Construction and properties of the modules for patching (Modularity 5.20.10)

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July 1, 2010

1 Introduction/Motivation

Recall that our ultimate goal is to prove a modularity lifting theorem (stated in Andrew's talk "Overview of the Taylor-Wiles method"), which we have reduced to showing an $R = \mathbf{T}$ theorem. That is, we have a surjection from a deformation ring R' to a Hecke algebra $\mathbf{T}_\mathfrak{m}$, and we want to prove it is an isomorphism.

Brandon will prove this next time. The idea of the proof – the patching argument – is to use framed versions $R_n^\square$ of the rings R_{Q_n}. Here the Q_n form a Taylor-Wiles system; Mike established the existence of such last time. These each live over $R_0^\square$ (which is very close to the R' we care about):

$$R_n^\square \twoheadrightarrow R_0^\square.$$

On the $\mathbf{T}$ side we will have certain modified Hecke algebras $\mathbf{T}_n^\square$ which I will describe later. We'll construct maps $R_n^\square \twoheadrightarrow \mathbf{T}_n^\square$ lifting $R_0^\square \to \mathbf{T}_0^\square$, and certain $R_n^\square$-modules $M_n^\square$. By a pigeonhole principle sort of thing, we'll be able to pass to an inverse limit

$$R_\infty^\square \twoheadrightarrow \mathbf{T}_\infty^\square.$$

We'll have an inverse limit $M_\infty^\square$. We'll be able to show that $M_\infty^\square[\frac{1}{p}]$ is a faithful $R_\infty^\square[\frac{1}{p}]$-module, and some other nice things. Then we'll deduce that this faithfulness must have been true at level 0, where the $R_0^\square[\frac{1}{p}]$-action was through (a framed version of) our map $R' \to \mathbf{T}_\mathfrak{m}$, which we can therefore conclude is injective, hence an isomorphism.

Phew! This argument is clearly a technological marvel on par with my iPhone. For such a thing to work, we need precise control over what happens on each level n as we go up the tower. Specifically, Brandon will need to show that a certain collection of rings and moduled cooked up from the $R_n^\square$ and $M_n^\square$ form a "patching datum", meaning that they satisfy a collection of technical axioms that make the patching argument go through.

The goal of this talk is two-fold. One thing I need to do is define the relevant Hecke algebras $\mathbf{T}_n^\square$ and modules $M_n^\square$. To ruin the surprise, the module $M_n^\square$ will arise from a space of modular forms (which remember, are just functions on a finite set because we cleverly set things up that way) of suitable level, depending on the Taylow-Wiles set of primes Q_n. The other thing I need to do is show that the modules $M_n^\square$ satisfy nice properties, so that Brandon can show that this his patching data are actually patching data. So this is all really an expansion of section 4 of the notes from Andrew's overview talk, if you're following along at home.

Remark 1.0.1. Although ultimately we'll use framed stuff for patching, mostly we'll deal with unframed stuff in this talk.

2 Setup and statment of what we're going prove

Throughout, $\mathcal{O}$ is the ring of integers of a p-adic field with residue field k (where $\overline{\rho}$ lives).

Recall that the quaternion algebra D is ramified exactly at the places St and all the archimedean places. Let's fix some notation. For a compact open subgroup $U \subset (D \otimes_F \mathbf{A}_F^f)^\times$ set

$$X(U) = D^\times \backslash (D \otimes_F \mathbf{A}_F^f)^\times / (U \cdot (\mathbf{A}_F^f)^\times).$$

Let

$$S(U) = \text{Functions}(X(U), \mathcal{O}).$$

This is the space of automorphic forms on $D^\times$ of weight 2 and level U. Let the bad primes for U be

$$\Sigma(U) = S_p \cup \mathrm{St} \cup \{v|\infty\} \cup \{v : U_v \text{ is non-maximal}\},$$

and define the Hecke algebra $\mathbf{T}(U)$ to be the $\mathcal{O}$-subalgebra of $\text{End}(S(U))$ generated by the T_v for $v \notin \Sigma(U)$. (It comes from the double coset $U_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) U_v$.)

Now let's fix the "ground level" $U^\circ \subset (\mathbf{A}_F^f \otimes D)^\times$ [a compact open subgroup] for the construction. Picking up on a technical point Andrew mentioned, which will be relevant today, we need to choose a huge prime v_{aux} which has nothing to do with anything. In other words it should be outside of $\mathrm{St} \cup S_p \cup \{v|\infty\} \cup \bigcup_{n \geq 1} Q_n$. We can certainly arrange this because, for example, all the primes in the Q_ns satisfy $\mathbf{N}v \equiv 1 \bmod p$. Now take U° to be the maximal compact for all places $v \neq v_{aux}$; we will specify $U^\circ_{v_{aux}}$ later.

Let $\mathbf{T} = \mathbf{T}(U^\circ)$

Recall that we have a modular lift ρ_f of our residual representation $\overline{\rho}$ which satisfies a bunch of nice properties. Via Jacquet-Langlands and our assumptions on ρ_f, the Hilbert modular form f gives rise to an element of $S(U^\circ)$ which is an eigenform for $\mathbf{T}$, and hence we get a map $\mathbf{T} \to \mathcal{O}$. Set $\mathfrak{m}$ to be the unique maximal ideal of $\mathbf{T}$ containing the kernel of this map.

Now let Q be a Taylor-Wiles set of primes (disjoint from v_{aux}). Let R_Q be what it was in Mike's talk, with the additional caveat that we permit ramification at v_{aux}. Thus

$$R_Q = \widetilde{R} \otimes_{B_0} B$$

where $\widetilde{R}$ is the universal global deformation ring of $\overline{\rho}$ with determinant χ_p, unramified outside $\mathrm{St} \cup S_p \cup Q \cup \{v_{aux}\}$; the ring B_0 is the product of the universal local deformation rings with the right determinant, at the places in $\mathrm{St} \cup S_p$; the modification B is the product of the universal Steinberg deformation rings at places in St, and "suitably modified" universal ord-cryst deformation rings at places in S_p.

Recall that the universal deformation

$$\rho_Q : G_F \to \mathrm{GL}_2(R_Q)$$

when restricted to G_{F_w} for $w \in Q$, has the form

$$\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$$

for tamely ramified characters η_i. As Mike discussed, via class field theory, this endows R_Q with the structure of an $\mathcal{O}[\Delta_Q]$-algebra; here Δ_Q is the product (over the primes $v \in Q$) of the maximal p-power quotients of the cyclic groups $(\mathcal{O}_{F_v}/\mathfrak{p}_v)^\times$. So

$$\mathcal{O}[\Delta_Q] \approx \frac{\mathcal{O}[[y_1, \ldots, y_h]]}{((y_1+1)^{p^{a_1}} - 1, \ldots, (y_h+1)^{p^{a_h}} - 1)},$$

where $h = \#Q$ and the a_i are integers ≥ 1, and in fact $\geq n$ if $Q = Q_n$ is part of a Taylor-Wiles system. Note that $\mathcal{O}[\Delta_Q]$ is a local ring, with maximal ideal $\langle p, y_1, \ldots, y_h \rangle$, because all the $y_1 + 1$ have p-power order, i.e. because Δ_Q is a p-group. We'll use this fact later.

Moreover this $\mathcal{O}[\Delta_Q]$-algebra structure on R_Q is essentially canonically determined by Q, as long as we include in the data of Q a *choice* of one of the two distinct eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$ for each $v \in Q$; this lets us pick out one of the characters η_i comprising $G_F \to \mathrm{GL}_2(R_Q)$ to be "η_1", to which we can then apply class field theory and get the map $\Delta_Q \to R_Q^\times$ as Mike explained. Call this distinguished eigenvalue $\alpha_v \in k$.

We let $\mathfrak{a}_Q \lhd \mathcal{O}[\Delta_Q]$ be the **augmentation ideal** – recall that in any group ring $A[G]$, this is the kernel of the map $A[G] \to A$ which sends each $g \in G$ to 1. This ideal will show up later, in relating the modules $M_n^\square$ to $M_0^\square$. In the presentation above, it is generated by the y_is, one for each element of the TW set Q. For now, let's see how R_Q is related to $R_\varnothing = R'$.

Lemma 2.0.2. The canonical map $R_Q \to R_\varnothing$ is surjective with kernel $\mathfrak{a}_Q R_Q$.

Proof. We show that $G_F \to \mathrm{GL}_2(R_Q) \to \mathrm{GL}_2(R_Q/\mathfrak{a}_Q)$ is universal for the appropriate deformation problem. Fix a deformation $\rho_A : G_F \to \mathrm{GL}_2(A)$ of $\overline{\rho}$, which is ordinary-crystalline at places over p, Steinberg at places in St, and ramified only in $St \cup S_p \cup \{v_{aux}\}$. Then we certainly get a map $\varphi_A : R_Q \to A$ such that $\varphi_A \circ \rho_Q = \rho_A$. But since ρ_A is unramified at any $w \in Q$, when composed with φ_A the distinguished character $\eta_{1,w}$ is trivial on inertia. Thus if Δ_w is the maximal p-power quotient of $(\mathcal{O}_{F_w}/\mathfrak{p}_w)^\times$ and $\delta : \Delta_Q \to R_Q^\times$ the map Mike discussed, we have $\varphi_A \circ \delta(\sigma) = 1$ for all $\sigma \in \Delta_w$. This holds for all $w \in Q$, so the elements $1 - \delta(\sigma)$ for $\sigma \in \Delta_Q$ are all killed by φ_A. These elements generate the augmentation ideal $\mathfrak{a}_Q$, so φ_A factors through $R_Q/\mathfrak{a}_Q R_Q$. But if $\varphi : R_Q/\mathfrak{a}_Q R_Q \to A$ were another map lifting ρ_A, then the composition of φ with the projection from R_Q would have to agree with φ_A by universality of R_Q. Since said projection is, of course, surjective, this shows that $(R_Q/\mathfrak{a}_Q R_Q, \rho_Q \bmod \mathfrak{a}_Q)$ is universal for the type of deformations we want. □

The lemma shows that we know exactly how to relate R_Q to its level zero version $R_\varnothing$. For patching, we now want to set certain R_Q-modules M_Q of automorphic forms, which will be free over $\mathcal{O}[\Delta_Q]$ and related to $M_\varnothing$ in the same manner as the lemma.

For this we need to specify some new compact open subgroups $U_Q \subset V_Q \subset U^\circ$, by shrinking (physically speaking) the level at $w \in Q$. These will all agree except for those $w \in Q$. For $w \in Q$ set $V_{Q,w}$ to be the Iwahori I_w:

$$V_{Q,w} = \mathrm{I}_w = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_w}) : c \in \mathfrak{p}_w \}.$$

Set
$$U_{Q,w} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathcal{O}_{F_w}) : c \in \mathfrak{p}_w, d = a^{-1} \in \Delta_w \right\},$$
i.e. the image of $ad^{-1} \bmod \mathfrak{p}_w$ should map to 1 in the maximal p-power quotient, for all $w \in Q$.

The following is clear.

Lemma 2.0.3. U_Q is normal in V_Q, and $V_Q/U_Q = \Delta_Q$. $\square$

This means that the induced map of sets $X(U_Q) \to X(V_Q)$ "wants to be" a Δ_Q-torsor. The role of the auxiliary prime v_{aux} will be to ensure that this is the case, as will be discussed below. This Galois property of the aforementioned cover will then be used to show that the module M_Q we shall define next, is actually $\mathcal{O}[\Delta_Q]$-free.

Now we have Hecke algebras $\mathbf{T}(V_Q)$ and $\mathbf{T}(U_Q)$, which, as we have defined them, contain only T-operators for places away from $\mathrm{St} \cup S_p \cup \{v|\infty\} \cup Q \cup \{v_{aux}\}$ and nothing else. We also want some U-operators for places in Q. So set

$$\mathbf{T}(U_Q)^+ = \langle \mathbf{T}(U_Q), \{U_w : w \in Q\} \rangle \subset \mathrm{End}(S(U_Q)),$$

$$\mathbf{T}(V_Q)^+ = \langle \mathbf{T}(V_Q), \{U_w : w \in Q\} \rangle \subset \mathrm{End}(S(V_Q)).$$

(Note that we've avoided notational ambiguity, since the U_w is distinct from the w-component of the level zero compact open subgroup U_w°. Still, I'm sorry that they look so similar.)

I'll remind you that the U_w operator is given by the Iwahori-double coset

$$I_w \left(\begin{smallmatrix} \varpi_w & 0 \\ 0 & 1 \end{smallmatrix} \right) I_w.$$

Now let us define some ideals in Hecke algebras. As before we set $\mathfrak{m}[= \mathfrak{m}_\varnothing = \mathfrak{m}^\circ]$ to be the maximal ideal of $\mathbf{T} = \mathbf{T}(U^\circ)$ containing the kernel of the eigenvalue map for the automorphic form on $(D \otimes \mathbf{A}_F^f)^\times$ corresponding to our modular form f. There is a natural map $\mathbf{T}(V_Q) \to \mathbf{T}$ sending the T-operators to themselves; set $\mathfrak{m}_Q$ to be the contraction of $\mathfrak{m}$ along this homomorphism. This is a maximal ideal of $\mathbf{T}(V_Q)$. Now set $\mathfrak{n}_Q$ to be the ideal of $\mathbf{T}(V_Q)^+$ generated by $\mathfrak{m}_Q$ plus $U_w - \widetilde{\alpha_w}$ for all $w \in Q$, where $\widetilde{\alpha_w}$ is any lift of $\alpha_w \in k$ to $\mathcal{O} \subset \mathbf{T}(V_Q)^+$. Similarly we have a map $\mathbf{T}(U_Q)^+ \to \mathbf{T}(V_Q)^+$. So we can contract $\mathfrak{n}_Q$ to get an ideal $\mathfrak{m}_Q^+$ of $\mathbf{T}(U_Q)^+$.

Here is the picture:

$$\begin{tikzcd}
 & \mathfrak{m}_Q \lhd \mathbf{T}(V_Q) \arrow[dl] \arrow[dr] & & \mathfrak{m}_Q^+ \lhd \mathbf{T}(U_Q)^+ \arrow[dl] \\
\mathfrak{m}_\varnothing \lhd \mathbf{T}(V_\varnothing) = \mathbf{T} & & \mathfrak{n}_Q \lhd \mathbf{T}(V_Q)^+ &
\end{tikzcd}$$

Now $\mathfrak{n}_Q$ is clearly maximal, provided it is not the unit ideal. (We're just setting all the generators equal to constants.) Why isn't it the unit ideal? In fact, this will come out of what we ultimately prove about $S(V_Q)_{\mathfrak{n}_Q}$, effectively that there is a modular form of level V_Q with action of the Hecke algebra $\mathbf{T}(V_Q)^+$ specified by $\mathfrak{n}_Q$, so the quotient $\mathbf{T}(V_Q)^+/\mathfrak{n}_Q$ is not the zero ring.

Remark 2.0.4. As Andrew explained to me, this can also be seen directly, as follows. (I'm kind of confused about this, and I couldn't really work out the details, but it shouldn't be too important for what follows...) Take a modular form f on D of level U°, hence level 1 at some $w \in Q$. Then f and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_w \end{smallmatrix}\right) f$ span the space of forms where we ramp up to Γ_0 (Iwahori) level at w. We need to check that the U_w operator has an eigenvector in this space of forms, with eigenvalue matching our chosen lift $\widetilde{\alpha_w}$. Since f has level 1 at w, the corresponding automorphic representation is unramified principle series at w, i.e. it is of the form $V = \pi(\mu, \nu) = \{\varphi : \mathrm{GL}_2(F_w) \to \mathbf{C} : \varphi(\left(\begin{smallmatrix} a & * \\ & b \end{smallmatrix}\right) g) = \mu(a)\nu(b)|a/b|_w^{1/2} f(g)\}$ for some unramified characters $\mu, \nu : F_w^\times \to \mathbf{C}^\times$. The dimension of the spherical fixed vectors in V is 1; this space is spanned by f itself Then f and $\left(\begin{smallmatrix} 1 & \\ & \varpi_w \end{smallmatrix}\right) f$ span the Iwahori fixed vectors V^{I_w}. The U_w operator is given by the double coset

$$I_w \left(\begin{smallmatrix} \varpi_w & \\ & 1 \end{smallmatrix}\right) I_w = \coprod_{x \in (\mathcal{O}_{F_w}/\mathfrak{p}_w)} \left(\begin{smallmatrix} \varpi_w & \tilde{x} \\ & 1 \end{smallmatrix}\right) I_w.$$

Using this decomposition and the basis of indicator functions for the two cells in

$$G = \mathrm{GL}_2(F_w) = B \sqcup B \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) I_w, \qquad (B = \text{borel})$$

one can compute the action of U_w explicitly. The eigenvalues should be $\mu(\varpi_w), \nu(\varpi_w)$, one of which should match $\widetilde{\alpha_w}$ somehow (?).

So we have a maximal ideal $\mathfrak{m}_Q^+$ in $\mathbf{T}(U_Q)^+$. We will localize at this to define the rings of Hecke operators T_Q and modules of automorphic forms M_Q which we will use for patching: set

$$T_Q = \mathbf{T}(U_Q)^+_{\mathfrak{m}_Q^+}, \qquad M_Q = S(U_Q)_{\mathfrak{m}_Q^+}.$$

There is a natural Galois representation $G_F \to \mathrm{GL}_2(T_Q)$ which induces a surjection $R_Q \twoheadrightarrow T_Q$, and hence an R_Q-module structure on M_Q (which is naturally a T_Q-module). Since R_Q is an $\mathcal{O}[\Delta_Q]$-algebra, this makes M_Q an $\mathcal{O}[\Delta_Q]$-module as well.

Now we can state our main results.

Theorem 2.0.5. *The module M_Q is $\mathcal{O}[\Delta_Q]$-free. Moreover,*

$$M_Q/\mathfrak{a}_Q M_Q \cong M_\varnothing.$$

This should imply more or less directly that certain quotients of appropriate framed versions $M_{Q_n}^\square$ of the M_{Q_n}s form a "patching datum".

3 Proof of Main Theorem

To prove the first part of the theorem (freeness of M_Q over $\mathcal{O}[\Delta_Q]$), we will argue sort of topologically. We will show that the action on $\mathcal{O}[\Delta_Q]$ on M_Q via the map $\mathcal{O}[\Delta_Q] \to R_Q \to T_Q \subset \mathrm{End}(M_Q)$, agrees with another action which is a bit easier to understand. Specifically, we will show that the map of double coset spaces

$$X(U_Q) \to X(V_Q)$$

is a Galois cover with deck group $\Delta_Q = V_Q/U_Q$, and deduce from this that Δ_Q acts freely on M_Q.

Remark 3.0.6. If you look in DDT for the proof of the analgous freeness assertion in their setup (Thm 4.16), you'll see that they invoke the fact that a certain map of modular curves corresponding to an inclusion of congruence subgroups, is unramified, and they proceed using Riemann-Hurwitz. Our strategy here is thus sort of similar to theirs.

To ensure that said map is a Galois cover, i.e. a Δ_Q-torsor, we need to specify the group level U° at the auxiliary prime v_{aux} I mentioned earlier. Essentially, by adding enough level at that place, we can ensure that the "stackiness" – the order of the automorphism group $N_{U_Q,x}$ for various $x \in X(U_Q)$ – is prime to p, hence prime to the index $\#\Delta_Q = [V_Q : U_Q]$, for all x. This is what we turn to now.

3.1 The smallness condition and v_{aux}

First let us relate the $V_Q/U_Q = \Delta_Q$ action on the fibers of $X(U_Q) \to X(V_Q)$, to the amount of stackiness. This requires a bit of group theory.

3.1.1 Some group theory

First note the following trivial fact.

Lemma 3.1.1. Suppose we have subgroups A, B, C of a group G, and suppose $C \lhd B$. Then $(A \cap B)/(A \cap C)$ is naturally a subgroup of B/C via $(A \cap B)/(A \cap C) \hookrightarrow B/(A \cap C) \twoheadrightarrow B/C$. □

Lemma 3.1.2. Suppose we have groups and subgroups

$$K' \lhd K \subset G \supset H.$$

Consider the obvious map of double coset spaces

$$\pi : H\backslash G/K' \twoheadrightarrow H\backslash G/K.$$

Let $S(g_0)$ be the fiber $\pi^{-1}(Hg_0K)$. Set $J(g_0) = (K \cap g_0^{-1}Hg_0)/(K' \cap g_0^{-1}Hg_0)$. By the previous lemma we can regard $J(g_0)$ as a subgroup of K/K'. Then $S(g_0)$ is naturally in bijection with the left coset space $J(g_0)\backslash(K/K')$.

Proof. Define $\alpha : J(g_0)\backslash(K/K') \to S(g_0)$ by

$$\alpha : J(g_0) \cdot (kK') \mapsto Hg_0kK'.$$

We can see α is a well-defined map of sets as follows. If $J(g_0) \cdot (k_1K') = J(g_0) \cdot (k_2K')$ then there exists $j \in J \subset K/K'$ such that $j(k_1K') = k_2K'$. Now $j = k(K' \cap g_0^{-1}Hg_0)$ for some $k \in K \cap g_0^{-1}Hg_0$. Say $k = g_0^{-1}hg_0$. The map of the previous lemma regards j as an element of K/K' as the coset kK'. The condition $j(k_1K') = k_2K'$ says that $kk_1K' = k_2K'$. So $k_2 = kk_1k'$ for some $k' \in K'$. Now $\alpha(J(g_0) \cdot (k_1K')) = Hg_0k_1K'$ while $\alpha(J(g_0) \cdot (k_2K')) = Hg_0k_2K'$. So we must show these agree. But $Hg_0k_2K' = Hg_0kk_1k'K' = Hg_0(g_0^{-1}hg_0)k_1k')K' = Hhg_0k_1k'K' = Hg_0k_1K'$, so they do.

Conversely, define $\beta : S(g_0) \to J(g_0)\backslash(K/K')$ by $Hg_0kK' \mapsto J(g_0)(kK')$. Again, we must check this is well-defined. If $Hg_0k_1K' = Hg_0k_2K'$ then $g_0k_1 = hg_0k_2k'$ for some

$h \in H, k' \in K'$. so $k_1 = g_0^{-1} h g_0 k_2 k'$. In particular this means $g_0^{-1} h g_0 \in K \cap g_0^{-1} h g_0$. Call this element k. So we need to show that $J(g_0)(k_2K') = J(g_0)(kk_2k'K')$. Clearly we can ignore the k'. We leave to the reader to check that $kk_2K' = (k(K' \cap g_0^{-1} H g_0)) \cdot (k_2K')$ when we regard $k(K' \cap g_0^{-1} H g_0) \in J$ as an element of K/K'. So $kk_2k'K' = kk_2K'$ is a translate of k_2K' on the left by an element of $J(g_0)$, so we win.

Finally, it is clear that α and β are mutually inverse bijections. $\square$

Remark 3.1.3. Note that the lemma identifies $S(g_0)$ with $J(g_0)\backslash(K/K')$ note merely as sets, but as (K/K')-sets. Hence to prove that $H\backslash G/K' \to H\backslash G/K$ is a K/K'-torsor, it suffices to ensure that $J(g_0)$ vanishes for all $g_0 \in G$.

3.1.2 Application to our setup

In our setup, the previous lemma identifies the fiber of $X(U_Q) \to X(V_Q)$ over $D^\times x V_Q(\mathbf{A}_F^f)^\times$ as a Δ_Q-set with the quotient

$$\frac{(V_Q \cdot (\mathbf{A}_F^f)^\times)/(U_Q \cdot (\mathbf{A}_F^f)^\times)}{(x^{-1}D^\times x \cap V_Q \cdot (\mathbf{A}_F^f)^\times)/(x^{-1}D^\times x \cap U_Q \cdot (\mathbf{A}_F^f)^\times)}.$$

(We think of this quotient as a space of left-cosets of the denominator.) We'd like to show the denominator is trivial, so that this is simply the quotient

$$V_Q(\mathbf{A}_F^f)^\times / U_Q(\mathbf{A}_F^f)^\times = V_Q/U_Q = \Delta_Q,$$

so the action on fibers is simply transitive as required.

Now the denominator is itself a quotient of

$$\frac{x^{-1}D^\times x \cap V_Q(\mathbf{A}_F^f)^\times}{x^{-1}D^\times x \cap (\mathbf{A}_F^f)^\times}.$$

The denominator of the latter is simply $F^\times$.

So we want to ensure that

$$(x^{-1}D^\times x \cap V_Q(\mathbf{A}_F^f)^\times)/F^\times = \{1\}.$$

Since $V_Q \subset U^\circ$, we can simply impose conditions on the ground level U° so that

$$\mathcal{G} := (x^{-1}D^\times x \cap U^\circ(\mathbf{A}_F^f)^\times)/F^\times = \{1\}.$$

Now $x^{-1}D^\times x$ is discrete, and $U^\circ(\mathbf{A}_F^f)^\times/F^\times$ is compact. (Because U° is compact, and the finite part of the idéle class group is compact.) So this is a finite group.

Now choose v_{aux} lying over some prime $\ell_{aux} \geq 5$, sufficiently large so that v_{aux} is unramified for both F and D (and is outside $St \cup S_p$). Set

$$U^\circ_{v_{aux}} = \{m \in \mathrm{GL}_2(\mathcal{O}_{F_{v_{aux}}}) : m \equiv 1 \bmod \mathfrak{p}_{v_{aux}}\}.$$

Observe that $U^\circ_{v_{aux}}$ is pro-ℓ_{aux}.

Proposition 3.1.4. $\mathrm{GL}_2(F_{v_{aux}})$ has no elements of order ℓ_{aux}.

This is because $F_{v_{aux}}$ is an unramified extension of $\mathbf{Q}_{\ell_{aux}}$, so it does not contain ℓ_{aux} roots of unity. But if $M \in \mathrm{GL}_2(F_{v_{aux}})$ had order ℓ_{aux}, it would have an ℓ_{aux} root of unity as an eigenvalue, lying in some quadratic extension of $F_{v_{aux}}$. But for $\ell_{aux} \geq 5$, $[F_{v_{aux}}(\mu_{\ell_x}) : F_{v_{aux}}]$ cannot be two.

By the proposition, it follows that $D^\times$ has no elements of ℓ_{aux}-power order, since $D^\times \hookrightarrow (D \otimes_F F_{v_{aux}})^\times \approx \mathrm{GL}_2(F_{v_{aux}})$ as v_{aux} splits D. Since the finite group $\mathcal{G}$ we want to show is trivial is a subquotient of (a conjugate of) $D^\times$, we know that its order is thus prime to ℓ_{aux}.

To prove it is trivial, we will show our group $\mathcal{G}$ is an ℓ_{aux}-group.

So suppose $\overline{g} \in \mathcal{G}$ had order n prime to ℓ_{aux}. Fix a representative $g \in x^{-1}D^\times x \cap U^\circ(\mathbf{A}_F^f)^\times$. Consider the v_{aux} component $g_{v_{aux}}$. Inside $(D \otimes_F F_{v_{aux}})^\times$, $g_{v_{aux}}$ sits in the subgroup

$$x_{v_{aux}}^{-1} D^\times x_{v_{aux}} \cap U^\circ_{v_{aux}} F^\times_{v_{aux}} \subset (D \otimes F_{v_{aux}})^\times \approx \mathrm{GL}_2(F_{v_{aux}}).$$

Write $g_{v_{aux}} = uj$ as product of something in $U^\circ_{v_{aux}}$ and someting in $F^\times_{v_{aux}}$ (here j stands for "idele") Now $(uj)^n = u^n j^n \in F^\times$, so $u^n \in F^\times_{v_{aux}}$. Thus the image of u in $U^\circ_{v_{aux}}/U^\circ_{v_{aux}} \cap F^\times_{v_{aux}}$ has order prime to ℓ_{aux}. But this group is pro-ℓ_{aux}, being a quotient of $U^\circ_{v_{aux}}$, and hence $u = 1$. Thus $g_{v_{aux}} \in F^\times_{v_{aux}} \cap x^{-1}_{v_{aux}} D^\times x_{v_{aux}}$. In particular, $g_{v_{aux}}$ commutes with $x_{v_{aux}}$, so $g_{v_{aux}} \in F^\times_{v_{aux}} \cap D^\times = F^\times$. This shows that in fact $g \in F^\times$ so $\overline{g} = 1 \in \mathcal{G}$ as desired.

3.2 Proof of freeness

OK great, so now we have seen that $X(U_Q) \to X(V_Q)$ is a Δ_Q-torsor, provided we impose "principal congruence subgroup" level in U° at a well-chosen place v_{aux}. In particular, this shows that $X(U_Q)$ is (non-canonically) the same as $X(V_Q) \times \Delta_Q$. So $S(U_Q) = S(V_Q) \otimes_{\mathcal{O}} \mathcal{O}[\Delta_Q]$. Since $S(V_Q)$ is $\mathcal{O}$-finite free, this means that $S(U_Q)$ is $\mathcal{O}[\Delta_Q]$-finite free. (That is, with the action of $\mathcal{O}[\Delta_Q]$ as deck transformations of $X(U_Q)$.)

Now the localization $M_Q = S(U_Q)_{\mathfrak{m}_Q^+}$ is a summand of $S(U_Q)$ as a T_Q-module. (This is because T_Q is finite semilocal over the p-adically complete ring $\mathcal{O}$.) So provided we know that the deck transformation action of $\mathcal{O}[\Delta_Q]$ agrees with the action coming from the homomorphism

$$\mathcal{O}[\Delta_Q] \to R_Q \to T_Q \subset \mathrm{End}(M_Q)$$

this shows that that M_Q is a summand of a finite free $\mathcal{O}[\Delta_Q]$-module. Since $\mathcal{O}[\Delta_Q]$ is local, that would force M_Q to be $\mathcal{O}[\Delta_Q]$-free. (Projective = flat = locally free = free, for finitely generated modules over a local Noetherian ring.)

Thus we are reduced to showing that these two actions of the "diamond operators" Δ_Q agree. For this we will need to know the following.

Lemma 3.2.1. The Hecke algebra T_Q is reduced.

Proof. (FIXME: Cf. Taylor Cor. 1.8(3) in "On the Meromorphic Continuation...")

The rough idea is the following. It suffices to consider the generic fiber of the Hecke algebra, i.e. to consider the space of forms after inverting p in our coefficients. Reducedness of the Hecke algebra says that eigenforms are determined by their Hecke eigenvalues at all but finitely many places. This is because we defined the Hecke ring as subring of the endomorphisms of the module of module forms; consequently the modular forms are faithful over the Hecke algebra. Thus if some local quotient of the Hecke algebra is not a field, the corresponding quotient module of modular forms will have positive dimension. This translates directly to the existence of several eigenforms with exactly the same Hecke eigenvalues for almost all places.

So to prove reducedness, it's enough to check that in our situation, the Hecke eigenvalues are actually determined by the corresponding Galois representation which we know (huh?!?), which is determined by the knowledge of what's happening at all but finitely many places. $\square$

By the lemma, the module M_Q is spanned by Hecke eigenforms $f \in S(U_Q)_{\mathfrak{m}_Q^+} = M_Q$, corresponding to the various irreducible components $T_Q/\mathfrak{p}$ ($\mathfrak{p}$ a minimal prime) of T_Q. So for each f (i.e. for each minimal $\mathfrak{p}$) we have two actions of Δ_Q on $M_Q/\mathfrak{p}M_Q$:

- The one coming from the $\Delta_Q = V_Q/U_Q$ action on $X(U_Q)$;
- and the one coming from the morphisms

$$\mathcal{O}[\Delta_Q] \to R_Q \to T_Q \to T_Q/\mathfrak{p}.$$

If we know these agree, for each f, then it follows that the two actions of Δ_Q on all of M_Q agree. So fix such an f. Let π be the representation of $(D \otimes_F \mathbf{A}_F^f)^\times$ generated by f, after tensoring with $E = \mathrm{Frac}(\mathcal{O})$. Fix one of the TW primes $w \in Q$, and consider the local component π_w of π at w. Now f itself is a $U_{Q,w}$-fixed vector. (I'll remind you that $U_{Q,w} = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_{F_w}) : c \equiv 0 \bmod \varpi_w, ad^{-1} = 1 \in \Delta_Q\}$. This is like $\Gamma_1(w)$, sort of.) Moreover we know how the Hecke operator U_w (sorry for the notation!) acts on f: as a lift of one of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_w)$, which are distinct, and whose product is equal to $\mathbf{N}w \equiv 1 \bmod p$, because w is a Taylor-Wiles prime. These stringent conditions on the action of U_w rule out the possibility that π_w is Steinberg, but I am not sure why. Now by a classification result, this implies that π_w is in fact a tamely ramified principal series

$$\pi_w = \pi(\mu, \nu), \qquad \mu, \nu : F_v^\times \to \mathbf{C}^\times \text{ tame.}$$

Now $\Delta_Q = V_Q/U_Q$ acts on the right (in the way we like) on

$$D^\times \backslash (D \otimes \mathbf{A}_F^f)^\times$$

by translation by a representative for the coset $vU_Q \in \Delta_Q$. This induces an action of the w-part Δ_w = maximal p-power quotient of $(\mathcal{O}_{F_w}/\varpi_w)^\times$ of Δ_Q, on the invariants $\pi_w^{U_{Q,w}}$. Moreover, if we write

$$\pi_w = \pi(\mu, \nu) = \{\varphi : \mathrm{GL}_2(F_w) \to \mathbf{C} \mid \varphi(\left(\begin{smallmatrix} a & * \\ & b \end{smallmatrix}\right) g) = \mu(a)\nu(b)|a/b|^{1/2}\varphi(g)\}$$

as a space of functions on $D_w^\times$, this action of Δ_w agrees by definition with the usual (= right regular) action of

$$\{\langle x \rangle = \left(\begin{smallmatrix} \tilde{x} & \\ & 1 \end{smallmatrix}\right) : x \in \Delta_w, \tilde{x} \text{ a lift to } \mathcal{O}_{F_w}\} \subset \{\langle x \rangle : x \in (\mathcal{O}_{F_w}/\varpi_w)^\times\}.$$

(We really only care about these diamond operators for $x \in \Delta_w$, but they make perfect sense for any $x \in (\mathcal{O}_{F_w}/\varpi_w)^\times$.)

"The following lemma is well-known":

Lemma 3.2.2 (Similar to Taylor, "On the meromorphic continuation...", Lemma 1.6)**.** If μ and ν are tame, then $\pi(\mu,\nu)^{U_{Q,w}}$ is two dimensional, with a basis e_μ, e_ν of U_w-eigenvectors, such that

$$U_w e_\mu = \mu(\varpi_w)e_\mu, \qquad U_w e_\nu = \nu(\varpi_w)e_\nu, ^1$$

and the diamond operators act by

$$\langle x\rangle e_\mu = \mu(\widetilde{x})e_\mu, \qquad \langle x\rangle e_\nu = \nu(\widetilde{x})e_\nu, \qquad (x \in (\mathcal{O}_{F_w}/\varpi_w)^\times).$$

(Does anyone know a real reference? Probably it's not too hard prove; it should just be some explicit computation.) □

Now f is a U_w-eigenvector with eigenvalue $\widetilde{\alpha_w}$, by the way we set things up, so we can say that μ is determined by α_w plus the action of Δ_w on $\pi_w^{U_{Q,w}}$.

Local Langlands implies that

$$\rho_{f,p} : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

corresponding to $\pi = \pi_f$, satisfies

$$\rho_{f,p}|_{G_{F_w}} \sim \begin{pmatrix} \mu & \\ & \nu \end{pmatrix}$$

where μ, ν are the Galois characters corresponding to μ, ν via class field theory. [2] But this representation is precisely the one we know as

$$G_{F_w} \to \mathrm{GL}_2(R_Q) \to \mathrm{GL}_2(T_Q) \to \mathrm{GL}_2(T_Q/\mathfrak{p}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p).$$

So the action of the diamond operators $\langle x\rangle$ for $x \in \Delta_w$ act on f by the values of the character $\mu(\widetilde{x})$, agrees with action of x via the value of the character "$\eta_1(\widetilde{x})$" we picked out when originally defining $\mathcal{O}[\Delta_Q] \to R_Q$.

This essentially proves the desired compatibility between the two actions, modulo all the details I've omitted or gotten wrong. Therefore we have completed the proof of the freeness of M_Q over $\mathcal{O}[\Delta_Q]$.

3.3 Proof of relation of level Q with level $\varnothing$

The remaining part of the theorem is the "moreover", namely:

$$M_Q/\mathfrak{a}_Q M_Q \cong M_\varnothing.$$

The key ingredient in the proof of the "moreover" will be the following.

Proposition 3.3.1. There is an isomorphism

$$M_\varnothing := S(U^\circ)_\mathfrak{m} \xrightarrow{\sim} S(V_Q)_{\mathfrak{n}_Q}.$$

---

[1]This may not be quite right...

[2]Clearly I've been sloppy somewhere regarding $\overline{\mathbf{Q}}_p$ and $\mathbf{C}$....

Assuming the proposition, let us deduce what we want. By the proposition, it is sufficient to prove that

$$M_Q/\mathfrak{a}_Q M_Q \cong S(V_Q)_{\mathfrak{n}_Q},$$

since the latter is the same as $M_\varnothing$. Since we've already "shown" that the two Δ_Q actions are the same, we may as well compute the Δ_Q-coinvariants of M_Q, namely $M_Q/\mathfrak{a}_Q M_Q$, using the description of M_Q as $\mathcal{O}$-valued functions "upstairs" in the Δ_Q torsor

$$X(U_Q) \overset{\Delta_Q}{\twoheadrightarrow} X(V_Q).$$

But with this description, it is more or less obvious that the desired isomorphism holds: we have (non-canonically) that

$$S(U_Q) = S(V_Q) \otimes_{\mathcal{O}} \mathcal{O}[\Delta_Q]$$

so

$$S(U_Q)_{\Delta_Q} = S(V_Q).$$

Since quotients commute with localization, the same equality holds when we localize at $\mathfrak{m}_Q^+$, resp. $\mathfrak{n}_Q$.

It remains only to prove the relationship between $S(V_Q)_{\mathfrak{n}_Q}$ and $S(U^\circ)_{\mathfrak{m}}$.

4 Proof of Proposition

We will prove the proposition by induction on the size of the TW-set Q, reducing to the case when $Q = \{w\}$ is a singleton.

This argument is due to Andrew, the basic outline being from Taylor's "On the meromorphic continuation..." paper (Lemma 2.2).

4.1 Inductive setup

Specifically, let V be any compact open subgroup of U°. Let $\Phi : \mathbf{T}(V) \to k$ be a homomorphism with kernel $\mathfrak{m}$, and let $\widetilde{\Phi} : \mathbf{T}(V) \to \mathcal{O}$ be any set-theoretic lift. Let w be a TW prime, meaning the following.

a) $w \notin \Sigma(V)$ (which, recall, is just the bad set of places: $S_p \cup St \cup \{v | \infty\} \cup \{v : V_v \text{ nonmaximal}\}$). In practice, i.e. for our inductive argument, this means w is nonarchimedean and outside $S_p \cup St$ and the TW primes we already added.

b) $\mathbf{N}w = 1 \bmod p$.

c) $X^2 - \Phi(T_w)X + \mathbf{N}w$ has distinct roots $\alpha, \beta \in k$.

By Hensel's lemma, we obtain a factorization in $\mathbf{T}(V)_{\mathfrak{m}}[X]$:

$$X^2 - T_w X + \mathbf{N}w = (X - A)(X - B)$$

where $\Phi(A) = \alpha, \Phi(B) = \beta$.

Now let V' be the compact open subgroup of V obtained by replacing $V_w = \mathrm{GL}_2(\mathcal{O}_{F_w})$ with $V'_w = \mathrm{I}_w$ the Iwahori. Let U_w be the Hecke operator on $S(V')$ given by $\mathrm{I}_w \left(\begin{smallmatrix}\varpi_w & \\ & 1\end{smallmatrix}\right) \mathrm{I}_w$ as before. Let $\mathbf{T}(V')^+$ be the subalgebra of $\mathrm{End}(S(V'))$ generated by $\mathbf{T}(V')$ and U_w. Let $\mathfrak{m}'$ be the ideal of $\mathbf{T}(V')^+$ generated by $\mathfrak{p}, T_v - \widetilde{\Phi}(T_v)$ for $v \notin \Sigma(V')$, and $U_w - \widetilde{\alpha}$.

Proposition 4.1.1 (Induction step)**.** There is an isomorphism

$$\eta : S(V)_{\mathfrak{m}} \to S(V')_{\mathfrak{m}'}$$

given by

$$f \mapsto Af - \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right) f.$$

Granting this, the proof of the main theorem is complete. For using this induction step, we can build a chain of isomorphisms from $S(U^\circ)_{\mathfrak{m}}$ to $S(V_Q)_{\mathfrak{m}_Q^+}$ by adding the Taylow-Wiles primes $w \in Q$ one at a time, invoking the induction step each time.

4.2 Proof of induction step

4.2.1 Well-definedness of η

Note that *a priori* it is not clear that η lands in the localization $S(V')_{\mathfrak{m}'}$. (Recall that $\mathbf{T}(V')_{\mathfrak{m}'}$ is semilocal and finite over $\mathcal{O}$, hence a direct sum of its localizations at its maximal ideals, so in particular we can regard those localizations as subs rather than quotients.) We can characterize $S(V')_{\mathfrak{m}'}$ as precisely the $\mathbf{T}(V')^+$-submodule of $S(V')$ on which $\mathfrak{m}'$ acts topologically nilpotently. (Also, $S(V)_{\mathfrak{m}} \subset S(V)$ is characterized similarly.) As a first step, let us use this characterization to show that η actually lands where we want it to.

The following is a consequence of explicit computations done with double cosets.

Lemma 4.2.1. The identities

$$T_w f = U_w f + \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right) f, \qquad U_w \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right) f = \mathbf{N}_w \cdot f$$

hold for any $f \in S(V)$. $\square$

As a consequence we have

Lemma 4.2.2. $U_w \circ \eta = \eta \circ A$.

Proof. Using the previous lemma, we can expand

$$U_w \eta(f) = T_w(Af) - \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right)(Af) - \mathbf{N}w \cdot f.$$

Since A is a root of $X^2 - T_w X + \mathbf{N}w \in \mathbf{T}(V)_{\mathfrak{m}}[X]$, we have

$$A^2 = T_w A - \mathbf{N}w.$$

Hence

$$\eta(Af) = A^2 f - \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right)(Af) = T_w A - \mathbf{N}w \cdot f - \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right)(Af) = U_w \eta(f). \qquad \square$$

Now we deduce that η is well-defined. We need to show that $\mathfrak{p}$, $T_v - \widetilde{\Phi}(T_v)$ for $v \notin \Sigma(V')$, and $U_w - \widetilde{\alpha}$ all act topologically nilpotently on $\eta(f)$ for any $f \in S(V)_{\mathfrak{m}}$.

For $\mathfrak{p}$ this is clear.

For $T_v - \widetilde{\Phi}(T_v)$, one checks that T_v commutes with η. It is clear that T_v commutes with A, since $\mathbf{T}(V)$ is commutative. It is maybe not so obvious that $\mathbf{T}(V)$ commutes with the right regular action of $\left(\begin{smallmatrix} 1 & \\ & \varpi_w \end{smallmatrix}\right)$; that follows from a calculation with the appropriate double coset. Consequently the topological nilpotence of $T_v - \widetilde{\Phi}(T_v)$ on $\eta(f)$ follows from the topological nilpotence of the same operator acting on $f \in S(V)_{\mathfrak{m}}$.

Finally, by the previous lemma we have $(U_w - \widetilde{\alpha})(\eta f) = \eta(A - \widetilde{\alpha})(f)$. But this is topologically nilpotent since f is in $S(V)_{\mathfrak{m}}$, hence $A - \widetilde{\alpha}$, acts topologically nilpotently on f.

[FIXME: Explain the last sentence.]

4.2.2 Aside: the "integration pairing" on $X(U)$

Next we will show that η is injective. To do so, we will make use of an "integration pairing"

$$\langle , \rangle_U : S(U) \otimes S(U) \to \mathcal{O}$$

for any $U \subset U^\circ$. This is defined by

$$\langle f, g \rangle_U = \sum_{x \in X(U)} f(x) g(x).$$

It is just the "L^2 inner product" with respect to the counting measure on $X(U)$.

As I think Akshay mentioned several lectures ago, in principle we should use a different measure: we should weight a point x by the amount of "stackiness" of $X(U)$ at x:

$$N_{U,x} = [x^{-1} D^\times x \cap U \cdot (\mathbf{A}_F^f)^\times : F^\times].$$

But we arranged U° so that $N_{U,x}$ is automatically 1.

4.2.3 Injectivity

Lemma 4.2.3. For $f, g \in S(V)$, we have

$$\langle f, T_w g \rangle_V = \langle f, \left(\begin{smallmatrix} 1 & \\ & \varpi_w \end{smallmatrix}\right) g \rangle_{V'}.$$

Proof. An explicit computation with double cosets, which we omit. □

Lemma 4.2.4. Let $\pi : S(V') \to S(V)$ be the adjoint to the inclusion $S(V) \hookrightarrow S(V')$. Then the composition

$$S(V)_{\mathfrak{m}} \xrightarrow{\eta} S(V')_{\mathfrak{m}'} \xrightarrow{\pi} S(V)_{\mathfrak{m}}$$

equals $\mathbf{N}w \cdot A - B$.

Proof. A similar argument to what we did above using topological nilpotence shows that the composition above is well-defined, i.e. lands in $S(V)_{\mathfrak{m}}$. It uses the fact that the adjoint π respects the action of the T_vs.

Now fix $g \in S(V)$. The adjoint $\pi(\eta(g))$ is characterized by

$$\langle f, \pi(\eta(g)) \rangle_V = \langle f, \eta(g) \rangle_{V'}, \qquad \forall f \in S(V).$$

So

$$\langle f, \pi\eta(g) \rangle_V = \langle f, \eta g \rangle_{V'} = \langle f, Ag - \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right) g \rangle_{V'}.$$

Now

$$\langle f, Ag \rangle_{V'} = \sum_{x \in X(V')} f(x)(Ag)(x) = \sum_{y \in X(V)} \sum_{X(V') \ni x \mapsto y} f(x)(Ag)(x)$$

$$= [V : V'] \sum_{y \in X(V)} f(y)(Ag)(y) = [V : V'] \langle f, Ag \rangle_V.$$

On the other hand by the last lemma

$$\langle f, \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right) g \rangle_{V'} = \langle f, T_w g \rangle_V.$$

So we see

$$\langle f, \pi\eta g \rangle_V = \langle f, [U : U']Ag - T_w g \rangle_U.$$

This shows that $\pi\eta g = ([U : U']A - T_w)g$. But $[U : U'] = \#\mathbf{P}^1(\Bbbk(w)) = \mathbf{N}w + 1$, so $\pi\eta g = (\mathbf{N}w \cdot A + A - (A + B))g = (\mathbf{N}wA - B)g$. $\square$

To conclude that η is injective, by the last lemma it suffices to show that $\mathbf{N}w \cdot A - B$ is a unit. We just need to show it is not in $\mathfrak{m}$. But $\mathbf{N}w = 1 \bmod p$, so

$$\mathbf{N}w \cdot A - B \equiv \alpha - \beta \bmod \mathfrak{m},$$

and this is nonzero because α and β were assumed distinct.

So we crucially used the fact that we are at a TW-prime!

4.2.4 Cokernel is torsion-free

In fact, the last proof gives us a bit more. If we scale π by the inver of $\mathbf{N}wA - B$, we get a genuine section of η. So the image of η is a summand of $S(V')_{\mathfrak{m}'}$, and hence the cokernel of η is torsion free.

4.2.5 Surjectivity

It remains to prove that η is surjective.

The first key point is to show that $S(V')_{\mathfrak{m}'}$ (and hence the image of η) is contained in the space of old-forms, i.e. those coming from $S(V)$ either by the inclusion or by $f \mapsto \left(\begin{smallmatrix}1 & \\ & \varpi_w\end{smallmatrix}\right) f$. Call this space $\mathrm{Old}(V')$.

Lemma 4.2.5. $S(V')_{\mathfrak{m}'} \subset \mathrm{Old}(V')$.

Proof. This is a representation theoretic argument, which I sketch, following Andrew's writeup of it.

Fix $f \in S(V')_{\mathfrak{m}'}$. After tensoring with $E = \mathrm{Frac}(\mathcal{O})$, f generates an automorphic representation of $(D \otimes \mathbf{A}_F^f)^\times$, perhaps reducible, say $\pi = \bigoplus \pi^{(i)}$. We can decompose $f = \sum f^{(i)}$ according to this decomposition. The component $f^{(i)}$ is an I_w-fixed vector in $\pi_w^{(i)}$, for each i. By the representation theory of $\mathrm{GL}_2(F_w)$, this forces $\pi_w^{(i)}$ to be an irreducible unramified principle series, or a "special representation with trivial central character".

The latter case will be ruled out the fact that since $f \in S(V')_{\mathfrak{m}'}$, we know how the U_w operator acts on it. Indeed, U_w acts on the Iwahori invariants $(\pi_w^{(i)})^{I_w}$ of a special representation $\pi_w^{(i)}$ by ± 1. In the special case, the Iwahori fixed vectors are 1-dimensional, so they are spanned by $f^{(i)}$. So U_w acts on $f^{(i)}$ by ± 1. Thus $U_w \pm 1$ acts topologicaly nilpotently on $f^{(i)}$. Now $\alpha\beta = \mathbf{N}w = 1 \bmod p$. Since $\alpha \neq \beta$, it cannot be the case that $\alpha = \pm 1 \in k$. So since $U_w \pm 1$ acts topologically nilpotently on $f^{(i)}$, $U_w - \widetilde{\alpha}$ cannot, as the difference $\widetilde{\alpha} \pm 1$ acts invertibly. This holds for each i, so $U_w - \widetilde{\alpha}$ cannot act topologically nilpotently on f. But this contradicts the fact that $f \in S(U')_{\mathfrak{m}'}$.

It follows that each $\pi_w^{(i)}$ is an irreducible unramified principle series. This means the spherical fixed vectors are one-dimensional – spanned by sph, say – and the Iwahori fixed vectors are spanned by

$$\mathrm{sph}, \qquad \begin{pmatrix} 1 & \\ & \varpi_w \end{pmatrix} \cdot \mathrm{sph}.$$

These are all oldforms. In particular, the Iwahori fixed vector $f^{(i)}$ is in $\mathrm{Old}(V')$. So the linear combination $f = \sum f^{(i)}$ is in $\mathrm{Old}(V')$ too. ∎

But in fact we want more: we want $\mathrm{im}(\eta)$ to be in the old-forms *coming from* $S(V)_{\mathfrak{m}}$!. Call this space $\mathrm{Old}_{\mathfrak{m}}(V')$. It's now convenient to introduce some notation: write F for the level-raising map

$$S(V) \oplus S(V) \to S(V')$$

$$(f_1, f_2) \mapsto f_1 + \begin{pmatrix} 1 & \\ & \varpi_w \end{pmatrix} f_2.$$

Lemma 4.2.6. $S(V')_{\mathfrak{m}'} \subset \mathrm{Old}_{\mathfrak{m}}(V') = F(S(V)_{\mathfrak{m}}^{\oplus 2})$.

Proof. F respects the action of $\mathbf{T}(V')$. This implies that for any maximal ideal $\mathfrak{n}$ of $\mathbf{T}(V')$, we have $F(S(V)_{\mathfrak{n}}^{\oplus 2}) \subset S(V')_{\mathfrak{n}}$ and $F^{-1}(S(V')_{\mathfrak{n}}) = S(V)_{\mathfrak{n}}^{\oplus 2}$.

Apply this with $\mathfrak{n} = \mathfrak{m}_0$, the maximal ideal of $\mathbf{T}(V')$ generated by the $T_v - \widetilde{\Phi}(T_v)$ for $v \notin \Sigma(V')$. By definition, $S(V')_{\mathfrak{m}'} \subset S(V')_{\mathfrak{m}_0}$. By the last lemma,

$$F(S(V)^{\oplus 2}) \supset S(V')_{\mathfrak{m}'},$$

so

$$S(V')_{\mathfrak{m}'} \subset S(V')_{\mathfrak{m}_0} \subset F(F^{-1}(S(V')_{\mathfrak{m}_0})) = F(S(V)_{\mathfrak{m}_0}^{\oplus 2}).$$

But by multiplicity one, $S(V)_{\mathfrak{m}_0} = S(V)_{\mathfrak{m}}$, since the only difference between $\mathfrak{m}_0$ and $\mathfrak{m}$ is the presence of the single Hecke operator $T_w - \widetilde{\Phi}(T_w)$. ∎

An easy computation shows:

Lemma 4.2.7. $U_w F\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = F\begin{pmatrix} T & \mathbf{N}w \\ -1 & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. ∎

Lemma 4.2.8. $S(V)_{\mathfrak{m}}^{\oplus 2}$ is the direct sum of submodules $\mathcal{A} = \{(Af, -f) : f \in S(U)_{\mathfrak{m}}\}$ and $\mathcal{B} = \{(Bf, -f) : f \in S(U)_{\mathfrak{m}}\}$. Moreover we have

$$\left(\begin{smallmatrix} T_w & \mathbf{N}_w \\ -1 & 0 \end{smallmatrix}\right) a = Aa, \qquad a \in \mathcal{A}$$

$$\left(\begin{smallmatrix} T_w & \mathbf{N}_w \\ -1 & \end{smallmatrix}\right) b = Bb, \qquad b \in \mathcal{B}.$$

Proof. We compute

$$T_w Af - \mathbf{N}wf = A^2 f$$

by a previous calculation. So

$$\left(\begin{smallmatrix} T_w & \mathbf{N}w \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} Af \\ -f \end{smallmatrix}\right) = \left(\begin{smallmatrix} A^2 f \\ -Af \end{smallmatrix}\right) = A\left(\begin{smallmatrix} Af \\ -f \end{smallmatrix}\right).$$

The computation for B is similar. The decomposition $S(V)_{\mathfrak{m}}^{\oplus 2} = \mathcal{A} \oplus \mathcal{B}$ is true because the determinant of the "change of basis matrix" is

$$\det\left(\begin{smallmatrix} A & B \\ -1 & -1 \end{smallmatrix}\right) = A - B \equiv \alpha - \beta \neq 0 \bmod \mathfrak{m}$$

which is a unit in $\mathbf{T}(V)_{\mathfrak{m}}$. □

Lemma 4.2.9. F restricts to a surjection $\mathcal{A} \twoheadrightarrow S(V')_{\mathfrak{m}'}$.

Proof. For $a \in \mathcal{A}, b \in \mathcal{B}$, the last lemmas show

$$U_w F(a+b) = F\left(\begin{smallmatrix} T_w & \mathbf{N}w \\ -1 & 0 \end{smallmatrix}\right)(a+b) = F(Aa + Bb).$$

So

$$(U_w - \widetilde{\alpha})F(a+b) = F((A-\widetilde{\alpha})a + (B-\widetilde{\alpha})b).$$

Iterating this gives

$$(U_w - \widetilde{\alpha})^n F(a+b) = F((A-\widetilde{\alpha})^n a + (B-\widetilde{\alpha})^n b).$$

As $n \to \infty$ this goes to $(B-\widetilde{\alpha})^n F(x')$, since $A - \widetilde{\alpha}$ is topologically nilpotent on $S(V)_{\mathfrak{m}}$. But $B - \widetilde{\alpha}$ is invertible. Consequently if $U_w - \widetilde{\alpha}$ is topologically nilpotent on $F(a+b)$, then $F(b) = 0$. We know that $S(V')_{\mathfrak{m}'} \subset F(\mathcal{A} \oplus \mathcal{B})$, so this implies that in fact $S(V')_{\mathfrak{m}'} \subset F(\mathcal{A})$. □

Finally we can prove the surjectivity of η. For $\eta(f) = F(a)$ where $a = (Af, -f) \in S(V)_{\mathfrak{m}}^{\oplus 2}$. Since everything in $S(V')_{\mathfrak{m}'}$ is of the form $F(a)$ for some a, by the last lemma, and since every $a \in \mathcal{A}$ is of the form $(Af, -f)$ for some $f \in S(V)_{\mathfrak{m}}$, we are done.