are the minimal closed connected normal L-subgroups of G_L with positive dimension, and they commute pairwise as subgroups of G_L. By canonicity, for each $\sigma \in \mathrm{Gal}(L/k)$ the descent data isomorphism $\sigma^*(G_L) \simeq G_L$ must carry each $\sigma^*(G_{L,j})$ to a $G_{L,j'}$, and in this way $\mathrm{Gal}(L/k)$ acts on the set of $G_{L,j}$'s. By Galois descent, the product of the factors from a single orbit descends to a k-factor of the k-simple semisimple G, so the Galois action must be transitive. Let $\Gamma \subseteq \mathrm{Gal}(L/k)$ be the isotropy group of $G_{L,1}$ for this action, so if $k' \subseteq L$ is the fixed field of Γ then by Galois descent the direct factor $G_{L,1}$ of G_L descends to a direct factor G' of $G_{k'}$ that must be absolutely simple over k' (since $G'_L = G_{L,1}$ is absolutely simple over L). The projection $G_{k'} \to G'$ corresponds to a morphism of k-groups $h : G \to \mathrm{Res}_{k'/k}(G')$. We shall prove that this is an isomorphism, or equivalently that h_L is an isomorphism. Since the extension field L/k splits k'/k, the L-group $\mathrm{Res}_{k'/k}(G')_L \simeq \mathrm{Res}_{k'\otimes_k L/L}(G'_L)$ canonically decomposes into the product of the $\mathrm{Gal}(L/k')$-twists of $G'_L = G_{L,1}$ (in accordance with the splitting of $k' \otimes_k L$, whose factors are in canonical bijection with the k-embeddings of k' into L, which is to say in bijection with the elements of $\mathrm{Gal}(L/k') = \Gamma$). For each $\gamma \in \Gamma$, h_L restricts to the identity map on the canonical factor of $\gamma^*(G_{L,1})$ in the source and target.

That h_L is an isomorphism is a special case of the more general claim that if $\{G_i\}$ and $\{G'_i\}$ are finite collections of connectged almost k-simple semisimple groups with positive dimension over a field k and

$$h : G = \prod G_i \to G'_i = G'$$

is a k-group morphism such that $h|_{G_i}$ composed with projection to G'_i is an isogeny for all i, then h is a product of isogenies $h_i : G_i \to G'_i$ (so h is an isomorphism if and only if all h_i's are isomorphisms). If $\ker h$ is not finite then $\ker(h_{\overline{k}})^0_{\mathrm{red}}$ is a closed smooth normal connected $\overline{k}$-subgroup of $G_{\overline{k}}$, so by [2, 14.10(2)] it has to contain one of the $\overline{k}$-simple factors of some $(G_i)_{\overline{k}}$. But each map $h|_{G_i} : G_i \to G' \to G'_i$ has finite kernel, so we conclude that indeed $\ker h$ is finite. Since $\dim G = \sum \dim G_i = \sum \dim G'_i = \dim G$, it follows that h is an isogeny. Thus, the $h(G_i)$'s are pairwise commuting closed k-subgroups of G' with positive dimension that generate this group and hence are normal subgroups. These are certainly minimal such k-subgroups, since each G_i is almost k-simple, so by [2, 22.10] the $h(G_i)$'s must be the G'_j's up to rearrangement as k-subgroups of G'. But since the projection $G' \to G'_i$ carries G_i onto G'_i by hypothesis, it follows that h carries G_i isogenously to $G'_i \subseteq G'$ for each i. Thus, h is the product of these isogenies $G_i \to G'_i$. ■

Choose a compact open subgroup K in $G(\mathbf{A}_F^S)$ and let K' be its image in $\overline{G}(\mathbf{A}_F^S)$. Since H may not be smooth, the compact subgroup K' may not be open. However, by shrinking K if necessary we can make K' small enough so that it lies in a compact open subgroup $\overline{K} \subseteq \overline{G}(\mathbf{A}_F^S)$. By Theorem 5.2, $\overline{G}$ has finite class numbers. Thus, there exist finitely many $y_i \in \overline{G}(\mathbf{A}_F^S)$ such that

$$\overline{G}(\mathbf{A}_F) = \bigcup_i \overline{G}(F) y_i \overline{G}(F_S)\overline{K}; \tag{3}$$

note that $\overline{K}$ and $\overline{G}(F_S)$ commute since $\mathbf{A}_F = F_S \times \mathbf{A}_F^S$. Since the natural map $\mathrm{pr}^0 : G(F)\backslash G(\mathbf{A}_F)^0 \to \overline{G}(F)\backslash\overline{G}(\mathbf{A}_F)$ is proper by Theorem 4.1, each preimage

$$(\mathrm{pr}^0)^{-1}(\overline{G}(F)\backslash\overline{G}(F)y_i\overline{K}) \subseteq G(F)\backslash G(\mathbf{A}_F)^0 \tag{4}$$

is compact in $G(F)\backslash G(\mathbf{A}_F)^0$.

For our choice of compact open subgroup K in $G(\mathbf{A}_F^S)$, the product set $G(F_S)K = K \cdot G(F_S)$ is an open subgroup of $G(\mathbf{A}_F)$. Thus, for each y_i in (3), the compact preimage in (4) is contained in a union of finitely subsets $G(F)\backslash G(F)z_{ij}G(F_S)K \subseteq G(F)\backslash G(\mathbf{A}_F)$. But $K \subseteq G(\mathbf{A}_F)^0$ since K is compact, so $G(\mathbf{A}_F)^0 G(F_S)$ is the union of the finitely many double cosets

$$G(F)z_{ij}G(F_S)K.$$

(Keep in mind that $G(F_S)$ commutes with K.) Since G is smooth, by Lemma 3.8 there is a finite set $\{x_r\}$ of representatives in $G(\mathbf{A}_F^S)$ for $G(\mathbf{A}_F)^0 \cdot G(F_S)\backslash G(\mathbf{A}_F)$. Let $K_0 \subseteq G(\mathbf{A}_F^S)$ denote the compact open subgroup $\cap_r x_r K x_r^{-1}$. If we go through the preceding argument again using K_0 in the role of K (so the set of z_{ij}'s will change: $G(\mathbf{A}_F)^0 G(F_S)$ is a union of double cosets $G(F)z_{ij}G(F_S)K_0$), then for any $g \in G(\mathbf{A}_F)$ we can write $g = g_r x_r$ for a unique r and $g_r \in G(\mathbf{A}_F)^0 G(F_S)$, and since $g_r \in G(F)z_{ij}G(F_S)K_0$ for some z_{ij} we have

$$g \in G(F)z_{ij}G(F_S)K_0 x_r \subseteq G(F)z_{ij}G(F_S)x_r K = G(F)z_{ij}x_r G(F_S)K$$

since $x_r \in G(\mathbf{A}_F^S)$ commutes with $G(F_S)$. Thus, the finite set of products $z_{ij}x_r$ represents all elements of the double coset space $\Sigma_{G,S,K}$.

Appendix A. Topology on adelic points

In [15, Ch. 1], Weil defines a process of "adelization" of algebraic varieties over global fields. We shall prove in the most general setting (i.e., without affineness hypotheses) that Weil's adelization process coincides with the set of adelic points in the sense of Grothendieck. Although we only need the affine case in the proof of Theorem 1.4, we include the generalization to the non-affine case since it is useful in other settings to consider the adelic points of G/P for connected reductive groups G and parabolic subgroups P. The reader who is familiar with the equivalence of the constructions of Weil and Grothendieck in the affine case (including Weil restriction) may wish to skip this appendix.

Let F be a global field and let S be a finite (non-empty) set of places of F (always understood to contain the archimedean places). We let $\mathbf{A}_{F,S} \subseteq \mathbf{A}_F$ denote the open subring of adeles that are integral at all places away from S, so the topological ring $\mathbf{A}_F$ is the direct limit of the open subrings $\mathbf{A}_{F,S}$ over increasing S. For a separated finite type F-scheme X, we would like to endow the set $X(\mathbf{A}_F)$ with a natural structure of Hausdorff locally compact topological space in a manner that is functorial in $\mathbf{A}_F$ and compatible with the formation of fiber products (for topological spaces and F-schemes). For affine X the coordinate ring $\Gamma(X, \mathscr{O}_X)$ is F-isomorphic to $F[t_1, \ldots, t_n]/I$, so as a set $X(\mathbf{A}_F)$ is identified with the closed subset of the adelic Euclidean space $\mathbf{A}_F^n$ where the functions $f : \mathbf{A}_F^n \to \mathbf{A}_F$ for $f \in I$ all vanish. This zero set has a locally compact subspace topology. To see that this topology transferred to $X(\mathbf{A}_F)$ is independent of the choice of presentation of $\Gamma(X, \mathscr{O}_X)$, it is more elegant to uniquely characterize this construction by means of functorial properties. The starting point is:

Theorem A.1. *Let R be a topological ring. There is a unique way to topologize $X(R)$ for affine finite type R-schemes X in a manner that is functorial in X, compatible with the formation of fiber products, carries closed immersions to topological embeddings, and for $X = \operatorname{Spec} R[t]$ gives $X(R) = R$ its usual topology.*

If R is Hausdorff, then $X(R)$ is Hausdorff and closed immersions $X \hookrightarrow X'$ induce closed embeddings $X(R) \to X'(R)$. If in addition R is locally compact, then $X(R)$ is locally compact.

The Hausdorff property is necessary to require if we want closed immersions to go over to closed embeddings. Indeed, by considered the origin in the affine line we see that such a topological property forces the identity point in R to be closed, and this forces R to be Hausdorff since (viewing R as an additive group) a topological group whose identity point is closed must be Hausdorff (because the diagonal morphism for a group object is a base change of the identity section in any category admitting fiber products).

Proof. To see uniqueness, we pick a closed immersion $i : X \hookrightarrow \operatorname{Spec} R[t_1, \dots, t_n]$. By forming the induced map on R-points and using compatibility with products (view affine n-space as product of n copies of the affine line), as well as the assumption on closed immersions, the induced set map $X(R) \hookrightarrow R^n$ is a topological embedding into R^n endowed with its usual topology. This proves the uniqueness, and that $X(R)$ has to be Hausdorff when R is Hausdorff. Likewise, we see that $X(R)$ is closed in R^n in the Hausdorff case, so when R is also locally compact then so is $X(R)$.

There remains the issue of existence, and for this we pick an R-algebra isomorphism

$$\Gamma(X, \mathscr{O}_X) \simeq R[t_1, \dots, t_n]/I \tag{5}$$

for an ideal I, and identify $X(R)$ with the subset of R^n on which the elements of I (viewed as functions $R^n \to R$) all vanish. The main issue is to check that this construction is intrinsic to X and enjoys all of the desired properties. For independence of the choice of (5), observe that if $f : X \to X'$ is a map of affine finite type R-schemes (e.g., the identity map) and we choose respective closed immersions i and i' of X and X' into affine n-space and affine n'-space over R, then f lifts to a morphism between the affine spaces carrying $i(X(R))$ into $i'(X'(R))$ on R-points of the affine spaces. Hence, the continuity of polynomial maps $R^n \to R^{n'}$ implies the well-definedness of this construction. The same argument establishes that this construction is functorial in X.

If $i : X \hookrightarrow X'$ is a closed immersion and we pick a closed immersion of X' into an affine space and use it to make the topologies on both $X(R)$ and $X'(R)$ (this being legitimate in view of the well-definedness that we have already checked), then it is clear that $i : X(R) \to X'(R)$ is an embedding of topological spaces (even a closed embedding when R is Hausdorff). By forming products of closed immersions into affine spaces, we see that $(X \times_{\operatorname{Spec} R} X')(R) \to X(R) \times X'(R)$ is a topological isomorphism via reduction to the trivial special case when X and X' are affine spaces.

Finally, to see that $(X \times_Y Z)(R) \to X(R) \times_{Y(R)} Z(R)$ is a topological isomorphism (for given R-scheme maps $X \to Y$ and $Z \to Y$), consider the isomorphism

$$X \times_Y Z \simeq (X \times_R Z) \times_{Y \times_R Y} Y$$

and its topological counterpart. Since we have already checked compatibility with absolute products (over the final object in the category), the separatedness of Y over R reduces us to the case in which one of the structure maps of the scheme fiber product is a closed immersion. But we have already seen that closed immersions are carried into topological embeddings, so we are done. ∎

Example A.2. If $R \to R'$ is a continuous map of topological rings (e.g., the inclusion of F into $\mathbf{A}_F$ or of $\mathscr{O}_{F,S}$ into $\mathbf{A}_{F,S}$, with the subring having the discrete topology in both cases), then for any affine finite type R-scheme X with base change X' to R', the natural map $X(R) \to X(R') = X'(R')$ is continuous, and when $R \to R'$ is a topological embedding then so is $X(R) \to X(R')$. Moreover, if R' is closed (resp. open) in R then $X(R) \to X(R')$ is a closed (resp. open) embedding. These claims are immediate from the construction of the topologies by means of closed immersions of X into an affine space over R (and the base change on this to give a closed immersion of X' into an affine space over R'). The same argument shows that if R is discrete in R' then $X(R)$ is discrete in $X(R')$.

Example A.3. Since F is discrete in $\mathbf{A}_F$, so F^n is discrete in $\mathbf{A}_F^n$, it follows that for any affine finite type F-scheme X, $X(F) \to X(\mathbf{A}_F)$ is a topological embedding onto a discrete subset. Similarly, if X_S is affine of finite type over $\mathscr{O}_{F,S}$, then $X(\mathscr{O}_{F,S})$ is a discrete subset of $X(\mathbf{A}_{F,S})$. If X is affine of finite type over a discrete valuation ring R with fraction field L then $X(R)$ is open in $X(L) = X_L(L)$.

Example A.4. Let $R \to R'$ be a finite ring extension that makes R' locally free as an R-module. Assume that R and R' are endowed with topological ring structures such that R' has the quotient topology from one (equivalently, any) presentation as a quotient of a finite free R-module. In particular, R has the subspace topology from R' because R' is projective as an R-module (so the inclusion $R \to R'$ admits a splitting). The main examples of interest are a finite extension of complete discrete valuation rings, local fields, or adele rings of global fields. For an affine R-scheme X of finite type with associated base change X' over R', consider the Weil restriction $\mathscr{X} = \mathrm{Res}_{R'/R}(X)$ that is an affine R-scheme of finite type [5, §7.6]. There is a canonical bijection of sets $X'(R') = \mathscr{X}(R)$, and by viewing X' and $\mathscr{X}$ as an R'-scheme and R-scheme respectively we get topologies on both sides of this equality.

To prove that these two topologies agree, using a closed immersion of X into an affine space over R reduces us to the case when X is such an affine space because Weil restriction carries closed immersions to closed immersions in the affine case. With $X = \mathbf{A}^n_R = \mathrm{Spec}(\mathrm{Sym}_R(M))$ for $M = R^{\oplus n}$, we have $X' = \mathrm{Spec}(\mathrm{Sym}_{R'}(M'))$ for $M' = R' \otimes_R M$ and $\mathscr{X}$ is naturally a closed subscheme of $\mathrm{Spec}(\mathrm{Sym}_R(M \otimes_R P))$ for a finite free R-module P equipped with a surjection onto the dual module $R'^{\vee} = \mathrm{Hom}_R(R', R)$. The set $X'(R') = \mathrm{Hom}_{R'}(M', R') = \mathrm{Hom}_R(M, R')$ is endowed with its natural topology as a finite free R'-module, and via the inclusion $R' \hookrightarrow P^{\vee}$ the set $\mathscr{X}(R)$ is $\mathrm{Hom}_R(M, R') = M^{\vee} \otimes_R R'$ with the subspace topology from $M^{\vee} \otimes_R P^{\vee}$. Thus, the agreement of topologies comes down to R' inheriting its given topology as a subspace of $P^{\vee}$. But the inclusion $R' \hookrightarrow P^{\vee}$ is split, so R' is a direct summand of $P^{\vee}$. Thus, the subspace topology on R' coincides with the quotient topology via a surjection from $P^{\vee}$, and by hypothesis such a quotient topology is the given topology on R'.

When attempting to generalize Theorem A.1 beyond the affine case, an immediate problem is that if U is an open affine in an affine X of finite type over R, then $U(R) \to X(R)$ need not be an open embedding; it may even fail to be a topological embedding. For example, if X is the affine line over F and U is the complement of the origin, then $U(R) \hookrightarrow X(R)$ is the map $R^{\times} \to R$ where R has its usual topology but $R^{\times}$ has a structure of topological group coming from the affine model $U = \mathbf{G}_m \simeq \mathrm{Spec}\, R[x,y]/(xy-1)$ inside of the plane (i.e., $r, r' \in R^{\times}$ are close when r is near r' in R and r^{-1} is near ${r'}^{-1}$ in R). The example of adele rings shows that the unit group of a topological ring need not be a topological group with respect to the induced topology from the ring. Since the topology on $R^{\times} = \mathbf{G}_m(R)$ is a topological group structure, we see that in such examples the inclusion $R^{\times} \to R$ cannot be a topological embedding.

More generally, if $X = \mathrm{Spec}\, A$ and $U = \mathrm{Spec}\, A_f$ with $f \in A$, then $U(R) \subseteq X(R)$ is the subset where $f : X(R) \to R$ is *unit*-valued – the preimage of the subset $R^{\times}$ – and this preimage might not be open. Such openness in general (for a fixed R) is equivalent to the set of non-units in R being closed, but this fails for adele rings (in which one can find sequences of non-units that converge to 1).

If R is a topological (Hausdorff) field, such as a local field, then the set of non-units consists of a single closed point $\{0\}$. More generally, since failure of openness of units is the only obstacle to basic open affines inducing open embeddings on spaces of R-points, globalization of the topology can be done when the units are open. The following result is therefore an easy exercise (using that any map from a local scheme S to a scheme X must factor through an open affine in X):

Theorem A.5. *If $R^{\times}$ is open in R, then for any open immersion $U \hookrightarrow X$ of affine finite type R-schemes, the induced map $U(R) \hookrightarrow X(R)$ is an open embedding. In such cases, if R is local then there is a unique way to topologize $X(R)$ for arbitrary locally finite type R-schemes X subject to the requirements of functoriality, carrying closed (resp. open) immersions of schemes into embeddings (resp. open embeddings) of topological spaces, compatibility with fiber products, and giving $X(R) = R$ its usual topology when X is the affine line over R.*

This agrees with the earlier construction for affines, and if R is Hausdorff then $X(R)$ is Hausdorff when X is separated over R. If R is locally compact and Hausdorff, then $X(R)$ is locally compact.

Remark A.6. If X is a locally finite type scheme over a local field k (such as $\mathbf{C}$ or $\mathbf{Q}_p$), then $X(k)$ is a locally compact topological space via Theorem A.5. The same goes for $X(\mathscr{O})$ with a compact discrete valuation ring $\mathscr{O}$ and a locally finite type $\mathscr{O}$-scheme X.

Remark A.7. If Z is closed in X and U is its open complement, then the disjoint subsets $Z(R)$ and $U(R)$ in $X(R)$ may not cover $X(R)$, even if X is affine. The problem is that "Zariski open" corresponds to a unit condition on R-points whereas "Zariski closed" corresponds to a nilpotence condition on R-points. Thus, if R contains elements that are neither nilpotent nor units then $X(R)$ may fail to be the union of $U(R)$ and $Z(R)$. More geometrically, if we consider maps $\operatorname{Spec} R \to X$ then the image might hit both Z and U. For local artinian R this does not happen since $\operatorname{Spec} R$ is one point; this is why the construction of a topology on $X(R)$ is straightforward when R is a field.

In view of the above discussion, it is a remarkable fact that when $R = \mathbf{A}_F$ is the adele ring of a global field, one can (following a method due to Weil) naturally topologize $X(R)$ for arbitrary finite type R-schemes X. It is not true in such generality that immersions of schemes are carried into topological embeddings, but the topology is functorial and compatible with fiber products, it gives closed embeddings when applied to closed immersions, and it recovers the earlier construction in the affine case. Since we will find it useful later to apply Weil's viewpoint (in the affine setting), we now present a Grothendieck-style development of Weil's construction.

The key to Weil's construction in the affine case is that if X is a finite type affine F-scheme (for a global field F) then by chasing denominators in a finite presentation of the coordinate ring of X we can find a finite set S of places and a finite-type algebra over $\mathscr{O}_{F,S}$ whose generic fiber is the coordinate ring of X. Geometrically, we can find an affine finite type $\mathscr{O}_{F,S}$-scheme X_S whose generic fiber is X. Grothendieck's technique of limits of schemes [8, IV$_3$, §8–§11] shows that an analogous result holds for all finite type F-schemes, not just the affine ones. Since $F = \varinjlim \mathscr{O}_{F,S}$ and $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S}$ with the limit taken over increasing S, the next theorem thereby provides an essential step in Weil's construction (beyond the affine case). In the statement of this theorem, a scheme over a ring R is *finitely presented* if it is covered by finitely many open affines of the form $\operatorname{Spec}(R[t_1, \dots, t_n]/(f_1, \dots, f_m))$ with quasi-compact overlaps. This coincides with the notion of finite type when R is noetherian, but the adele ring $\mathbf{A}_F$ is not noetherian.

Theorem A.8. *Let $\{A_i\}$ be a directed system of rings, with direct limit A. Let X be a finitely presented A-scheme. Then there exists some i_0 and a finitely presented A_{i_0}-scheme X_{i_0} whose base change to A is isomorphic to X.*

Moreover, if X_{i_0} and Y_{i_0} are two finitely presented A_{i_0}-schemes for some i_0, and we write X_i and Y_i to denote their base changes over A_i for all $i \geq i_0$ (and likewise for X and Y over A), then the natural map of sets

$$\varinjlim \operatorname{Hom}_{A_i}(X_i, Y_i) \to \operatorname{Hom}_A(X, Y)$$

is bijective. A map $f_{i_0} : X_{i_0} \to Y_{i_0}$ acquires property $\mathbf{P}$ upon base change to some A_i if and only if the induced map $f : X \to Y$ over A has property $\mathbf{P}$, where $\mathbf{P}$ is any of the following properties: closed immersion, separated, proper, smooth, affine, flat, open immersion, finite, fibers geometrically connected of pure dimension d.

The "smearing out" X_{i_0} of a finitely presented A-scheme X is essentially unique up to essentially unique isomorphism in the following sense: for finitely presented A_{i_0}-schemes X_{i_0} and X'_{i_0} with A-fiber X, there exists some $i \geq i_0$ and an isomorphism $h_i : X_i \simeq X'_i$ compatible with the common identification with X upon base change to A, and if h_i and H_i are two such isomorphisms then for some $i' \geq i$ the induced isomorphisms $h_{i'}$ and $H_{i'}$ are equal.

Proof. See [8, IV$_3$, §8–§11] for complete details. The full list of properties $\mathbf{P}$ that descend through limits is much longer, but we only need the ones mentioned above. ■

We now apply Theorem A.8 to a finite type F-scheme X: pick a finite set of places S so that there is a finite type $\mathscr{O}_{F,S}$-scheme X_S with generic fiber X. For any finite set of places S' of F containing S, we define $X_{S'}$ over $\mathscr{O}_{F,S'}$ by base change. Note that if we are given a morphism of $\mathscr{O}_{F,S'}$-schemes $\operatorname{Spec} \mathbf{A}_{F,S'} \to X_{S'}$ for some S', then for any finite set of places S'' of F containing S' we get an induced map of $\mathscr{O}_{F,S''}$-schemes $\operatorname{Spec} \mathbf{A}_{F,S''} \to X_{S''}$ by base change, and likewise by passing to generic fibers we get an F-scheme map $\operatorname{Spec} \mathbf{A}_F \to X$. Putting this together, we get a natural map of sets

$$\varinjlim X_{S'}(\mathbf{A}_{F,S'}) = \varinjlim X_S(\mathbf{A}_{F,S'}) \to X_S(\mathbf{A}_F) = X(\mathbf{A}_F) \tag{6}$$

(which is readily checked to equal the limit of the base change maps). In this limit we are working only with S' containing S, and increasing S at the outset has no impact. Theorem A.8 makes precise the sense in which this direct limit is intrinsic to X. For example, we leave it as a pleasant exercise (using Theorem A.8) to endow the left side of (6) with a structure of (set-valued) functor in X.

We can do better: the left side of (6) is naturally a topological space in a manner that respects functoriality in X, and (6) is bijective. Before explaining this, we note that the left side of (6) is what Weil defines to be the *adelization* of a finite type F-scheme X. It is by means of this bijection that we shall trasport a topological structure to the right side of (6) for general X, recovering the earlier topological construction for affine X.

Bijectivity of (6) is obvious for affine X, due to the obvious assertion that if $F[t_1, \dots, t_n]/(f_1, \dots, f_m) \to \mathbf{A}_F$ is a map of F-algebras then for some finite set of places S of F, the t_j's lands in some $\mathbf{A}_{F,S}$ and the f_j's have coefficients in $\mathscr{O}_{F,S}$. In order to establish bijectivity in a more global situation (i.e., not assuming X to be affine) the key point is that since $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S'}$ and X_S is of finite type over the noetherian ring $\mathscr{O}_{F,S}$, we can rewrite (6) as the natural map

$$\varinjlim \mathrm{Hom}_{\mathbf{A}_{F,S'}}(\mathrm{Spec}\, \mathbf{A}_{F,S'}, (X_S)_{\mathbf{A}_{F,S'}}) \to \mathrm{Hom}_{\mathbf{A}_F}(\mathrm{Spec}\, A_F, X_{\mathbf{A}_F}),$$

and this is a bijection by Theorem A.8 (applied over $\mathbf{A}_F = \varinjlim \mathbf{A}_{F,S'}$).

Now let us establish the topological properties of (6) that we asserted above. First we need some notation. For an $\mathscr{O}_{F,S}$-scheme X_S and a place v of F not in S (i.e., v is a maximal ideal of $\mathscr{O}_{F,S}$), we will write $X_{S,v}$ to denote the base change of X_S to the completion $\mathscr{O}_v$ at v. For any v, we write X_v to denote the base change of X_S (or $X_{S,v}$) to the fraction field F_v of $\mathscr{O}_v$.

Theorem A.9. *Let X_S be a finite type $\mathscr{O}_{F,S}$-scheme. Using the projections from $\mathbf{A}_{F,S}$ to F_v for $v \in S$ and to $\mathscr{O}_v$ for $v \notin S$, the natural map of sets*

$$X_S(\mathbf{A}_{F,S}) \to \prod_{v \in S} X_v(F_v) \times \prod_{v \notin S} X_{S,v}(\mathscr{O}_v) \tag{7}$$

is a bijection. When X is affine and we give both sides their natural topologies, using product topology on the right side, this is a homeomorphism.

In general, if we use the above bijection to define a topology on $X_S(\mathbf{A}_{F,S})$, then for any finite sets of places $S' \subseteq S''$ containing S, the natural map $X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$ is an open continuous map of topological spaces and it is injective when X_S is separated over $\mathscr{O}_{F,S}$.

In the statement of this theorem, we are using Remark A.6 to give the $X_v(F_v)$'s and $X_{S,v}(\mathscr{O}_v)$'s their natural topologies.

Proof. The bijectivity aspect amounts to the claim that a morphism of $\mathscr{O}_{F,S}$-schemes $\mathrm{Spec}\, \mathbf{A}_{F,S} \to X_S$ is uniquely determined by its restriction to the open subschemes $\mathrm{Spec}\, F_v$ ($v \in S$) and $\mathrm{Spec}\, \mathscr{O}_v$ ($v \notin S$), and that it may be constructed from such arbitrary given data. Note that the quasi-compact $\mathrm{Spec}\, \mathbf{A}_{F,S}$ is not the disjoint union of these infinitely many open affine subschemes.

To prove the injectivity of (7), consider $f, g \in X_S(\mathbf{A}_{F,S})$ that induce the same F_v-points for all $v \in S$ and the same $\mathscr{O}_v$-points for all $v \notin S$. The product map

$$(f, g) : \mathrm{Spec}\, \mathbf{A}_{F,S} \to X_S \times_{\mathscr{O}_{F,S}} X_S$$

must be factored through the diagonal morphism. Since the diagonal of X_S is an immersion, we can pick an open subscheme U in $X_S \times X_S$ around the diagonal through which $\Delta_{X_S/\mathscr{O}_{F,S}}$ factors as a closed immersion (if we assume X_S to be separated, then we can take $U = X_S \times X_S$). We will first prove that (f, g) lands in U, and that pullback along this factorized map through U carries the ideal sheaf of the diagonal closed subscheme to the ideal sheaf 0 on $\mathrm{Spec}\, \mathbf{A}_{F,S}$. This will provide a factorization through the diagonal, so $f = g$ as desired.

In order to factor (f, g) through the open U, pick a coherent ideal sheaf $\mathscr{I}$ on $X_S \times X_S$ whose zero locus is the complement of U. The desired factorization of (f, g) through the open subscheme U is a set-theoretic property, and amounts to the claim that the pullback of the coherent $\mathscr{I}$ to a quasi-coherent ideal sheaf on

$\operatorname{Spec} \mathbf{A}_{F,S}$ with no zeros: it is the unit ideal sheaf. By functoriality, the pullback of this ideal sheaf to each $\operatorname{Spec} F_v$ $(v \in S)$ and $\operatorname{Spec} \mathscr{O}_v$ $(v \notin S)$ is the unit ideal. Also, because $\mathscr{I}$ arises by pullback from a coherent sheaf, it is locally finitely generated. Under the dictionary between quasi-coherent ideal sheaves on an affine scheme and ideals in the coordinate ring, the ideal $I \subseteq \mathbf{A}_{F,S}$ associated to $\mathscr{I}$ must therefore be finitely generated.

We are reduced to the task of proving that a finitely generated ideal I in $\mathbf{A}_{F,S}$ is the unit ideal if it induces the unit ideal in each F_v $(v \in S)$ and $\mathscr{O}_v$ $(v \notin S)$. The finiteness hypothesis on I is crucial; it is easy to construct ideals in $\mathbf{A}_{F,S}$ that are not finitely generated but generate the unit ideal in each standard factor ring (consider the ideal generated by elements that have a uniformizer component in all but finitely many places). Let $a_1, \ldots, a_n$ be generators of I. By hypothesis, for each $v \in S$ one of the $a_{i,v} \in F_v$ is nonzero, and for each $v \notin S$, one of the $a_{i,v} \in \mathscr{O}_v$ is a unit. Since there are just finitely many of these a_i's, it is easy to make an $\mathbf{A}_{F,S}$-linear combination which is equal to 1. Thus, $I = (1)$.

This proves that (f, g) factors through U, and now we run through a similar argument with pullback of the coherent ideal sheaf of the diagonal closed subscheme in U: the algebraic problem is to show that if I is an ideal in $\mathbf{A}_{F,S}$ that projects to 0 in F_v for $v \in S$ and projects to 0 in $\mathscr{O}_v$ for each $v \notin S$, then $I = 0$. But this is trivial, since $\mathbf{A}_{F,S}$ is the product of such F_v's and $\mathscr{O}_v$'s. This completes the proof that (7) is injective.

Now we prove that (7) is surjective. Assume we are given maps $x_v : \operatorname{Spec} F_v \to X_v$ for $v \in S$ and $x_v : \operatorname{Spec} \mathscr{O}_v \to X_{S,v}$ for all $v \notin S$ (all maps over $\mathscr{O}_{F,S}$). We claim that there exists $x \in X_S(\mathbf{A}_{F,S})$ inducing the given local data. Let $\{U_i\}$ be a finite open affine covering of X_S. Since each x_v is a point valued in a local ring, its image lands in some U_i (chase the closed point). Pick one for each v, and let V_i be the set of v's for which we have selected U_i as an open affine through which x_v factors. We have a natural finite product decomposition $\mathbf{A}_{F,S} = \prod_i \mathbf{A}_{F,V_i}$, where $\mathbf{A}_{F,V_i}$ is the subproduct of the product ring $\mathbf{A}_{F,S}$ corresponding to local factors for places $v \in V_i$. Since the Spec functor carries finite products into disjoint unions, we may focus on each $\mathbf{A}_{F,V_i}$ separately. Since U_i is affine, we are reduced to the claim that if $\phi_\alpha : \operatorname{Spec} R_\alpha \to \operatorname{Spec} B$ are maps of affine schemes over some affine base $\operatorname{Spec} C$, then there exists a map of C-schemes $\phi : \operatorname{Spec}(\prod R_\alpha) \to \operatorname{Spec} B$ inducing each ϕ_α. By restating in terms of ring maps, this is obvious.

Now that (7) is proved to be a bijection, we may use the product topology on its target to endow $X_S(\mathbf{A}_{F,S})$ with a topology. For affine X_S, this recovers the topology constructed earlier: by using a finite presentation of the coordinate ring of X_S as an $\mathscr{O}_{F,S}$-algebra, and recalling how the topology on points of affine schemes (of finite type) was defined by means of embeddings into affine spaces, the problem down to the claim that the product topology on $\mathbf{A}^n_{F,S}$ agrees with the product topology on

$$\prod_{v \in S} F_v^n \times \prod_{v \notin S} \mathscr{O}_v^n.$$

This is trivial.

Finally, we have to check that if $S' \subseteq S''$ is an inclusion of finite sets of places of F containing S, then the map $X_{S'}(\mathbf{A}_{F,S'}) \to X_{S''}(\mathbf{A}_{F,S''})$ is an open embedding of topological spaces. Via (7), this map is (topologically) the product of three maps: the identity maps on $\prod_{v \in S'} X_v(F_v)$ and on $\prod_{v \notin S''} X_{S,v}(\mathscr{O}_v)$, and the base change map

$$\prod_{v \in S''-S'} X_{S,v}(\mathscr{O}_v) \to \prod_{v \in S''-S'} X_v(F_v).$$

Thus, we are reduced to show that for $v \notin S$, the natural map $X_{S,v}(\mathscr{O}_v) \to X_v(F_v)$ is continuous and open, and even injective when X_S is separated. The injectivity for separated X_S follows from the valuative criterion for separatedness, so we just have to check continuity and openness.

In general, for a finite type scheme X over a complete discrete valuation ring $\mathscr{O}$ with fraction field K given its natural topology, we claim that $X(\mathscr{O}) \to X_K(K)$ is a continuous open map. If U is an open subscheme of X, then by Theorem A.5, $U(\mathscr{O})$ is open in $X(\mathscr{O})$. Since $X(\mathscr{O})$ is the union of the $U_i(\mathscr{O})$'s for $\{U_i\}$ an open covering of X, our problem is of local nature on X. Hence, we may assume X is affine. By picking a closed immersion of X into an affine space over $\mathscr{O}$, the fact that $\mathscr{O}^n$ is open in K^n then provides what we need. ∎

Using Theorem A.9 to topologize $X_S(\mathbf{A}_{F,S})$ for finite type $\mathscr{O}_{F,S}$-schemes X_S, it is immediate from the construction that this topology is functorial in X_S, has a countable base of opens, carries fiber products into fiber products, and carries closed immersions into closed embeddings (use Theorem A.5 and the fact that an arbitrary product of closed embeddings is a closed embedding). For open immersions $U_S \hookrightarrow X_S$ it is not true in general that $U_S(\mathbf{A}_{F,S}) \to X_S(\mathbf{A}_{F,S})$ is an open embedding, though it is a topological embedding. Indeed, an arbitrary product of open embeddings is a topological embedding but usually does not have open image. This is the reason that the construction of the topology on $X_S(\mathbf{A}_{F,S})$ in the non-affine case had to be done globally via the product decomposition in (7), without trying to glue topologies coming from open affines in X_S.

Corollary A.10. *Let X_S be a finite type $\mathscr{O}_{F,S}$-scheme. The topological space $X_S(\mathbf{A}_{F,S})$ is locally compact, and is Hausdorff when X_S is separated.*

Proof. Since our topology construction does commute with products and carries closed immersions to closed embeddings, it is clear that if X_S is separated then $X_S(\mathbf{A}_{F,S})$ is Hausdorff. As for local compactness, we want the infinite product space $X_S(\mathbf{A}_{F,S})$ to be locally compact. Since the factor spaces $X_v(F_v)$ are locally compact for $v \in S$, we just have to check that $X_{S,v}(\mathscr{O}_v)$ is compact for $v \notin S$. More generally, for any compact discrete valuation ring R and any finite type R-scheme X, we claim $X(R)$ is compact. Indeed, Theorem A.5 shows that for a finite open affine covering $\{U_i\}$ of X the spaces $\{U_i(R)\}$ form a finite open covering of $X(R)$, so the problem comes down to the affine case, which in turn is reduced to the trivial case of affine space (R^n is compact since R is compact). ∎

Let X be a finite type F-scheme. We use Theorems A.8 and A.9 with the bijection (6) to give $X(\mathbf{A}_F)$ a functorial topological structure. To make sense of this, we need to briefly recall how one topologizes direct limits. If $\{T_\alpha\}$ is a directed system of topological spaces, with direct limit set T as sets, we declare $U \subseteq T$ to be open if and only if the preimage of U in each T_α is open. This is readily checked to be a direct limit in the topological category. In general such abstract topologies are hard to handle. However, the case when transition maps are open involves no subtlety: if $T_\alpha \to T_{\alpha'}$ is an open continuous map for all $\alpha' \geq \alpha$, then T is the directed union of the images U_α of the T_α's, and by giving each U_α the quotient topology from T_α it is clear that the topology on T is characterized by declaring the topological spaces U_α to be open subspaces.

Since the behavior of quotient topologies with respect to fiber products (or even absolute products) is subtle in general, the topology on $X(\mathbf{A}_F)$ is probably rather hard to work with unless we impose a hypothesis on X to ensure injectivity of the transition maps in the limit of $X_{S'}(\mathbf{A}_{F,S'})$'s. As we see from the final part of Theorem A.9, as well as Theorem A.8, assuming X to be separated over F ensures such injectivity. It is a pleasant exercise to check that when X is F-separated, then (6) expresses $X(\mathbf{A}_F)$ as a direct limit of locally compact Hausdorff spaces with transition maps that are open embeddings. In this way, we see that $X(\mathbf{A}_F)$ is locally compact and Hausdorff (with a countable base of opens) when X is F-separated, and moreover that this topology is compatible with fiber products. The functor $X \rightsquigarrow X(\mathbf{A}_F)$ does not generally carry open immersions into topological embeddings, but closed immersions do go over to closed embeddings of topological spaces (due to openness of the transition maps in the above topological direct limits).

To summarize: for a finite type separated F-scheme X, the set $X(\mathbf{A}_F)$ acquires a functorial structure of locally compact Hausdorff topological space with a countable base of opens, and this topological construction is compatible with fiber products and carries closed immersions between such F-schemes into closed embeddings of topological spaces. Moreover, if X is the generic fiber of a separated finite type $\mathscr{O}_{F,S}$-scheme X_S, then $X_S(\mathbf{A}_{F,S})$ is naturally an open subset of $X(\mathbf{A}_F)$. As a special case, when X is a group scheme of finite type over F (automatically separated), the set $X(\mathbf{A}_F)$ is naturally a locally compact Hausdorff topological group.

Example A.11. In [12] it is said that if $\{U_i\}$ is an open affine cover of X then $\{U_i(\mathbf{A}_F)\}$ covers $X(\mathbf{A}_F)$ set-theoretically. This is false because the image of a morphism $\operatorname{Spec} \mathbf{A}_F \to X$ need not be contained in any of the U_i's. Moreover, the set $\cup U_i(\mathbf{A}_F)$ inside of $\prod_v X(F_v)$ is not independent of $\{U_i\}$ in general, and in particular it is not intrinsic to X.

Example A.12. Let $F \to F'$ be a finite extension of global fields, and X a quasi-projective F-scheme. Let X' be the base change of X, and let $\mathscr{X}$ denote the Weil restriction $\mathrm{Res}_{F'/F}(X')$, which exists and is separated and finite type over F [5, pp. 194–196]. In this case we have $\mathscr{X}(\mathbf{A}_F) = X'(\mathbf{A}_{F'})$ as sets, so it is natural to ask if we have equality as topological spaces. In the affine case this follows from Example A.4 (applied to the base changes of X' and $\mathscr{X}$ over $R' = \mathbf{A}_{F'}$ and $R = \mathbf{A}_F$ respectively). In the general case, we fix a finite set S_0 of places of F such that X extends to a quasi-projective $\mathscr{O}_{F,S_0}$-scheme X_{S_0}, and we let S_0' be the preimage of S_0 in F'. We write $X'_{S_0'}$ to denote the base change of X_{S_0}. Thus, $\mathrm{Res}_{\mathscr{O}_{F',S_0'}/\mathscr{O}_{F,S_0}}(X'_{S_0'})$ exists as a finite type and separated $\mathscr{O}_{F,S_0}$-scheme $\mathscr{X}_0$, and $\mathscr{X}_0(\mathbf{A}_{F,S}) = X'_{S_0'}(\mathbf{A}_{F',S'})$ as sets for any finite set of places S of F containing S_0 and for its preimage S' in F'. By the definition of the topology on the adelic points as a direct limit with open transition maps, the problem of topological equality is reduced to checking that the equality $\mathscr{X}_0(\mathbf{A}_{F,S_0}) = X'_{S_0'}(\mathbf{A}_{F,S_0'})$ (for general S_0) is a homeomorphism.

These topologies are defined as product topologies, and so the problem reduces to checking that for each place $v \in S_0$ the equality of sets $\prod_{v'|v} X'(F'_{v'}) = \mathrm{Res}_{F'/F}(X')(F_v)$ is a homeomorphism and for each place v of F not in S_0 the equality of sets

$$\prod_{v'|v} X'_{S_0'}(\mathscr{O}_{F',v'}) = \mathrm{Res}_{\mathscr{O}_{F',S_0'}/\mathscr{O}_{F,S_0}}(X'_{S_0'})(\mathscr{O}_{F,v})$$

is a homeomorphism. Since any finite subset of $X'_{S_0'}$ lies in an open affine and $\prod_{v'|v} \mathscr{O}_{F',v'}$ is semi-local, the construction of these Weil restrictions in terms of affine opens reduces us the case when $X'_{S_0'}$ is affine. We can then apply Example A.4 with the ring extensions $F_v \to F' \otimes_F F_v = \prod_{v'|v} F'_{v'}$ and

$$\mathscr{O}_{F,v} \to \mathscr{O}_{F',S_0'} \otimes_{\mathscr{O}_{F,S_0}} \mathscr{O}_{F,v} = \prod_{v'|v} \mathscr{O}_{F',v'}.$$

This concludes the verification that Weil restriction is compatible with the topology on adelic points (in the quasi-projective case).

Though Example A.2 shows that $X(F)$ is a discrete closed set in $X(\mathbf{A}_F)$ for finite type affine F-schemes X (as F is discrete and closed in $\mathbf{A}_F$), globalizing to the non-affine case usually destroys such properties. For example, consider the sequence (x_n) in $\mathbf{P}^1(\mathbf{A}_\mathbf{Q})$ where $x_n = [a_n, b_n]$ with $a_n, b_n \in \mathbf{Z} - \{0,1\}$ having a_n converging to zero in $\mathbf{Q}_p$ for half the primes, and converging to 1 (though local unit values) in the other non-archimedean factors, and vice-versa for b_n. Using the bijection in Theorem A.9 we see

$$x_n \in \mathbf{P}^1(\mathbf{Q}) \subseteq \mathbf{P}^1(\mathbf{A}_{\mathbf{Q},\infty}) = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}^1(\widehat{\mathbf{Z}}) = \mathbf{P}^1(\mathbf{R}) \times \prod_p \mathbf{P}^1(\mathbf{Z}_p) = \mathbf{P}^1(\mathbf{R}) \times \prod_p \mathbf{P}^1(\mathbf{Q}_p)$$

for all n, with the infinite product describing the topology. This is a sequence of rational points that converges to $[0,1]$ in half of the non-archimedean factors, and to $[1,0]$ in the others (and we may pass to a subsequence to also get convergence in the compact archimedean factor $\mathbf{P}^1(\mathbf{R})$). Thus, the limit is a point $[e, e']$ with $e, e' \in \mathbf{A}_\mathbf{Q}$ mutually orthogonal non-trivial idempotents away from ∞. No $\mathbf{A}_\mathbf{Q}^\times$-scaling can bring this to a pair of homogeneous coordinates in $\mathbf{Q}$. Thus, the limit is not a $\mathbf{Q}$-rational point. This shows that the set of rational points inside of the adelic points need not be closed in general (for non-affine X). However, the induced topology on the set of rational points is always discrete:

Theorem A.13. *Let X be a separated finite type F-scheme. The natural map $X(F) \to X(\mathbf{A}_F)$ is injective and the induced topology on $X(F)$ is the discrete topology.*

The failure of $X(F)$ to be closed in $X(\mathbf{A}_F)$ is analogous to the discrete subspace $\{1/2^n\}$ inside $\mathbf{R}$.

Proof. The injectivity is obvious (e.g., since $\mathbf{A}_F$ is faithfully flat over F). Since $X(\mathbf{A}_F)$ is a Hausdorff space with a countable base of opens, the discreteness of the induced topology on $X(F)$ amounts to the fact that if (x_n) is a sequence in $X(F)$ converging in $X(\mathbf{A}_F)$ to some $x \in X(F)$, then $x_n = x$ for large n. To verify this fact, pick an open affine U in X with $x \in U(F)$. Choose a place v_0. The projection $X(\mathbf{A}_F) \to X(F_{v_0})$ is clearly continuous, so $x_n \to x$ in $X(F_{v_0})$. But $U(F_{v_0})$ is open in $X(F_{v_0})$, so for large n we have $x_n \in U(F_{v_0}) \cap X(F)$ inside of $X(F_{v_0})$. This overlap is obviously equal to $U(F)$, so by dropping some initial terms we have $x_n \in U(F)$ for all n. Thus, the convergence $x_n \to x$ in $X(\mathbf{A}_F)$ takes place inside of the subset $U(\mathbf{A}_F)$. But $U(\mathbf{A}_F) \to X(\mathbf{A}_F)$ is a topological embedding, so $x_n \to x$ inside the topological space $U(\mathbf{A}_F)$. Since U is affine, $U(F)$ is discrete and closed in $U(\mathbf{A}_F)$. Thus, $x_n = x$ for large n. ∎

Theorem A.14. *Assume X is a proper F-scheme. The locally compact Hausdorff topological space $X(\mathbf{A}_F)$ is compact. In particular, if X_S is a finite type $\mathscr{O}_{F,S}$-scheme with generic fiber X, then $X(\mathbf{A}_F) = X_{S'}(\mathbf{A}_{F,S'})$ for every sufficiently large finite set of places S' of F that contains S.*

The subset $X(F)$ is closed in $X(\mathbf{A}_F)$ if and only if it is finite.

Proof. By increasing S if necessary, Theorem A.8 shows that we can assume X_S is proper over $\mathscr{O}_{F,S}$. For any $v \notin S$, the valuative criterion for properness ensures that the map $X_{S,v}(\mathscr{O}_v) \to X_v(F_v)$ is bijective. Near the end of the proof of Theorem A.9 we saw that this map is also an open embedding, so it is a homeomorphism. It then follows that for any finite S' containing S, the natural map $X_S(\mathbf{A}_{F,S}) \to X_{S'}(\mathbf{A}_{F,S'})$ is a homeomorphism. Thus, passing to the direct limit on these homeomorphisms shows that $X_S(\mathbf{A}_{F,S}) \to X(\mathbf{A}_F)$ is a homeomorphism. It remains to prove that $X_S(\mathbf{A}_{F,S})$ is compact. Since $X_S(\mathbf{A}_{F,S})$ is a topological product of the spaces $X_v(F_v)$ for $v \in S$ and the spaces $X_{S,v}(\mathscr{O}_v) \simeq X_v(F_v)$ for $v \notin S$, we just have to prove that if X is a proper scheme over a locally compact field K, then the topological space $X(K)$ is compact.

First assume K is non-archimedean. If $\mathscr{O}$ denotes the compact valuation ring of K and there is a closed immersion $X \hookrightarrow X'_K$, where X'_K is the generic fiber of a proper $\mathscr{O}$-scheme X', then topologically $X'_K(K) = X'(\mathscr{O})$, and $X(K)$ is closed in here. Since $X'(\mathscr{O})$ is compact (being covered by $U_i(\mathscr{O})$ for finitely many open affines U_i, for which $U_i(\mathscr{O})$ is closed in some compact $\mathscr{O}^{n_i}$), the compactness of $X(K)$ would follow.

Thus, the problem comes down to finding a closed immersion of X into the generic fiber of a proper $\mathscr{O}$-scheme X'. If X is projective, then we can take X' to be a projective space over $\mathscr{O}$. In general, the Nagata compactification theorem [9] ensures that any separated finite type scheme over a noetherian base Y admits an open immersion into a proper Y-scheme. Thus, there is an open immersion of X into a proper $\mathscr{O}$-scheme X'. By replacing X' with the scheme-theoretic image of this quasi-compact open immersion, we may assume X is a schematically dense open in X'. This property is preserved by flat base change to the generic fiber, so X is a schemetically dense open in the K-proper X'_K. But by hypothesis X is K-proper, so by separatedness of X'_K we see that the open immersion $X \hookrightarrow X'_K$ is a closed immersion. This is also schematically dense, so the kernel ideal sheaf must vanish. Thus, $X \simeq X'_K$, as desired.

In the archimedean case, if $K = \mathbf{R}$ then $X(\mathbf{R})$ is a closed subset of $X_{\mathbf{C}}(\mathbf{C})$, so we may reduce to the case $K = \mathbf{C}$. In this case, Chow's Lemma provides a projective $\mathbf{C}$-scheme X' and a surjection of schemes $X' \to X$. The continuous map $X'(\mathbf{C}) \to X(\mathbf{C})$ is surjective, so it suffices to prove that X' is compact. The closed immersion $X' \hookrightarrow \mathbf{P}^n_{\mathbf{C}}$ induces a closed embedding of $X'(\mathbf{C})$ into $\mathbf{P}^n(\mathbf{C})$, which is compact.

With $X(\mathbf{A}_F)$ now proved to be compact Hausdorff, if $X(F)$ is finite then it is certainly a closed subset, and if $X(F)$ is a closed subset then it is compact with the discrete topology, so it has to be finite. ∎

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