The Patching Argument

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1 Motivating the Patching Argument

My main references for this talk were Andrew's overview notes and Kisin's paper "Moduli of Finite Flat Group Schemes." I also would like to thank Andrew for his help and letting me incorporate some of his Tex code which saved me time and energy.

Since this is the final lecture in this seminar, we begin by stating the theorem we set out to prove.

Theorem 1.1. *Let $F/\mathbb{Q}$ be a totally real number field and let $\rho : G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be a continuous representation of its absolute Galois group, with $p > 5$. Assume that ρ satisfies the following conditions:*

- *ρ ramifies at only finitely many places.*
- *ρ is odd, i.e., $\det \rho(c) = -1$ for all complex conjugations $c \in G_F$.*
- *ρ is potentially crystalline and ordinary at all places above p.*
- *$\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.*
- *There exists a parallel weight two Hilbert modular form f such that ρ_f is potentially crystalline and ordinary at all places above p and $\bar{\rho} = \bar{\rho}_f$.*

Then there exists a Hilbert modular form g such that $\rho = \rho_g$.

As explained in Andrew's Lecture 18, we can make a solvable totally real base changes to arrange so that ρ is crystalline and ordinary at all places dividing p and that ρ is Steinberg at all places where it is ramified. We are careful to choose a solvable extension preserving the absolute irreducibility condition. Solvable base change ensures that modularity of this new ρ implies modularity of the one we started with. All the other conditions are preserved. For convenience, we also base change so that the number of real places of F and the number of Steinberg places of ρ are both even.

The techniques of level lowering and level raising as discussed in Akshay's most recent talk allows us find a new Hilbert modular form f' with the same reduction mod p whose level exactly matches with ρ, that is, f' is ramified only where ρ is Steinberg and is Steinberg there, and further f' is ordinary at all places dividing p. Some further reductions are discussed in Andrew's notes to get ourselves to the following situation:

Theorem 1.2. *We have a representation*

$$\bar{\rho} : G_F \to \mathrm{GL}_2(k)$$

where k is a finite field of characteristic p, a finite set St of places of F away from p and a modular representation ρ_f lifting $\bar{\rho}$. Let S_p denote the places of F above p. We assume the following hypotheses:

(A1) ρ_f is crystalline and ordinary at all places in S_p, Steinberg at all places in St and unramified at all other places.

(A2) $\det \rho_f = \chi_p$.

(A3) $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.

(A4) $\bar{\rho}|_{G_{F_v}}$ is trivial for $v \in S_p \cup St$.

(A5) F has even degree over $\mathbb{Q}$ and St has even cardinailty.

Then ρ is modular.

As we go through the patching argument, the question may arise: why would one think to do it this way? The argument feels very unnatural. The best I can do is to point to similar argument from Iwasawa theory which arose much more naturally and undoubtably inspired this one. If you are unfamiliar with Iwasawa theory or not interested, you may skip to the next section.

Everything I say can be found in Washington's book *Cyclotomic Fields* in much more detail. The important first case in Iwasawa theory is the study of the p-part of the class group of the cyclotomic fields $\mathbb{Q}(\zeta_{p^n}) = K_n$. Let M_n be the p-part of the class group K_n. One first observes that M_n comes with an action of $\mathrm{Gal}(K_n/\mathbb{Q})$. For our brief discussion, the relevant action is that of $\mathrm{Gal}(K_n/K_1)$ which is a cyclic group of order p^{n-1}. Furthermore, we think of $\mathrm{Gal}(K_n/K_1)$ as a quotient of $\Gamma = \mathrm{Gal}(K_\infty/K_1)$. So that Γ acts on all the M_n's.

Now, there exists maps of abelian groups

$$M_{n+1} \to M_n.$$

One can think interpret this map as norm map K_{n+1}/K_n either at the level of ideal class groups or at the level of ideles. Or an alternative description exists in terms of Hilbert class fields. It is not hard to show the map is both surjective and Γ-equivariant.

Each M_n is an abelian p-group with the structure of a $\mathbb{Z}_p[\Gamma/\Gamma^{p^{n-1}}]$-module. Hence the projective limit

$$M_\infty := \lim_{\leftarrow} M_n$$

is naturally a module over the completed group ring

$$\lim_{\leftarrow} \mathbb{Z}_p[\Gamma/\Gamma^{p^{n-1}}] = \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$$

the last isomorphism being given by sending a topological generator for Γ to $1+T$.

The value of this limiting process is that while the M_n themselves may be very mysterious, there is a nice structure theory for $\mathbb{Z}_p[[T]]$-modules which can be applied to M_∞. Knowing this and the fact that M_n can be recovered as quotient of M_∞ by an augmentation ideal, allows one to derive strong results about how $|M_n|$ grows as we go up the tower.

Idea of the Proof: As both Mike and Sam have discussed, given a set of Taylor-Wiles primes Q, the corresponding deformation ring and space of modular forms have an action of $\mathcal{O}[\Delta_Q]$. The naive idea would be to take a limit where the Taylor-Wiles sets become congruent to 1 modulo higher and higher powers of p. The limiting "deformation ring" and "space of forms" then become modules over a power series in $|Q|$ variables. One could then use commutative algebras results which only work over domains to deduce that $R = T$.

One big problem not present in the Iwasawa setup is that these TW-sets have absolutely no relationship to each other. There will be no obvious maps from M_{n+1} to M_n, and this will lead to the "miracle" that is patching argument. While the Iwasawa construction moves vertically, the Taylor-Wiles construction is "horizontal" one and will involve many more choices, but nevertheless, in the end, it works and we can recover facts about our original R and T in exactly the same way one does in Iwasawa theory.

2 Setup/Recollection of Previous Results

There is unfortunately a huge amount of notation to set up here so let's get started.

2.1 Deformation Rings

All our deformation rings will be algebras over $\mathcal{O}$ which is the ring of integers of its fraction field E, a p-adic field. We will denote the residue field of $\mathcal{O}$ by $\mathbb{F}$ and assume the $\bar{\rho}$ is representation over $\mathbb{F}$.

In this section, we add in the framings we need and recall the necessary dimension formulas.

Let $\tilde{R}^\square$ be the universal deformation ring of $\bar{\rho}$ unramified outside $S_p \cup St$ together with framings at each place in $S_p \cup St$. The essential thing is that $\tilde{R}^\square$ is naturally an algebra over the universal local framed deformation ring $\tilde{R}_v$ of $\bar{\rho}|_{G_{F_v}}$ for all $v \in S_p \cup St$. All our local deformation rings will be framed so I leave off the box.

The functor of forgetting the framings makes $\tilde{R}^\square$ an algebra over the plain old global deformation ring $\tilde{R}$ which exists since $\bar{\rho}$ is absolutely irreducible.

We now state the first important dimension formula which was discussed in the fall:

(R1) $\tilde{R}^\square$ is smooth over $\tilde{R}$ of relative dimension $j = 4|S_p \cup St| - 1$.

Note that we need these framings at $v \in S_p \cup St$ because otherwise the local deformation rings may not be representable. In fact, as Andrew points out in his overview we can actually assume that $\bar{\rho}|_{G_{F_v}}$ is trivial.

Now let R_v be the quotient of $\tilde{R}_v$ which represents the crystalline-ordinary (respectively Steinberg component) of the local deformation ring at $v \in S_p \cup St$ as discussed in Rebecca and Brian's (Corollary 4.3) talks earlier this quarter.

Set $\tilde{B} = \hat{\otimes}_{v \in St \cup S_p} \tilde{R}_v$ and $B = \hat{\otimes}_{v \in St \cup S_p} R_v$, and note that B is a quotient of $\tilde{B}$.

The relevant properties of the R_v are

- R_v is a flat $\mathcal{O}$-algebra (also complete local with residue field $\mathbb{F}$).
- R_v is a domain (i.e. $\mathrm{Spec}\, R_v$ is connected).
- $R_v[1/p]$ is a regular E-algebra.
- R_v has relative dimension 3 for $v \in St$ and $3 + [F_v : Q_p]$ for $v \in S_p$ over $\mathcal{O}$.

These were all discussed or proved in previous lectures except I believe the dimension formula which was only discussed for $R_v[1/p]$ but should be in the notes. For later applications, we will also want to know that R_v has the same dimension at all maximal ideals. This is Lemma 4.6 in Brian's notes from this quarter.

Proposition 2.1. *(B1) B is a flat $\mathcal{O}$-algebra (also complete local with residue field $\mathbb{F}$).*

(B2) B is a domain of relative dimension $3|S_p \cup St| + [F : Q]$ over $\mathcal{O}$.

(B3) $B[1/p]$ is a regular E-algebra.

Proof. Since we can build B up one step at a time, for simplicity, let R_1 and R_2 complete local $\mathcal{O}$-algebras that are domains and which become formally smooth after inverting p. Because the formation of the local deformation rings commutes with any finite extension $\mathcal{O}'$ of $\mathcal{O}$, we can further assume that same properties hold for $R_i \otimes_{\mathcal{O}} \mathcal{O}'$.

Let $A = R_1 \hat{\otimes} R_2$. For (B1), it suffices to show that A if flat over R_1 since R_1 is flat over $\mathcal{O}$. Note that $R_1 \to A$ is a local map of complete local Noetherian rings. By Prop 5.1, it suffices to show that $A/m_1^n A$ is flat over R_1/m_1^n where m_1 is the maximal ideal of R_1. However, once you quotient by m_1^n the completed tensor product goes away and we get

$$A/m_1^n A \cong R_1/m_1^n \otimes_{\mathcal{O}} R_2$$

which is clearly flat over R_1/m_1^n since R_2 is flat over $\mathcal{O}$.

Since A is $\mathcal{O}$-flat, it is p-torsion free and so $A \hookrightarrow A[1/p]$. Thus for (B2) and (B3), it suffices to show $A[1/p]$ is a regular domain. Intuitively, one might think of $R_i[1/p]$ as being bounded functions on the corresponding rigid analytic space which we will call X_i. This is not quite true, but in any case, if it were, then $A[1/p]$ would be bounded functions on the rigid analytic product space $X_1 \times X_2$. And we are reduced to the statement that the product of smooth spaces is smooth and product of geometrically connected is geometrically connected. This motivates why one might believe modulo lots of technical details that it would be true. Now I give the more hands-on algebraic proof.

Let $X_1 = \mathrm{Spec}\ R_1[1/p], X_2 = \mathrm{Spec}\ R_2[1/p]$, and $X = \mathrm{Spec}\ A[1/p]$. The rough idea is that while X is very far from being the product of X_1 and X_2, it looks like a product at the level of MaxSpec and this turns out to be enough. This is made precise in Lemma 5.2.

Recall also the following essential facts from Brian's lecture "Generic Fibers of Deformation Rings":

1. $A[1/p], R_1[1/p], R_2[1/p]$ are all Noetherian and Jacobsen rings. In particular, their closed points are dense in their spectrum.

2. All maximal ideal of $A[1/p], R_1[1/p], R_2[1/p]$ have a residue field a finite extension of E.

3. Under any homomorphism $R_i[1/p] \to E'$ where E' is finite extension of E, R_i lands in the ring of integers of E'.

Regularity (B3) for Noetherian ring over a field can be checked by a functorial criterion on Artin local E-algebras (see Remark 2.3 in Lecture 21). Thus, it follows immediately from Lemma 5.2. Knowing regularity, we get the dimension count by applying Lemma 5.2 to the dual numbers over E. It remains to show the X is connected.

There are natural projections $\pi_i : X \to X_i$ given by the evident ring inclusions. Further, given any rational point $x \in X_2(E)$, we get a section $s_x : X_1 \to X$. Our point x corresponds to a map $R_2[1/p] \to E$ which from fact (3), gives rise to a map

$$R_2 \to \mathcal{O}$$

which induces a map

$$R_1 \hat{\otimes}_{\mathcal{O}} R_2 \to R_1$$

which after inverting p yields s_x. Note that by construction s_x is a closed immersion onto the fiber $\pi_2^{-1}(x)$.

We can now show X is irreducible. Assume that U and V were two disjoint non-empty open subsets of X. After extending the field if necessary, we can assume U and V contain rational points u and v. Let s_u and s_v be sections passing through u and v respectively. Since X_1 is irreducible, $s_u^{-1}(U) \cap s_v^{-1}(V)$ is non-empty. Again extending the field, we can assume it contains a rational point y. Consider the fiber $X_y = \pi_1^{-1}(y)$. Both U and V intersect X_y, a closed subset of X. By remark above, X_y is the image of some section s_y and hence irreducible because X_2 is. $\square$

The universal deformation ring for $\bar{\rho}$ unramified outside $S_p \cup St$, crystalline and ordinary in S_p, and Steinberg in St with local framings is given by

$$R^{\square} = \tilde{R}^{\square} \hat{\otimes}_{\tilde{B}} B.$$

Intuitively, one should just think the B is gotten from $\tilde{B}$ by the univeral equations forcing the desired local properties and all that we are doing here is just applying those universal conditions to the global deformation ring.

Proposition 2.2. *Let*

$$g = \dim(\ker(H^1(G_{F_{S_p \cup St}}, \mathrm{ad}^0) \to \oplus_{v \in S_p \cup St} H^1(G_{F_v}, \mathrm{ad}^0)) + \sum_{v \in S_p \cup St} \dim H^0(F_v, \mathrm{ad}) - \dim H^0(G_{F_{S_p \cup St}}, \mathrm{ad}) \tag{1}$$

Then, $R^{\square}$ can be written as a quotient of $B[[x_1, \dots, x_g]]$.

Proof. The quantity g is exactly the number of generators of $\tilde{R}$ over $\tilde{B}$. This can be shown by a slight modification (to take into account framings) of the argument given by Samit in Lecture 6. Taking any presentation for $\tilde{R}$ over $\tilde{B}$ and then tensoring with B over $\tilde{B}$ gives the desired presentation. $\square$

2.2 Taylor-Wiles Sets

Just as in Mike's lecture, a *TW set of primes* is a set Q of places of F satisfying the following conditions:

- Q is disjoint from S_p and St.
- $N(v) \equiv 1 \pmod{p}$ for all $v \in Q$.
- The eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$ are distinct and belong to k.
- The map
$$H^1(G_{F_{S_p \cup St \cup Q}}, \mathrm{ad}^0(\bar{\rho})(1)) \to \oplus_{v \in Q} H^1(F_v, \mathrm{ad}^0(\bar{\rho})(1))$$
is an isomorphism.

Note that the last conditions implies that TW-sets always have the same size h.

Given a TW-set Q, we define $R_Q^{\square}$ to be the universal deformations ring unramified outside $S_p \cup St \cup Q$ which is ord-cryst at S_p and Steinberg at St and with local framings at $S_p \cup St$ but not at Q. So $R_Q^{\square}$ is exactly the same as $R^{\square}$ except we allow ramification now at the auxiliary set Q. Note that there exists a natural map $\varphi_Q : R_Q^{\square} \to R^{\square}$ since $R^{\square}$ is the unramified at Q quotient of $R_Q^{\square}$. We now recall a series of important properties of these $R_Q^{\square}$.

(Q1) All the conditions together combined with a duality result for Selmer groups imply $R_Q^{\square}$ is a quotient of $B[[x_1, \dots, x_g]]$ just as $R^{\square}$ is.

(Q2) $R_Q^\square$ is an algebra over the group ring $\mathcal{O}[\Delta_Q]$, where Δ_Q is the maximal pro-p quotient of $\prod_{v\in Q}\mathcal{O}_v$.

(Q3) The kernel of φ_Q is the augmentation ideal $\mathfrak{a}_Q R_Q^\square$ where $\mathfrak{a}_Q$ is the augmentation ideal of $\mathcal{O}[\Delta_Q]$.

Let me say a word or two about these results. (Q2) involves a choice of root of the characteristic polynomial of Frob_v for each $v \in Q$ or equivalently a choice of one of the two univeral tame characters mapping I_v into $R_Q^{\square,\times}$. We will resolve this ambiguity by including the choice in our data below. Since the action of $\mathcal{O}[\Delta_Q]$ is exactly the "univeral" action of inertia, the unramified at Q quotient is given by the co-invariants under the Δ_Q action, that is,

$$R^\square \cong R_Q^\square / < g-1 > R_Q$$

where $< g-1 >$ is ideal generated by running over all $g \in \Delta_Q$. This gives (Q3).

Definition 2.3. A *TW-datum of depth n* is a TW-set Q_n together with a choice of root α_v of the characterstic polynomial of $\bar{\rho}(\mathrm{Frob}_v)$ for each $v \in Q_n$ and such that for all $v \in Q_n$, $Nv \equiv 1 \mod p^n$.

Mike showed in Lecture 24 the existence of TW-datum for any depth n given our assumptions on $\bar{\rho}$. For each $n \geq 1$, we fix once and for all a TW-datum Q_n of depth n. The end result being the existence of $R_{Q_n}^\square$ together with an $\mathcal{O}[\Delta_{Q_n}]$ structure. The condition on the norm of $v \in Q_n$ forces Δ_{Q_n} to grow with n.

Given (Q1), we now fix a surjection

$$B[[x_1, \ldots, x_g]] \to R_{Q_n}^\square$$

for all Q_n.

$R_{Q_n}^\square$ is an algebra over $\mathcal{O}[\Delta_{Q_n}]$ but choosing framing variables (i.e $R_{Q_n}[[y_1, \ldots, y_j]] \cong R_{Q_n}^\square$), we can make $R_{Q_n}^\square$ an algebra over $\mathcal{O}[[y_1, \ldots, y_j]][\Delta_{Q_n}]$. Choosing generators for the cyclic factors of Δ_{Q_n}, we get a homomorphism

$$\gamma_n : \mathcal{O}[[y_1, \ldots, y_j, T_1, \ldots, T_h]] \to R_{Q_n}^\square$$

where the kernel of γ_n contains $(T_i+1)^{p^{n_i}} - 1$ for some $n_i \geq n$.

The following key formula says that $B[[x_1, \ldots, x_g]]$ and $\mathcal{O}[[y_1, \ldots, y_j, T_1, \ldots, T_h]]$ have the same dimension.

Proposition 2.4. *Let g, h, j be defined as above with g generating global over local, h being size of TW-set, and j generating framed global over global. Then,*

$$h + j + 1 = \dim B + g.$$

Proof. In the end, we will only need an inequality ($\geq$). However, in this case, we know equality and so we might as well prove it. We have

$$h + j - g = h + (4|S_p \cup St| - 1) - (h - [F : \mathbb{Q}] + |S_p \cup St| - 1)$$

by (R1) and Proposition 1. Simplifying we get

$$h + j - g = [F : \mathbb{Q}] + 3|S_p \cup St| = \dim B - 1.$$

□

2.3 Hecke Modules

We now turn to the modular forms side. Let D be the unique quaternion algebra over F ramifying exactly at all infinite places and all places in St. Jacquet-Langlands tells us that any modular form f which could satisfy $\rho = \rho_f$ would come from a form on D. Thus, we lose nothing by working on D where certain things are much simpler.

Recall the following key result which Sam discussed last week:

Proposition 2.5. *Given a TW-datum Q of any depth, there exists a level U and direct summand M_Q of the space of automorphic forms $S(U)$ on D which is a Hecke-stable submodule such that the deformation ring $\tilde{R}_Q$ acts via a natural map $\tilde{R}_Q \to T_Q$. $\tilde{R}_Q$ comes with a $\mathcal{O}[\Delta_Q]$ structure which induces an action on M_Q. M_Q is a finite free $\mathcal{O}[\Delta_Q]$-module. Furthermore, $M = M_Q/\mathfrak{a}M_Q$.*

Let M_{Q_n} be the space of modular forms associated to our TW-datum Q_n.

It is a small technical point, but we have to pass to a framed version of M_{Q_n} to make the argument work. We use our chosen presentation

$$\tilde{R}_{Q_n}[[y_1, \dots, y_j]] \twoheadrightarrow \tilde{R}^{\square}_{Q_n}$$

which is a map of $\mathcal{O}[\Delta_Q]$-algebras. Define

$$M^{\square}_Q := M_Q \otimes_{\tilde{R}_{Q_n}} \tilde{R}^{\square}_{Q_n}$$

and similarly for M.

Proposition 2.6 (H1)**.** *Using the $\mathcal{O}[[y_1, \dots, y_j]]$-algebra structure coming from the framing, $M^{\square}_{Q_n}$ is a finite free over $\mathcal{O}[[y_1, \dots, y_j]][\Delta_{Q_n}]$ and $M^{\square} = M^{\square}_{Q_n}/\mathfrak{a}M^{\square}_{Q_n}$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}[[y_1, \dots, y_j]][\Delta_{Q_n}] \arrow[r] & \tilde{R}^{\square}_{Q_n} \\
\mathcal{O}[\Delta_{Q_n}] \arrow[r] \arrow[u] & \tilde{R}_{Q_n} \arrow[u]
\end{tikzcd}$$

which one can show is a Cartesian square by considering the chosen presentation of the unframed over the framed. Further, the vertical arrows are faithfully flat. The first statement follows from flat base change; the second from the fact that quotients commute with flat extension. □

3 Passing to the Limit

We recall now where we are headed.

A priori, we only started with a surjective map of $\mathcal{O}$-algebras

$$\varphi : \tilde{R} \to T$$

where T is some Hecke algebra acting faithfully on M a space of modular forms. $\tilde{R}$ is the deformation ring with no local conditions. We can pass to the framed version of this map:

$$\varphi^\square : \tilde{R}^\square \to T^\square$$

acting on $M^\square$. By how we chose $M^\square$ as a space of modular forms on a quaternion algebra $\varphi^\square$ will factor through the deformation ring with local conditions to give a map:

$$\varphi' : R^\square \to T^\square.$$

We will show this map is an isomorphism after inverting p. *Technical Aside*: To make everything work on the automorphic side one has to allow ramification at an auxiliary prime, this may cause the map φ' not to be surjective and so this has to be dealt with, but we won't worry about it here.

Observe that to show φ' is injective, it suffices to show that $R^\square$ acts faithfully on $M^\square$. Hence we forget T and focus on $M^\square$. We prove the following theorem:

Theorem 3.1. *The module $M^\square[1/p]$ is a finite projective (hence faithful) module over $R^\square[1/p]$. Further, $R^\square$ is finite over $\mathcal{O}[[y_1, \ldots, y_j]]$.*

Corollary 1. *The map*

$$\varphi'[1/p] : R^\square[1/p] \to T^\square[1/p]$$

is an isomorphism.

Note that by (H1) and (Q3) and some compatibilities, we can recover the action of $R^\square$ on $M^\square$ from any $(R^\square_{Q_n}, M^\square_{Q_n})$. This is the strategy we employ.

For each integer $n \geq 1$, with all the choices we have made we get a diagram:

$$\begin{tikzcd}
\mathcal{O}[[y_1, \ldots, y_j, T_1, \ldots, T_h]] \arrow[dr] & & \\
B[[x_1, \ldots, x_g]] \arrow[r, two heads] & R^\square_{Q_n} & \circlearrowleft M^\square_{Q_n}.
\end{tikzcd}$$

There is absolutely no a priori relationship between the diagrams for different n. However, for the sake of exposition, assume there existed compatible maps between the diagrams. Once we explain the patching technique the same argument will go through.

Set $R_\infty = \lim_{\leftarrow} R^{\square}_{Q_n}$ and $M_\infty = \lim_{\leftarrow} R^{\square}_{Q_n}$. We get the following diagram:

$$\begin{tikzcd}
A = \mathcal{O}[[y_1, \dots, y_j, T_1, \dots, T_h]] \arrow[d, dotted] \arrow[dr, two heads] & & \\
D = B[[x_1, \dots, x_g]] \arrow[r, two heads] & R_\infty & \curvearrowleft M_\infty.
\end{tikzcd}$$

where we pick a lift of the A-algebra structure on R_∞ to D, which we can do since its a power series ring.

Note that M_∞ is finite free over A. At each finite level, $M^{\square}_{Q_n}$ was finite free over $\mathcal{O}[[y_1, \dots, y_j]][\Delta_{Q_n}]$, but in the limit, because we demanded higher and higher congruences for Δ_{Q_n}, we get freeness over the power series ring A.

Proposition 3.2. *Let $\psi : A \to D$ be a map of domains of the same dimension and let V be a D-module which is finite free as an A-module. Then, ψ is finite and if A and D are regular then V is a projective D-module.*

Proof of Theorem 3.1. We assume the proposition and deduce the theorem as a corollary (modulo actually constructing compatible maps). Both A and D from the diagram above are clearly domains. Proposition 2.4 tells us they have the same dimension.

Setting $V = M_\infty$ first, we get that the map $A \to B[[x_1, \dots, x_g]]$ is finite. Since $B[[x_1, \dots, x_g]]$ surjects onto R_∞, we get that R_∞ is finite over A. This remains true after quotienting both sides by the augmentation ideal $\mathfrak{a}$ to get back down to $R_\square$.

A is already regular but D may not be. However, (B3) says that

$$D[1/p] = B[[x_1, \dots, x_g]][1/p] = (B \hat{\otimes}_{\mathcal{O}} \mathcal{O}[[x_1, \dots, x_g]])[1/p]$$

is regular. The second part of the proposition applies to $A[1/p], D[1/p], M_\infty[1/p]$ so $M_\infty[1/p]$ is projective over $D[1/p]$ hence faithful. Since $D[1/p]$ acts through $R^\infty[1/p]$ the map between them must be injective, hence an isomorphism.

Thus, $M_\infty[1/p]$ is projective over $R^\infty[1/p]$ and this property descends through the quotient by $\mathfrak{a}$ to $(R^{\square}[1/p], M^{\square}[1/p])$. $\square$

Proof of Prop 3.2. Each $d \in D$ acts on V and that action commutes with the action of A. This gives a map

$$D \to \mathrm{End}_A(V).$$

Let D' be the image of this map.

$$\begin{tikzcd}
D \arrow[r] \arrow[dr, two heads] & \mathrm{End}_A(V) \\
A \arrow[u] \arrow[r, "finite"'] & D' \arrow[u, hook]
\end{tikzcd}$$

It's clear that D' is finite over A so $\dim D' = \dim A = \dim D$. But D is a domain so any proper quotient of D has strictly smaller dimension so $D' = D$, and hence D is finite over A.

Now, we recall the Auslander-Buchsbaum Theorem (CRT Th'm 19.1): Let R be local Noetherian ring and M be finite module over R. Assume M has finite projective dimension. Then,

$$\text{pd}_R(M) + \text{depth}(M) = \text{depth}(R).$$

Assume A and D regular so that V has finite projective dimension over D. To show V is projective, it suffices to show the inequality

$$\text{depth}(V) \geq \text{depth}(D) = \text{dim}(D).$$

Take any regular sequence $a_1, \ldots, a_{\dim(A)}$ for V over A. The images of these in D form a regular sequence for V over D so we are done. $\square$

4 Patching Datum

The key idea in patching datum is that the deformation rings and auxiliary rings are determined by their finite artinian quotients. This leads to a pigeonhole argument to find compatible maps between diagrams.

We unfortunately begin with more notation:

- $m_A^{(n)}$ is the ideal generated by nth powers for any complete local ring A
- $M_n := M^{\square}_{Q_n}, M_0 := M^{\square}, R_n := R^{\square}_{Q_n}, R_0 := R^{\square}$
- $s =$ rank of M_n over $\mathcal{O}[[x_1, \ldots, x_j]][\Delta_{Q_n}] = \mathcal{O}[[x_1, \ldots, x_j, T_1, \ldots, T_h]]/\mathfrak{b}_n$
- $r_m := smp^m(h + j)$
- $c_m := (\pi_E^m, x_1^{p^m}, \ldots, x_j^{p^m}, (T_1 + 1)^{p^m} - 1, \ldots, (T_h + 1)^{p^m} - 1)$

Remark 4.1. For $m \leq n$, we have an inclusion of ideals $\mathfrak{b}_n \subset c_m$. This is simply because for $k \geq m$, $(T_i + 1)^{p^k} - 1$ is divisible by $(T_i + 1)^{p^m} - 1$.

Definition 4.2. A *patching datum* (D, L) of level m consists of:

1. A complete local noetherian ring D which is a B-algebra and such that $m_D^{(r_m)} = 0$ together with a D-module L which is finite free over $\mathcal{O}[[x_1, \ldots, x_j, T_1, \ldots, T_h]]/c_m$ of rank s;

2. A sequence of maps of complete local $\mathcal{O}$-algebras

$$\mathcal{O}[[x_1, \ldots, x_j, T_1, \ldots, T_h]]/c_m \to D \to R_0/(c_m R_0 + m_{R_0}^{(r_m)})$$

where the second map is a map of B-algebras;

3. A surjection $B[[y_1,\ldots,y_g]] \twoheadrightarrow D$;

4. And a surjection of $B[[y_1,\ldots,y_g]]$-modules

$$L \twoheadrightarrow M_0/c_m M_0.$$

This definition may seem arbitrary at first. However, the following two key properties illustrate the relevance of the definition.

Proposition 4.3. *For any $n \geq m$, we can construct a patching datum $(D_{m,n}, L_{m,n})$ of level m out of (R_n, M_n) by taking*

$$D_{m,n} = R_n/(c_m R_n + m_{R_n}^{(r_m)}) L_{m,n} = M_n/c_m M_n.$$

For each fixed level m, this yields an infinite sequence of patching datum, one for each $n \geq m$.

Proof. First, $D_{m,n}$ is quotient of R_n and so is a complete local Noetherian ring which inherits a B-algebra structure from R_n as well as a surjection

$$B[[y_1,\ldots,y_g]] \twoheadrightarrow D_{m,n}$$

which we fixed earlier for R_n.

The desired sequence of maps comes from reducing

$$\mathcal{O}[[x_1,\ldots,x_j,T_1,\ldots,T_h]] \to R_n \to R_0$$

modulo c_m, $(c_m R_n + m_{R_n}^{(r_m)})$, and $(c_m R_0 + m_{R_0}^{(r_m)})$ respectively.

Input (H1) tells us the M_n is finite free over $\mathcal{O}[[x_1,\ldots,x_j,T_1,\ldots,T_h]]/\mathfrak{b}_n$. Since $\mathfrak{b}_n \subset c_m$, $L_{m,n} = M_n/c_m M_n$ is finite free over $\mathcal{O}[[x_1,\ldots,x_j,T_1,\ldots,T_h]]/c_m$ of the same rank. The surjective map

$$L_{m,n} \twoheadrightarrow M_0/c_m M_0$$

comes from reducing the map $M_n \twoheadrightarrow M_0$ modulo c_m.

It turns out the one non-trivial check is that $L_{m,n}$ is actually a module over $D_{m,n}$. Since M_n is an R_n-module $L_{m,n} = M_n/c_m M_n$ is an $R_n/c_m R_n$ - module. It suffices to show that $m_{R_n}^{(r_m)}$ acts trivially on $M_n/c_m M_n$.

Let $a \in m_{R_n}$ then a acts on $M_n = M_{Q_n}^{\square}$ via the Hecke algebra $T_{Q_n}^{\square}$. Consider the action of a on the quotient

$$M_n/(\pi_E, x_1,\ldots,x_j,T_1,\ldots,T_h)M_n = M_0/(\pi_E, x_1,\ldots,x_j)M_0$$

which is a finite $\mathbb{F}$ vector space of rank s. Since a lies in the maximal ideal of the Hecke algebra it acts as a nilpotent endomorphism hence

$$a^s M_n \subset (\pi_E, x_1,\ldots,x_j,T_1,\ldots,T_h)M_n.$$

A standard pigeonhole argument implies that

$$a^{sp^m(h+j)}M_n \subset (\pi_E, x_1^{p^m}, \ldots, x_j^{p^m}, T_1^{p^m}, \ldots, T_h^{p^m})M_n.$$

Raising to m, to get necessary power of π_E, in there, we conclude that

$$a^{sp^m(h+j)m}M_n \subset c_m M_n.$$

If you are struggling like me to keep track of all the exponents, the important point is that there is a fixed power of a which only depends on m which lands you in $c_m M_n$. This is not hard to see once you have $a^s M_n \subset (\pi_E, x_1, \ldots, x_j, T_1, \ldots, T_h)M_n$. $\square$

Proposition 4.4. *There exist finitely many isomorphism classes of patching datum of level m.*

Proof. The number of elements in D is bounded above by the size of

$$B[[y_1, \ldots, y_g]]/m_{B[[y_1,\ldots,y_g]]}^{(r_m)}.$$

Also, L is free over finite ring. Its not hard to see from here that there are finitely many ways of putting the various structures on (D, L). $\square$

Finally, we come to the salvage for our earlier passing to the limit argument. Consider the following arrangement of the data:

$(D_{1,1}, L_{1,1})$

$(D_{1,2}, L_{1,2})$ $(D_{2,2}, L_{2,2})$

$(D_{1,3}, L_{1,3})$ $(D_{2,3}, L_{2,3})$ $(D_{3,3}, L_{3,3})$

$(D_{1,4}, L_{1,4})$ $(D_{2,4}, L_{2,4})$ $(D_{3,4}, L_{3,4})$ $(D_{4,4}, L_{4,4})$

$(D_{1,5}, L_{1,5})$ $(D_{2,5}, L_{2,5})$ $(D_{3,5}, L_{3,5})$ $(D_{4,5}, L_{4,5})$ $(D_{5,5}, L_{5,5})$

$(D_{1,6}, L_{1,6})$ $(D_{2,6}, L_{2,6})$ $(D_{3,6}, L_{3,6})$ $(D_{4,6}, L_{4,6})$ $(D_{5,6}, L_{5,6})$ $(D_{6,6}, L_{6,6})$

The columns correspond to patching datum of increasing levels. In the first column, we can choose a infinite subsequence of isomorphic patching datum of level 1. Call it (D_1, L_1). In the second column consider the subsequence already chosen and pick a sub-subsequence all of whose entries at level 2 are isomorphic. Call it (D_2, L_2). Repeating this process, we get a sequence (D_i, L_i) of patching datum of level i such that the reduction to a lower level $(\tilde{D}_i, \tilde{L}_i) \cong (D_{i-1}, L_{i-1})$.

Taking the inverse limit, we get a pair (D_∞, L_∞) which one checks has the same properties as the R_∞, M_∞ considered in the previous section. By this remarkable process, we manage to piece together seemingly disconnected pieces of information to build a tower which with some clever commutative algebra proves our modularity lifting theorem.

5 Appendix A: Algebra Lemmas

Proposition 5.1. *Let $R \to R'$ be a local homomorphism of complete local Noetherian rings. Then, R' is flat over R if and only if $R'/m^n R'$ is flat over $R/m^n R$ for $n \geq 1$.*

Proof. The slight difficulty here is that we are not assuming R' is finite type over R. The forward implication is clear. We can check flatness on finite type modules so assume

$$0 \to K \hookrightarrow M$$

is an injective map of finite type R-modules. To check that

$$K \otimes_R R' \to M \otimes_R R'$$

is injective, we would like to use that

$$K \otimes_R R' \otimes_R R/m^n R \to M \otimes_R R' \otimes_R R/m^n$$

is exact because its isomorphic to

$$K/m^n K \otimes_{R/m^n R} R'/m^n R' \to M/m^n M \otimes_{R/m^n R} R'/m^n R'.$$

The injectivity of the unquotiented map is thus equivalent to the statement that

$$\cap m^n (K \otimes_R R') = 0$$

i.e. that $V = K \otimes_R R'$ is separated as an R-module. Since K finite-type, V is finite-type as a R' module so standard Nakayama says that for any ideal $I \subset R'$ contained in the maximal ideal $\cap I^n V = 0$. Since the homomorphism is local $mR' \subset m_{R'}$. $\square$

Lemma 5.2. *Let R_1 and R_2 be complete local Noetherian $\mathcal{O}$-algebras. Let E be the fraction field of $\mathcal{O}$ and A be any local Artinian E-algebra. Then,*

$$\Psi : \mathrm{Hom}_E(R_1 \hat{\otimes}_{\mathcal{O}} R_2[1/p], A) \to \mathrm{Hom}_E(R_1[1/p], A) \times \mathrm{Hom}_E(R_2[1/p], A)$$

is a bijection.

Proof. Since p is invertible in A, we get

$$\mathrm{Hom}_E(R_1 \hat{\otimes}_{\mathcal{O}} R_2[1/p], A) = \mathrm{Hom}_{\mathcal{O}}(R_1 \hat{\otimes}_{\mathcal{O}} R_2, A).$$

Before completing, we have

$$\mathrm{Hom}_{\mathcal{O}}(R_1 \otimes_{\mathcal{O}} R_2, A) = \mathrm{Hom}_{\mathcal{O}}(R_1, A) \times \mathrm{Hom}_{\mathcal{O}}(R_2, A)$$

so the only question is does a homomorphism $f : R_1 \otimes_{\mathcal{O}} R_2 \to A$ extend to the completion. If it does, it does so uniquely.

Write $R_1 \cong \mathcal{O}[[x_1, \ldots, x_n]]/(g_1, \ldots, g_r)$ and $R_2 \cong \mathcal{O}[[y_1, \ldots, y_m]]/(h_1, \ldots, h_s)$. Let f_1, f_2 be the induced maps $R_1 \to A, R_2 \to A$ respectively. Now, A has both a reduction map $A \to E$ and section $E \to A$. We know from Brian's Lecture 6 that under the reduction maps $f_1(x_i)$ and $f_2(y_j)$ map to elements d_i, e_j respectively in the maximal ideal of $\mathcal{O}$. Considering d_i and e_j as elements of A under the section map, we see that $f_1(x_i - d_i) \in m_A$ and similarly $f_2(y_j - e_j) \in m_A$.

Now, let k be an integer such that $m_A^k = 0$. Then, its clear that

$$(x_1 - d_1, \ldots, x_r - d_r)^k \subset \ker f_1 \text{ and } (y_1 - e_1, \ldots, y_s - e_s)^k \subset \ker f_2.$$

Hence, the morphism f factors through $R_1/(x_1 - d_1, \ldots, x_r - d_r)^k \otimes_{\mathcal{O}} R_2/(y_1 - e_1, \ldots, y_s - e_s)^k$. Both quotients, however, are now polynomial rings over $\mathcal{O}$ so the completed tensor product is the same as the ordinary tensor product and so f trivially extends to the completion. $\square$