Lecture 21: Structure of ordinary-crystalline deformation ring for $\ell = p$

1. Basic problem

Let Λ be a complete discrete valuation ring with fraction field E of characteristic 0, maximal ideal $\mathfrak{m} = (\pi)$, and residue field k of characteristic $p > 0$; we will ultimately be interested in the case when k is finite (and in particular, perfect). Consider a complete local noetherian Λ-algebra R with residue field k, so

$$R = \Lambda[\![x_1, \dots, x_m]\!]/(f_1, \dots, f_s),$$

and suppose there is given a continuous representation

$$\rho : G_K \to \mathrm{GL}_n(R)$$

for a p-adic field K (i.e., K is a finite extension of $\mathbf{Q}_p$). Note that $R[1/p] = R[1/\pi] = R \otimes_\Lambda E$; we call this the "generic fiber" of R over Λ, but beware that as an E-algebra this is typically very far from being finitely generated. For shorthand, we write R_E to denote this generic fiber.

We are going to be interested in certain subsets of $\mathrm{MaxSpec}(R_E)$. Recall from the lecture in the fall on generic fibers of deformation rings that the maximal ideals of R_E are precisely the kernels of E-algebra homomorphisms $R_E \to E'$ into finite extensions E'/E, or equivalently Λ-algebra homomorphisms $R \to E'$, and that such maps are necessarily given by

$$h(x_1, \dots, x_m) \mapsto h(a_1, \dots, a_m)$$

for a_i in the maximal ideal of the valuation ring Λ' of E'. In more geometric terms, $\mathrm{MaxSpec}(R_E)$ is identified with the zero locus

$$\{(a_1, \dots, a_m) \in \overline{E}^m \mid |a_i| < 1, f_j(a_1, \dots, a_m) = 0 \text{ for all } j\}$$

taken up to the natural action by $\mathrm{Gal}(\overline{E}/E)$ on this locus.

Remark 1.1. Loosely speaking, we view R_E as an "algebraist's substitute" for working directly with the rigid-analytic space $\{f_1 = \cdots = f_s = 0\}$ inside of the open unit polydisk over E. There is a way to make this link more precise, by relating R_E to the algebra of bounded analytic functions on this analytic space, but we do not need such a result so we will pass over it in silence; nonetheless, trying to visualize $\mathrm{MaxSpec}(R_E)$ in terms of this analytic zero locus is a good source of intuition.

In the fall lecture on generic fibers of deformation rings, we recorded a few basic algebraic properties of R_E and we recall them now. First, R_E is noetherian and Jacobson; the latter means that every prime ideal is the intersection of the maximals over it, or equivalently the radical of any ideal is the intersection of the maximals over it. This ensures that focusing on MaxSpec does not lose a lot of information, much like algebras of finite type over a field (*the* classic example of a Jacobson ring). In contrast, a local ring of positive dimension (e.g., a discrete valuation ring, or R as above when not artinian!) is never Jacobson! An additional important property, already implicit in the preceding discussion, is that if $x \in \mathrm{MaxSpec}(R_E)$

then the corresponding residue field $E(x) = R_E/\mathfrak{m}_x$ is *finite* over E. It then makes sense to consider the *specialization* of ρ at x:

$$\rho_x : G_K \stackrel{\rho_E}{\to} \mathrm{GL}_n(R_E) \to \mathrm{GL}_n(E(x)).$$

Especially when E is finite over $\mathbf{Q}_p$, we visualize ρ as a "family" of p-adic representations $\{\rho_x\}$ with varying coefficient fields $E(x)$ of finite degree over E.

Remark 1.2. Note that each such ρ_x is continuous (and so *is* a p-adic representation of G_K) since x carries R into the valuation ring of $E(x)$ via a local map and ρ is continuous when R is given its local (i.e., max-adic) topology.

For a property $\mathbf{P}$ of (isomorphism classes of) G_K-representations over finite extensions of E and for any $x \in \mathrm{MaxSpec}(R_E)$, let $\mathbf{P}(x)$ denote the condition that ρ_x satisfies property $\mathbf{P}$. (In practice, $\mathbf{P}$ is always insensitive to finite scalar extension on the coefficient field over E.) It is useful to consider whether or not the locus

$$\mathbf{P}(R_E) = \{x \in \mathrm{MaxSpec}(R_E) \,|\, \mathbf{P}(x) \text{ holds}\}$$

is "analytic" in the sense that it is cut out by an ideal J of R_E. That is, for a Λ-algebra map $x : R \to E'$ to a finite extension E'/E, does ρ_x satisfy $\mathbf{P}$ if and only if $x(J) = 0$? A given ideal J in R_E satisfies this condition if and only if its radical does (since E' is reduced), so we may as well restrict attention to radical J. But since R_E is Jacobson, a radical ideal J in R_E is the intersection of the maximals over it, so in other words there is exactly one possibility for a radical J:

$$J_{\mathbf{P}} := \bigcap_{\mathbf{P}(x) \text{ holds}} \mathfrak{m}_x$$

where $\mathfrak{m}_x = \ker(x : R_E \to E(x))$. Note that if $\mathbf{P}(x)$ fails for all $x \in \mathrm{MaxSpec}(R_E)$ then $J_{\mathbf{P}} = (1)$ (either by logic, convention, or the utiliarian reason that it is consistent with what follows).

Turning this reasoning around, we take the above expression for $J_{\mathbf{P}}$ as a definition, so $V(J_{\mathbf{P}}) := \mathrm{Spec}(R_E/J_{\mathbf{P}})$ is the Zariski closure of the locus of $x \in \mathrm{MaxSpec}(R_E)$ such that $\mathbf{P}(x)$ holds. The analyticity question for $\mathbf{P}$ then amounts to the following question: *does every closed point of* $V(J_{\mathbf{P}})$ *satisfy* $\mathbf{P}$? It is by no means clear how one could answer this question, and in the early days of modularity lifting theorems this was a serious problem which had to be treated by ad hoc methods depending on the specific $\mathbf{P}$.

One of the big achievements of Kisin's introduction of integral p-adic Hodge theory into Galois deformation theory is to provide systematic techniques for proving an affirmative answer to this question for many interesting $\mathbf{P}$ involving conditions related to p-adic Hodge theory (e.g., crystalline with Hodge-Tate weights in the interval $[-2, 5]$). In any situation for which the $\mathbf{P}$-analyticity question has an affirmative answer, to exploit it one needs to answer a deeper question: *how can we analyze properties of* $R_E/J_{\mathbf{P}}$*, such as regularity, dimension, connectedness of spectrum, etc.*? Kisin's methods also gave a way to address this question. We will develop this for a special $\mathbf{P}$ that can be studied without the full force of p-adic Hodge theory: the "ordinary crystalline" deformation problem, to be defined later.

A key insight that underlies Kisin's strategy for answering both of these questions is to use $\mathbf{P}$ to define a new moduli problem on *arbitrary* R*-algebras* (forgetting the topological structure of R) which is shown to be represented by a proper (even projective) R-scheme $\Theta : \mathscr{X}_{\mathbf{P}} \to \operatorname{Spec} R$ such that:

(1) the map $\Theta_E : X_{\mathbf{P},E} \to \operatorname{Spec}(R_E)$ obtained by inverting p (equivalently, localizing by $\Lambda \to E$) is a closed immersion whose image has as its closed points precisely the ones which satisfy $\mathbf{P}$ (so this closed subscheme, after killing nilpotents, recovers $J_{\mathbf{P}}$ and provides an affirmative answer to the $\mathbf{P}$-analyticity question),

(2) the Λ-scheme $X_{\mathbf{P}}$ is "formally smooth" in a sense we will make precise later. (In practice $X_{\mathbf{P}}$ is very far from being finite type over Λ, just like R itself, so we cannot naively carry over the notion of smooth morphism from algebraic geometry in terms of a Jacobian criterion.)

An important consequence of condition (2) is that the generic fiber $X_{\mathbf{P},E}$ is "formally smooth" over E, which is to say that it is regular and hence reduced. (In geometric language, this says that the rigid-analytic space over E arising from $R_E/J_{\mathbf{P}}$ in the open unit polydisc is smooth.) In particular, $X_{\mathbf{P},E} = V(R_E/J_{\mathbf{P}})$ (affine!), so the E-algebra $R_E/J_{\mathbf{P}}$ that we wish to understand is the coordinate ring of the affine generic fiber $X_{\mathbf{P},E}$ of the (typically non-affine!) moduli scheme $X_{\mathbf{P}}$ over R which we can try to study by moduli-theoretic reasoning. In fact, we will study the structure of the generic fiber over E by using moduli-theoretic considerations with the schemes $X_{\mathbf{P}}$ mod πR and $X_{\mathbf{P}}$ mod $\mathfrak{m}_R$ which live in characteristic p!

Letting $I_{\mathbf{P}} = \ker(R \to R_E/J_{\mathbf{P}})$ be the ideal of the Zariski closure in $\operatorname{Spec} R$ of the $\mathbf{P}$-locus in $\operatorname{Spec} R_E$, the quotient $R/I_{\mathbf{P}}$ is reduced with generic $R_E/J_{\mathbf{P}}$. In practice we will think of $R/I_{\mathbf{P}}$ as an "integral parameter space for the property $\mathbf{P}$". In particular, the formal smoothness over Λ in (2) justifies viewing $X_{\mathbf{P}}$ as a "resolution of singularities" of $\operatorname{Spec}(R/I_{\mathbf{P}})$ (for which it has the same E-fiber).

2. Some commutative algebra and algebraic geometry

Before we launch into the definition and study of Kisin's moduli problems on R-algebras and their applications to the study of the $\mathbf{P}$-locus in $\operatorname{MaxSpec}(R_E)$, we digress to explain some general considerations in commutative algebra and algebraic geometry which will be used throughout his method. It will be clearer to carry out these general considerations now so that we will be ready for their applications later.

We consider the following general setup. Let (Λ, E, π, k, R) be as above, and let $f : X \to \operatorname{Spec} R$ be a proper R-scheme. There are two "reductions" of f that will be of interest: the reductions

$$\overline{f} : \overline{X} := X \bmod \pi \to \operatorname{Spec}(R/\pi R), \quad f_0 : X_0 := X \bmod \mathfrak{m}_R \to \operatorname{Spec} k$$

modulo πR and modulo $\mathfrak{m}_R$ respectively. In particular, X_0 is a proper (hence finite type) scheme over the residue field k. The quotient $R/\pi R$ is naturally a k-algebra (since $k = \Lambda/(\pi)$), but $\overline{X}$ is typically "huge" (not finite type) when thereby viewed as a k-scheme.

Since f is proper, it carries closed points of X into closed points of $\operatorname{Spec} R$. But there is only one closed point of the local $\operatorname{Spec} R$, and X_0 is closed in X, so we conclude that the closed points of X coincide with those of X_0. Moreover, if $x_0 \in X$ is a closed point then since it is closed in the scheme X_0 of finite type over k we see that the residue field $\kappa(x_0)$ of X at x_0 (or equivalently, of X_0 at x_0) is of *finite degree* over k, *Now assume that k is perfect* (e.g., finite). Consider a closed point $x_0 \in X$, so $\kappa(x_0)/k$ is a finite separable extension. Let $\Lambda(x_0)$ be the unique (up to unique isomorphism) finite unramified extension of Λ with residue field $\kappa(x_0)/k$. The completed local ring $\mathscr{O}^{\wedge}_{X,x_0}$ is a Λ-algebra with residue field $\kappa(x_0)/k$, so by Hensel's Lemma it admits a unique structure of $\Lambda(x_0)$-algebra over its Λ-algebra structure. (This application of Hensel's Lemma crucially uses that we are working with the completed local ring and not the usual algebraic local ring $\mathscr{O}_{X,x_0}$; this latter Λ-algebra is typically not a $\Lambda(x_0)$-algebra in a compatible manner.)

Hypothesis $(*)$: assume that $\mathscr{O}^{\wedge}_{X,x_0} \simeq \Lambda(x_0)[\![T_1, \dots, T_n]\!]$ as $\Lambda(x_0)$-algebras, for all closed points $x_0 \in X_0$.

This hypothesis can be checked by means of functorial criteria, and that is how it will be verified in later examples of interest. It follows from $(*)$ that the completion $\mathscr{O}^{\wedge}_{X,x_0}$ is regular, Λ-flat, and reduced modulo π for all x_0. These are the properties we will use to prove:

Proposition 2.1. *Under hypothesis $(*)$, the base change $\overline{X}$ over $R/(\pi)$ is reduced and the total space X is regular and Λ-flat.*

Proof. We first handle the Λ-flatness, and then turn to the claims concerning reducedness modulo π and regularity. The π-power torsion in $\mathscr{O}_X$ is a coherent ideal whose formation commutes with passage to stalks and completions thereof (by flatness of completion). For all closed points x_0, the completion of $\mathscr{O}_{X,x_0}$ is Λ-flat by inspection of its assumed structure. Hence, the π-power torsion ideal has vanishing stalks at all closed points, so it vanishes on an open subset of X which contains all closed points. Such an open subset must be the entire space, so X is Λ-flat.

By hypothesis $(*)$, each quotient $\mathscr{O}^{\wedge}_{\overline{X},x_0} = \mathscr{O}^{\wedge}_{X,x_0} \bmod \pi$ is reduced, so the proper scheme $\overline{X}$ over the complete local noetherian ring $\overline{R} = R/(\pi)$ has reduced local rings at the closed points. Thus, the coherent radical of the structure sheaf of $\mathscr{O}_{\overline{X}}$ has vanishing stalks at all closed points, so exactly as for the Λ-flatness above we conclude that $\overline{X}$ is reduced.

If the non-regular locus on X is closed then since the local rings on X at all closed points are regular (by inspection of their completions) it would follow that the non-regular locus is empty. That is, X is regular if the non-regular locus is closed. It remains to prove that the non-regular locus in X is Zariski-closed. The closedness of this locus in general locally noetherian schemes (and likewise for other properties defined by homological conditions) is a deep problem which was first systematically investigated by Grothendieck. His big discovery was that for a class of schemes called *excellent* the closedness always holds. He also proved that "most" noetherian rings which arise in practice are excellent.

We refer the reader to [4, Ch. 13] for an elegant development of the basic properties of excellence (including the definition!), and here we just record the main relevant points: excellence is a Zariski-local property, it is inherited through locally finite type maps, and every complete local noetherian ring (e.g., every field, as well as R above) is excellent. Hence, the scheme X is excellent, so its non-regular locus is Zariski closed. ■

Now we come to a very useful result which can be applied under the conclusions of the preceding proposition.

Lemma 2.2 (Reduced fiber trick)**.** *Let X be a Λ-flat R-scheme which is proper and for which $\overline{X} = X \bmod \pi$ is reduced. If X is connected and non-empty then $X_0 = X \bmod \mathfrak{m}_R$ and the generic fiber $X_E = X \otimes_\Lambda E$ are both connected and non-empty.*

In general (without connectedness hypotheses), there is a natural bijective correspondence between connected components C_0 of X_0 and C_E of X_E by the requirement that C_E is the E-fiber of the unique connected component of X with mod-$\mathfrak{m}_R$ fiber C_0.

Proof. Since X is non-empty and Λ-flat, X_E is non-empty. The theorem on formal functions, applied to the proper X over the complete local noetherian ring R, identifies the idempotents on X with those on X_0. In particular, X_0 is non-empty, and each connected component of X_0 uniquely lifts to a connected component of X. Hence, by passing to the connected components of X it suffices to prove that if X is connected then so is X_E.

The Λ-flatness of X implies that the ring $\mathscr{O}(X)$ of global functions on X injects into its localization $\mathscr{O}(X)[1/\pi] = \mathscr{O}(X)_E = \mathscr{O}(X_E)$ which is the ring of global functions on X_E. We assume that the latter contains an idempotent e and seek to prove $e = 0$ or $e = 1$. We can write $e = e'/\pi^n$ for a minimal $n \geq 0$ and a global function e' on X. If $n = 0$ then e' is idempotent on X and hence $e = e' \in \{0, 1\}$ since X is connected. Thus, we assume $n \geq 1$ and seek a contradiction to the minimality of n.

Since $e^2 = e$ on X_E, we can clear denominators (via Λ-flatness) to get $e'^2 = \pi^n e'$ on X. Thus, for $\overline{e}' = e' \bmod \pi$ we have $\overline{e}'^2 = 0$ on $\overline{X}$. But $\overline{X}$ is reduced, so $\overline{e}' = 0$ on $\overline{X}$. This says that e' is divisible by π locally on $\overline{X}$. Since X is Λ-flat, the local π-multiplier to get e' is unique and hence globalizes. That is, $e' = \pi e''$ for some $e'' \in \mathscr{O}(X)$. It follows that on X_E we have

$$e = \frac{e'}{\pi^n} = \frac{e''}{\pi^{n-1}},$$

contrary to the minimality of n. ■

Inspired by the two preceding results, we are led to wonder: how can we ever verify Hypothesis $(*)$? We now present a functorial criterion.

Proposition 2.3. *Hypothesis $(*)$ holds if and only if for every artin local finite Λ-algebra B, $X(B) \to X(B/J)$ is surjective.*

Proof. Fix a closed point $x_0 \in X$ and let k'/k be a finite Galois extension which splits $k_0 := \kappa(x_0)$. Let $\Lambda_0 = \Lambda(x_0)$, and let Λ' be the finite unramified extension of Λ corresponding to k'/k. Thus,

$$\Lambda' \otimes_\Lambda \Lambda_0 \simeq \prod_{j: k_0 \to k'} \Lambda'_j \tag{2.1}$$

where Λ'_j denotes Λ' viewed as a $\Lambda(x_0)$-algebra via the unique Λ-embedding $\Lambda_0 \to \Lambda'$ lifting the k-embedding $j : k_0 \to k'$.

Recall from above that $\mathscr{O}^\wedge_{X,x_0}$ is canonically a Λ_0-algebra. Let B be an artin local finite Λ_0-algebra. A complete local noetherian Λ_0-algebra with residue field k_0 is a formal power series ring over Λ_0 if and only if it is Λ_0-flat with residue field k_0 and is regular modulo π. These properties hold if and only if the finite étale scalar extension by $\Lambda_0 \to \Lambda'$ yields the analogous properties using the residue field k', so it is equivalent to prove that this scalar extension is a formal power series ring over Λ'. In particular, the functorial criterion for the latter condition is precisely that the natural map of sets

$$\mathrm{Hom}_{\Lambda_0}(\mathscr{O}^{\wedge}_{X,x_0}, B) \to \mathrm{Hom}_{\Lambda_0}(\mathscr{O}^{\wedge}_{X,x_0}, B/J) \tag{2.2}$$

is surjective for any artin local finite Λ'-algebra B with residue field of finite degree over k_0 and any square-zero ideal J in B.

We will reformulate this surjectivity in terms which are more easily related to the functor of points of X as we vary (B, J) with B an artin local finite Λ'-algebra. Since B is artin local and Λ-finite, the natural restriction map

$$\mathrm{Hom}_{\Lambda}(\mathscr{O}^{\wedge}_{X,x_0}, B) \to \mathrm{Hom}_{\Lambda}(\mathscr{O}_{X,x_0}, B) = X_{x_0}(B)$$

is bijective, where $X_{x_0}(B)$ denotes the set of Λ-maps $\mathrm{Spec}\, B \to X$ whose image is x_0. Using the Λ'-algebra structure on B and the canonical Λ'-algebra structure on $\mathscr{O}^{\wedge}_{X,x_0}$, we also have the alternative description

$$\mathrm{Hom}_{\Lambda}(\mathscr{O}^{\wedge}_{X,x_0}, B) = \mathrm{Hom}_{\Lambda'}(\Lambda' \otimes_{\Lambda} \mathscr{O}^{\wedge}_{X,x_0}, B) = \mathrm{Hom}_{\Lambda'}((\Lambda' \otimes_{\Lambda} \Lambda_0) \otimes_{\Lambda_0} \mathscr{O}^{\wedge}_{X,x_0}, B).$$

Using (2.1), this is identified with the disjoint union

$$\coprod_{j:k_0 \to k'} \mathrm{Hom}_{\Lambda'}(\Lambda' \otimes_{j,\Lambda_0} \mathscr{O}^{\wedge}_{X,x_0}, B) = \coprod_{j:k_0 \to k'} \mathrm{Hom}_{\Lambda_0}(\mathscr{O}^{\wedge}_{X,x_0}, B_j)$$

where B_j denotes B viewed as a Λ'-algebra via any $g \in \mathrm{Gal}(k'/k)$ lifting j on k_0.

The preceding identifications of Hom-sets are all functorial in B. In the final disjoint union above, as we vary through all pairs (B, J) with B an artin local finite Λ'-algebra and J a square-zero ideal in B, the simultaneous surjectivity of (2.2) for all pairs (B_j, J_j)'s is thereby identified with the surjectivity of the natural map

$$X_{x_0}(B) \to X_{x_0}(B/J)$$

as B varies through artin local finite Λ'-algebras. Recall that k'/k is an arbitrary but fixed finite Galois extension which splits $k_0 = \kappa(x_0)$. Thus, if $X(B) \to X(B/J)$ is surjective for all artin local finite Λ-algebras B and square-zero ideals $J \subset B$ then Hypothesis $(*)$ holds.

Conversely, if $(*)$ holds then for any such (B, J) we claim that $X(B) \to X(B/J)$ is surjective. Pick a point in $X(B/J)$. As a Λ-map $\mathrm{Spec}(B/J) \to X$ we claim that it hits a closed point x_0. Since B/J has residue field of finite degree over k, it suffices to show that this map lands in X_0 (as X_0 is a finite type k-scheme). Since the composite map $\mathrm{Spec}(B/J) \to \mathrm{Spec}(R)$ over Λ lifts a point of R valued in a *finite* extension of k, it suffices to check that the only such point is the evident one which kills $\mathfrak{m}_R$. Expressing $R/(\pi)$ as the quotient of a power series ring over Λ, it suffices to prove that the only local k-algebra map from $k[\![x_1, \ldots, x_m]\!]$ into a finite extension of k is "evaluation at the origin". This is verified by restriction to the k-subalgebras $k[\![x_i]\!]$ for $i = 1, \ldots, m$.

The chosen point in $X(B/J)$ identifies the residue field κ of B/J as a finite extension of the residue field k_0 at x_0, so B/J and hence B is thereby equipped with a natural structure of Λ_0-algebra over its Λ-algebra structure. The chosen point in $X(B/J)$ is thereby identified with a Λ-algebra map $\mathrm{Spec}(B/J) \to X$ hitting x_0 and lifting the specified extension structure

κ/k_0. This map corresponds to a local Λ-algebra map $\mathscr{O}_{X,x_0} \to B/J$ lifting $k_0 \to \kappa$, which in turn uniquely factors through such a local Λ-map

$$\mathscr{O}^{\wedge}_{X,x_0} \to B/J.$$

This latter map is a Λ_0-algebra map (as may be checked on residue fields). By $(*)$, the completion $\mathscr{O}^{\wedge}_{X,x_0}$ is a formal power series ring over Λ_0, so its map to B/J lifts to a Λ_0-algebra map $\mathscr{O}^{\wedge}_{X,x_0} \to B$. Running the procedure in reverse, this gives a Λ-map $\operatorname{Spec} B \to X$ which lifts the chosen point in $X(B/J)$. ■

Remark 2.4. The criterion in Proposition 2.3 is what was used in the verification of power series properties in the preceding lecture on the case $\ell \neq p$. In Kisin's papers, he expresses things in terms of a much more general theory of formal smoothness for maps of topological rings, and he thereby invokes some very deep results of Grothendieck in this theory.

For example, a noetherian algebra over a field of characteristic 0 is formally smooth (for the discrete topology) over that field if and only if it is regular. We will speak in the language of regularity and avoid any need for the theory of formal smoothness because we will appeal to general results in the theory of excellence (as was done in the proof of Proposition 2.1).

The reader who is interested in reading up on the general theory of formal smoothness (such as its flatness aspects) should look at [5, Ch. 28], §17.5 in EGA IV_4, and §19–22 (esp. 19.7.1 and 22.1.4) in Chapter 0_{IV} of EGA. Certainly if one goes deeper into Kisin's techniques (beyond the "ordinary crystalline" deformation condition to be considered below) then it becomes important to use formal smoothness techniques in the generality considered by Kisin.

The final topic we take up in this section is the algebro-geometric problem of giving a convenient criterion to prove that the proper map

$$f_E : X_E \to \operatorname{Spec}(R_E)$$

is a closed immersion. More specifically, we want to give a criterion involving points valued in *finite* E-algebras C. Keep in mind that even though R_E is a gigantic E-algebra in general, it is Jacobson and its maximal ideals have residue field of finite E-degree. In particular, the artinian quotients of R_E at its maximal ideals are examples of such E-algebras C. The same goes for the R_E-proper X_E at its closed points (which lie over $\operatorname{MaxSpec}(R_E)$, due to the properness of f_E).

Proposition 2.5. *If f_E is injective on C-valued points for all E-finite C then f_E is a closed immersion.*

Proof. We will first prove that f_E is a finite map (i.e., X_E is the spectrum of a finite R_E-algebra), so then we can use Nakayama's Lemma to check the closed immersion property. Since f_E is proper, it suffices to prove that it is quasi-finite. For any map of finite type between noetherian schemes, the locus of points on the source which are isolated in their fibers (i.e., the "quasi-finite locus") is an open set: this is a special case of semi-continuity of fiber dimension. Thus, if f_E has finite fibers over $\operatorname{MaxSpec}(R_E)$ then the open quasi-finite locus of f_E contains all closed points of X_E (as these are precisely the points over $\operatorname{MaxSpec}(R_E)$, due to properness of f_E). But X_E is a Jacobson scheme since it is finite type over the Jacobson ring R_E, and (as for any noetherian topological space) the only open set in X_E which contains all closed points is the entire space. Hence, if f_E has finite fibers over $\mathrm{MaxSpec}(R_E)$ then f_E is a quasi-finite and therefore *finite* map.

Letting C vary through the finite extension fields E'/E, the injectivity of f_E on E'-valued points implies that the fiber of f_E over each $y \in \mathrm{MaxSpec}(R_E)$ has only finitely many closed points. (Here we use that $f_E^{-1}(y)$ is closed in X_E and is of finite type over $E(y)$, with $E(y)$ finite over E.) But a scheme of finite type over a field has finitely many closed points if and only if it is finite. Thus, f_E indeed has finite fibers over $\mathrm{MaxSpec}(R_E)$. This argument even shows that such fibers have *at most one* physical point (since if a fiber contains two distinct points $x' \neq x$ then using E' containing $E(x)$ and $E(x')$ makes f_E fail to be injective on E'-valued points).

Now consider the finite map $f_E : X_E \to \mathrm{Spec}\, R_E$. To prove that the corresponding module-finite map of coordinate rings is surjective (so f_E is a closed immersion), it suffices to check surjectivity after localizing at maximal ideals of R_E. By Nakayama's Lemma, it is equivalent to check that the scheme-theoretic fiber $\mathrm{Spec}\, C \to \mathrm{Spec}\, E(y)$ of f_E over each $y \in \mathrm{MaxSpec}(R_E)$ satisfies $C = 0$ or $C = E(y)$. The two composite maps

$$\mathrm{Spec}(C \otimes_{E(y)} C) \rightrightarrows \mathrm{Spec}\, C \to \mathrm{Spec}\, E(y)$$

coincide, so for the E-finite algebra $C' = C \otimes_{E(y)} C$ we see that the composites

$$\mathrm{Spec}(C') \rightrightarrows \mathrm{Spec}\, C = X_E \times_{\mathrm{Spec}(R_E)} \mathrm{Spec}(E(y)) \hookrightarrow X_E$$

have the same composition with $f_E : X_E \to \mathrm{Spec}(R_E)$. By hypothesis, f_E is injective on C'-valued points! Hence, the projections $\mathrm{Spec}(C') \rightrightarrows \mathrm{Spec}(C)$ coincide, which is to say that the two inclusions $C \rightrightarrows C' = C \otimes_{E(y)} C$ coincide. This easily forces $C = E(y)$ if $C \neq 0$ (by consideration of an $E(y)$-basis of C containing 1). ■

3. The ordinary crystalline deformation problem

Now assume that k is finite! Let E'/E be a finite extension and Λ' its valuation ring. Fix a continuous representation $\rho : G_K \to \mathrm{GL}_2(E')$ with cyclotomic determinant χ. We already know what it means to say that ρ is *ordinary*: this means that there is a G_K-equivariant quotient line with action by an unramified character η. Such a quotient line is unique, as the $\Lambda^\times$-valued $\det \rho = \chi$ is ramified, so it is equivalent to say that ρ admits an I_K-equivariant quotient line with trivial I_K-action. This notion of ordinarity can be expressed in terms of a $\mathrm{GL}_2(\Lambda')$-valued conjugate of ρ by using saturated Λ'-lines.

In terms of a G_K-stable Λ'-lattice, we get an upper-triangular form for ρ, or equivalently for $\rho|_{I_K}$, and this extension structure identifies $\rho|_{I_K}$ with a class in

$$\mathrm{H}^1(I_K, \Lambda'(1)) = \varprojlim \mathrm{H}^1(I_K, (\Lambda'/(p^n))(1))$$

with $\Lambda'/(p^n)$ a finite free $\mathbf{Z}/(p^n)$-module since $[k : \mathbf{F}_p]$ is now assumed to be finite. This rank is equal to the $\mathbf{Z}_p$-rank of the finite free $\mathbf{Z}_p$-module Λ'. By computing with a $\mathbf{Z}_p$-basis of Λ', the natural map

$$\Lambda' \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mu_{p^n}) \to \mathrm{H}^1(I_K, (\Lambda'/(p^n))(1))$$

is an isomorphism, and we can pass this tensor product through an inverse limit to get

$$\mathrm{H}^1(I_K, \Lambda'(1)) = \Lambda' \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mathbf{Z}_p(1)).$$

The "crystalline" property of ρ is now going to be defined in terms of a description of $\mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ and the identification of $\rho|_{I_K}$ as a class in $\Lambda' \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mathbf{Z}_p(1))$. (After making the definition, we will make it more concrete in terms of matrices.) We note at the outset that the definition we will give is in fact equivalent to a special case of a general notion of "crystalline" defined in p-adic Hodge theory, but we have avoided any discussions of p-adic Hodge theory and so will likewise have no need to delve further into the justification for our choice of terminology. A reader who pursues the subject in greater depth will eventually meet the general concept of "crystalline", but it is logically unnecessary for our purposes.

Let K' denote the completion of the maximal unramified extension K^{un}, so

$$K' = W(\overline{\mathbf{F}}) \otimes_{W(\mathbf{F})} K$$

where $\mathbf{F}$ is the finite residue field of K. Thus, $I_K = G_{K'}$ and $\mathscr{O}_{K'}$ is a complete discrete valuation ring with uniformizer given by one for $\mathscr{O}_K$. In particular, $1 + \mathfrak{m}_{K'}$ is p-adically separated and complete as a multiplicative $\mathbf{Z}_p$-module. By Kummer theory,

$$\mathrm{H}^1(G_{K'}, \mathbf{Z}_p(1)) = \varprojlim K'^{\times}/K^{\mathrm{un}\times p^n} = \mathbf{Z}_p \times (1 + \mathfrak{m}_{K'}) \tag{3.1}$$

where the $\mathbf{Z}_p$-factor corresponds to powers of a fixed uniformizer of K (or K'). This direct product decomposition is not canonical: the direct factor of $\mathbf{Z}_p$ depends on a choice of uniformizer. However, the "$\mathbf{Z}_p$-hyperplane" of 1-units $1 + \mathfrak{m}_{K'}$ (a multiplicative $\mathbf{Z}_p$-module) is canonical.

Definition 3.1. The ordinary representation $\rho : G_K \to \mathrm{GL}_2(\Lambda')$ is *crystalline* if its class in $\mathrm{H}^1(I_K, \Lambda'(1))$ lies in the Λ'-hyperplane

$$(1 + \mathfrak{m}_{K'}) \otimes_{\mathbf{Z}_p} \Lambda'.$$

Equivalently, $\rho : G_K \to \mathrm{GL}_2(E')$ corresponds to a class in $\mathrm{H}^1(I_K, E'(1))$ lying in the E'-hyperplane $(1 + \mathfrak{m}_{K'}) \otimes_{\mathbf{Z}_p} E'$.

In view of the formulation over E', the crystalline condition is intrinsic to the E'-linear representation space for G_K, so it does not depend on a specific choice of G_K-stable Λ'-lattice.

Example 3.2. The concrete meaning of the crystalline condition is as follows. In terms of a choice of G_K-stable Λ'-lattice, consider $\rho|_{I_K} \bmod p^n$ for each $n \geq 1$. This is upper triangular unipotent, with upper-right entry given by a Λ'-linear combination of 1-cocycles $g \mapsto g(u^{1/p^n})/u^{1/p^n}$ on I_K, with $u \in \mathscr{O}_{K'}^{\times}$. Loosely speaking, Kummer theory shows that $\rho|_{I_K}$ is given by a Tate-curve type of construction with Λ'-coefficients, and the crystalline condition is that the "q-parameter" can be chosen to be a unit (or equivalent a 1-unit, since $\overline{\mathbf{F}}^{\times}$ is uniquely p-divisible).

Now we fix a residual representation $\rho_0 : G_K \to \mathrm{GL}_2(k)$ with cyclotomic determinant, and we seek to study its ordinary crystalline deformations with coefficients in p-adic fields or valuation rings thereof. In order to make good sense of these notions in deformation theory, we need to generalize the definition of "ordinary crystalline" to the case of more general coefficients.

Remark 3.3. We will later need to consider coefficients in rings like $R[T]$ that are *not* $\mathfrak{m}_R$-adically separated and complete, so this will create some delicate problems when we work with finite type R-schemes such as $\mathbf{P}^1_R$ that we want to use in the construction of a moduli scheme for an "ordinary crystalline" deformation problem. The creative use of proper (generally non-finite) R-schemes in Galois deformation theory is one of the innovations introduced by Kisin's work.

Consider the universal deformation $\rho : G_K \to \mathrm{GL}_2(R)$ or universal framed deformation $\rho^\square : G_K \to \mathrm{GL}_2(R^\square)$ of ρ_0 with cyclotomic determinant. We want to study the locus of "ordinary crystalline points" in $\mathrm{MaxSpec}(R_E)$. The formalism for this study will not really use the universality at all, so to keep the picture clear we now consider *any* continuous $\rho : G_K \to \mathrm{GL}_2(R)$ as at the outset such that $\det \rho$ is cyclotomic, and we continue to assume that the residue field k of R is *finite*. For any R-algebra A, let

$$\rho_A : G_K \to \mathrm{GL}_2(A)$$

denote the composition of ρ with $R \to A$ on matrix entries. Note that there is no meaningful continuity condition for ρ_A for general A, since we are not assuming that A carries an interesting topology compatible with the one on R.

In the special case that A is an $R/\mathfrak{m}_R^n$-algebra, ρ_A is continuous for the discrete topology on A (and the Krull topology on G_K) since $\rho \bmod \mathfrak{m}_R^n$ is continuous for the discrete topology on $R/\mathfrak{m}_R^n$. Beware that if we take $A = R/(p^n)$ with the discrete topology then ρ_A is typically *not* continuous. It will therefore be important that we can work modulo powers of the maximal ideal of R and bootstrap back up to geometric objects over R via limit procedures.

Example 3.4. Our work with ρ_A for $R/\mathfrak{m}_R^n$-algebras A will involve some Galois cohomology with A-coefficients viewed discretely, so we record here the useful fact that for any $\mathbf{Z}/(p^n)$-module M viewed discretely (such as $M = A$) the natural map

$$M \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mathbf{Z}_p(1)) \to \mathrm{H}^1(I_K, M(1))$$

is an isomorphism.

To prove this, we can use direct limits in M to reduce to the case when M is a finitely generated $\mathbf{Z}_p$-module. Hence, M is a finite direct sum of modules of the form $\mathbf{Z}/(p^r)$ with $r \leq n$, so it suffices to treat the case $M = \mathbf{Z}/(p^r)$. Then the assertion is that the natural map

$$\mathrm{H}^1(I_K, \mathbf{Z}_p(1))/(p^r) \to \mathrm{H}^1(I_K, \mu_{p^r})$$

is an isomorphism for all $r \geq 1$. For K' denoting the discretely-valued completion of K^{un} we have $I_K = G_{K'}$ and $1 + \mathfrak{m}_{K'}$ is p-adically separated and complete (as a multiplicative $\mathbf{Z}_p$-module), so Kummer theory and the description of $\mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ in (3.1) yields the result.

In the special case that A is the valuation ring of a finite extension of E, we have defined what it means to say that ρ_A is ordinary crystalline (in Definition 3.1). That definition involved the p-adic topology of the valuation ring. We wish to define this concept for $R/\mathfrak{m}_R^n$-algebras A, avoiding any use of nontrivial topologies on rings.

Definition 3.5. Let A be an R-algebra killed by $\mathfrak{m}_R^n$ for some $n \geq 1$. The representation ρ_A of G_K on $V_A := A^2$ is *ordinary crystalline* if there is a G_K-stable A-submodule $L_A \subset V_A$ such that L_A and V_A/L_A are locally free of rank 1 (equivalently, projective of rank 1) and:

(1) the I_K-action on V_A/L_A is trivial, or equivalently the continuous action of G_K on V_A/L_A is through an unramified character $\eta : G_K \to A^\times$ and on L_A is through $\chi\eta^{-1}$;
(2) under the "valuation" map $K^{\mathrm{un}\times} \twoheadrightarrow \mathbf{Z}$, the class in

$$\mathrm{H}^1(I_K, A(1)) = \mathrm{H}^1(I_K, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} A = (K^{\mathrm{un}\times}/(K^{\mathrm{un}\times})^{p^n}) \otimes_{\mathbf{Z}/(p^n)} A$$

describing $\rho_A|_{I_K}$ is carried to $0 \in A$ (i.e., the class arises from integral units of K^{un}).

We call the A-line $L_A \subset V_A$ an *ordinary crystalline structure* on ρ_A.

Remark 3.6. In the "crystalline" condition (2) in this definition, we have invoked the cohomology computation in Example 3.4. Also, in general there may be more than one choice of L_A (if any exist at all!). For example, if $A = k$ and ρ_k has trivial G_K-action (in particular, the mod-p cyclotomic character is trivial) then *every* line in $V_k = k^2$ is an ordinary crystalline structure on ρ_k.

A fundamental insight of Kisin is that rather than trying to parameterize deformations which admit an additional structure (such as an ordinary crystalline structure) that may not be unique, it is better to parameterize the space of *pairs* consisting of a deformation *equipped with* such an additional structure. To make reasonable sense of a parameter space for such enhanced objects, we will have to leave the framework of complete local noetherian rings and instead work with certain proper schemes over such rings.

The property in (2) in the preceding definition makes sense as a condition on classes in $\mathrm{H}^1(I_K, M(\chi))$ for any discrete $\mathbf{Z}/(p^n)$-module M for any $n \geq 1$. (Note that $M(\chi)$ is a discrete I_K-module, and even a discrete G_K-module.)

Definition 3.7. For any discrete $\mathbf{Z}/(p^n)$-module M equipped with a continuous unramified G_K-action (not necessarily trivial), the subgroup

$$\mathrm{H}^1_{\mathrm{crys}}(K, M(\chi)) \subset \mathrm{H}^1(K, M(\chi))$$

consists of classes whose restriction to $\mathrm{H}^1(I_K, M(\chi)) = M \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ is killed by the "valuation" mapping $\mathrm{H}^1(I_K, \mathbf{Z}_p(1)) \to \mathbf{Z}_p$ defined by Kummer theory. (In other words, the I_K-restriction is an M-linear combination of classes in $\mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ arising from integral units of the completion of K^{un}.)

It is immediate from the definition that if $M = \varinjlim M_i$ for a directed system of unramified discrete $(\mathbf{Z}/(p^n))[G_K]$-modules M_i then the equality

$$\varinjlim \mathrm{H}^1(K, M_i(\chi)) = \mathrm{H}^1(K, M(\chi))$$

carries $\varinjlim \mathrm{H}^1_{\mathrm{crys}}(K, M_i(\chi))$ isomorphically onto $\mathrm{H}^1_{\mathrm{crys}}(K, M(\chi))$. In other words, the formation of $\mathrm{H}^1_{\mathrm{crys}}(K, M(\chi))$ is compatible with direct limits in M. This will be very useful for reducing some general assertions to the special case of M which are $\mathbf{Z}_p$-finite (whereas in applications we will need to work with M which are not $\mathbf{Z}_p$-finite, such as $(R/\mathfrak{m}_R^n)[t]$).

Example 3.8. Isomorphism classes of pairs (ρ_A, L_A) as in Definition 3.5 correspond to elements in $\mathrm{H}^1_{\mathrm{crys}}(K, A(1))$.

Example 3.9. It is important to link Definition 3.5 and Definition 3.1. For $\rho : G_K \to \mathrm{GL}_2(\Lambda')$ as in Definition 3.1 we claim that it is ordinary crystalline in that initial sense (which can be checked over the fraction field E' of Λ') if and only if the artinian quotients $\rho \bmod \pi^n\Lambda'$ of ρ are ordinary crystalline in the sense of Definition 3.5 with $R = \Lambda'$ (i.e., each $\rho \bmod \pi^n\Lambda'$ admits an ordinary crystalline structure).

It is obvious that if ρ is ordinary crystalline in the initial sense then each artinian quotient $\rho \bmod \pi^n\Lambda'$ admits an ordinary crystalline structure. To go in reverse, suppose that every such artinian quotient admits an ordinary crystalline structure. Such structures are not unique in general, but since k is *finite* there are only finitely many such structures for each $n \geq 1$. These finite non-empty sets form an inverse system in an evident manner, and so the inverse limit is non-empty. (This is an elementary fact since the inverse system is indexed by positive integers and not a general index set.)

An element of the inverse limit is precisely the data of a saturated G_K-stable Λ'-line L in ρ such that (i) $\rho \bmod L$ has trivial I_K-action (as may be checked modulo π^n for all $n \geq 1$), and (ii) the class in $\mathrm{H}^1(I_K, \Lambda'(1)) = \Lambda' \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ corresponding to $(\rho|_{I_K}, L)$ has image under the "valuation mapping" $\mathrm{H}^1(I_K, \Lambda'(1)) \to \Lambda'$ which vanishes (as this can also be checked modulo π^n for all $n \geq 1$). The conditions (i) and (ii) say exactly that ρ is ordinary crystalline in the sense of Definition 3.1.

The proof of a later "formal smoothness" result over Λ will rest on:

Lemma 3.10. *For any $n \geq 1$, the functor $M \rightsquigarrow \mathrm{H}^1_{\mathrm{crys}}(K, M(\chi))$ on discrete unramified $(\mathbf{Z}/(p^n))[G_K]$-modules is right-exact.*

This is analogous to the fact that $\mathrm{H}^1(I_K, M(\chi)) = M \otimes_{\mathbf{Z}_p} \mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ (see Example 3.4) is right-exact in discrete p^n-torsion abelian groups M with trivial I_K-action.

Proof. By discreteness we can express any M as a direct limit of $\mathbf{Z}_p$-finite G_K-submodules, so any right exact sequence in M's is obtained as a direct limit of right-exact sequences of $\mathbf{Z}_p$-finite object. Thus, the compatibility with direct limits in M reduces the problem to right-exactness for M which are finite abelian p-groups.

There are two ways to settle the finite case. In [3, 2.4.2], Kisin does some work with cocycles to derive an explicit description of $\mathrm{H}^1_{\mathrm{crys}}(K, M(\chi))$ which makes the right-exactness evident by inspection. This is definitely the most elementary way to proceed.

For the reader who doesn't like cocycle arguments and is familiar with the fppf topology, here is an alternative explanation in such terms. This explanation is longer, but may be seen as more conceptual (and clarifies the role of finiteness of the residue field).

The finite discrete G_K-module $M(\chi)$ has unramified Cartier dual, so it is the generic fiber of a unique finite flat $\mathscr{O}_K$-group scheme $M(\chi)'$ with étale Cartier dual, and $M(\chi)'$ is functorial in M. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of such unramified G_K-modules then the complex of finite flat $\mathscr{O}_K$-group schemes

$$0 \to M_1(\chi)' \to M_2(\chi)' \to M_3(\chi)' \to 0$$

is short exact (in particular, short exact as abelian sheaves for the fppf topology over $\mathscr{O}_K$), as may be checked using the finite étale Cartier duals.

It can be shown by a Kummer-theoretic argument in the fppf topology that

$$\mathrm{H}^1_{\mathrm{crys}}(K, M(\chi)) = \mathrm{H}^1_{\mathrm{fppf}}(\mathscr{O}_K, M(\chi)'),$$

so it suffices to show that $\mathrm{H}^1_{\mathrm{fppf}}(\mathscr{O}_K, M(\chi)')$ is right exact in M. Equivalently, a short exact sequence in M induces a right-exact sequence in $\mathrm{H}^1_{\mathrm{fppf}}(\mathscr{O}_K, M(\chi)')$. The long-exactness of fppf cohomology then does the job provided that $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{O}_K, M(\chi)') = 0$ for any finite abelian p-group M equipped with an unramified continuous G_K-action.

The filtration by $\{p^m M\}$ reduces us to the case when M is p-torsion. If $r = \dim_{\mathbf{F}_p} M$ and k'/k is a finite Galois extension which splits M then for the corresponding finite unramified extension K'/K we have that $M(\chi)'_{\mathscr{O}_{K'}} = \mu_p^r$. Thus, we have an $\mathscr{O}_K$-subgroup inclusion

$$M(\chi)' \hookrightarrow \mathrm{Res}_{\mathscr{O}_{K'}/\mathscr{O}_K}(M(\chi)'_{\mathscr{O}_{K'}}) = \mathrm{Res}_{\mathscr{O}_{K'}/\mathscr{O}_K}(\mu_p^r),$$

where $\mathrm{Res}_{\mathscr{O}_{K'}/\mathscr{O}_K}$ denotes Weil restriction of scalars. This latter operation represented pushforward at the level of fppf sheaves, and it is an exact functor because $\mathscr{O}_K \to \mathscr{O}_{K'}$ is finite étale (and hence a split covering étale-locally over $\mathrm{Spec}(\mathscr{O}_K)$).

We conclude that $M(\chi)'$ is contained in the $\mathscr{O}_K$-group scheme $T' = \mathrm{Res}_{\mathscr{O}_{K'}/\mathscr{O}_K}(\mathbf{G}_m^r)$ which is an $\mathscr{O}_K$-torus (as we see by working étale-locally to split the covering $\mathrm{Spec}(\mathscr{O}_{K'}) \to \mathrm{Spec}(\mathscr{O}_K)$). Hence, we have a short exact sequence

$$1 \to M(\chi)' \to T' \to T'' \to 1$$

where $T'' := T'/M(\chi)'$ is another $\mathscr{O}_K$-torus. Thus, using the resulting long exact sequence in fppf cohomology, to prove $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{O}_K, M(\chi)') = 0$ it suffices to prove that $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{O}_K, T')$ vanishes and that any $\mathscr{O}_K$-torus (such as T'') has vanishing degree-1 cohomology. For the latter, first recall that degree-1 cohomology with affine coefficients classifies fppf torsors, so the degree-1 vanishing amounts to the triviality of such torsors when the coefficients are smooth and affine with connected fibers (such as a torus). To build a section splitting such a torsor over $\mathscr{O}_K$ it suffices (by smoothness of the coefficients, and the henselian property for $\mathscr{O}_K$) to find a section over the residue field. That is, we are reduced to proving the vanishing of degree-1 cohomology over k with coefficients in a smooth connected affine group (such as a torus). This is Lang's theorem, since k is finite.

Finally, to prove that $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{O}_K, T')$ is trivial, by using the definition of T' this amounts to vanishing of

$$\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{O}_K, \mathrm{Res}_{\mathscr{O}_{K'}/\mathscr{O}_K}(\mathbf{G}_m)).$$

The exactness of the Weil restriction functor in this case implies (by a δ-functor argument) that

$$\mathrm{H}^j_{\mathrm{fppf}}(\mathscr{O}_K, \mathrm{Res}_{\mathscr{O}_{K'}/\mathscr{O}_K}(\cdot)) \simeq \mathrm{H}^j_{\mathrm{fppf}}(\mathscr{O}_{K'}, \cdot)$$

for all $j \geq 0$. Hence, we just have to check the vanishing of $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{O}_{K'}, \mathbf{G}_m)$ By Grothedieck's work on Brauer groups, this is identified with $\mathrm{Br}(\mathscr{O}_{K'})$, and since $\mathscr{O}_{K'}$ is henselian this in turn is identified with $\mathrm{Br}(k')$. But k' is *finite*, so its Brauer group vanishes. ■

Now we can prove the main existence result for a proper (even projective) R-scheme that "classifies" ordinary-crystalline pushforwards of ρ.

Theorem 3.11. *For each $n \geq 1$, the functor on $R/\mathfrak{m}_R^n$-algebras given by*

$$A \rightsquigarrow \{L_A \subset V_A \,|\, L_A \text{ is an ordinary crystalline structure on } \rho_A\}$$

is represented by a closed subscheme $X_n \subseteq \mathbf{P}^1_{R/\mathfrak{m}_R^n}$.

There is a unique closed subscheme $X \subseteq \mathbf{P}^1_R$ such that $X \bmod \mathfrak{m}_R^n = X_n$ for all $n \geq 1$.

Proof. By the universal property of the projective line, $P_n := \mathbf{P}^1_{R/\mathfrak{m}_R^n}$ represents the functor carrying any $R/\mathfrak{m}_R^n$-algebra A to the set of locally free A-submodules $L_A \subset V_A = A^2$ of rank 1 such that V_A/L_A is also locally free of rank 1. Over P_n, consider the G_K-action on $\mathscr{O}^2_{P_n}$ defined by $\rho \bmod \mathfrak{m}_R^n$ and the $R/\mathfrak{m}_R^n$-algebra structure on $\mathscr{O}_{P_n}$.

For each $g \in G_K$ and any A-point of P_n, the condition that A-pullback of the g-action on $\mathscr{O}^2_{P_n}$ preserves the corresponding A-line in A^2 is represented by a closed subscheme Z_g of P_n. To prove this, we may work Zariski-locally on P_n so that the universal line subbundle is free and extends to a basis of $\mathscr{O}^2$. Then the vanishing of the resulting "lower left matrix entry" function over the open in the base is what cuts out the g-stability condition over such an open locus in P_n. These closed loci agree on overlaps and glue to the desired closed subscheme Z_g of P_n attached to g. Thus, the closed subscheme $Z_n = \cap_g Z_g \subseteq P_n$ representing the condition of G_K preserving the universal line subbundle of $\mathscr{O}^2$.

Consider the character $\eta_n : G_K \to \mathscr{O}(Z_n)^\times$ describing the G_K-action on the universal line subbundle over Z_n. The Zariski-closed conditions $\eta_n(g) = 1$ for all $g \in I_K$ cut out a closed subscheme $Z'_n \subseteq Z_n$ which represents the additional condition that the universal line subbundle is not only G_K-stable but has unramified G_K-action. In other words, Z'_n represents the functor of "ordinary structures" on ρ_n.

Over Z'_n, consider the further condition that the ordinary structure is crystalline. That is, for an A-point of Z'_n, we consider the property that the resulting A-line L_A in ρ_A is an ordinary crystalline structure. The map

$$\mathrm{H}^1(I_K, A(1)) \to A$$

defined by the valuation $K^{\mathrm{un}\times} \to \mathbf{Z}$ carries the class of (ρ_A, L_A) to an element $a \in A$, and this construction is functorial in A. Hence, by (the proof of) Yoneda's Lemma it defines a global function h_n on Z'_n. The zero scheme of h_n on Z'_n is clearly the desired X_n.

Having constructed $X_n \subset P_n$ for each $n \geq 1$, the behavior of moduli schemes with respect to base change implies that the isomorphism

$$P_n \simeq P_{n+1} \otimes_{R/\mathfrak{m}_R^{n+1}} (R/\mathfrak{m}_R^n)$$

carries X_n over to $X_{n+1} \bmod \mathfrak{m}_R^n$. In other words, $\{X_n\}$ is a system of compatible closed subschemes of the system $\{P_n\}$ of infinitesimal fibers of the proper morphism $\mathbf{P}^1_R \to \operatorname{Spec} R$ over the complete local noetherian ring R. Now comes the deepest step: by Grothendieck's "formal GAGA" (EGA III$_1$, §5), if R is any complete local noetherian ring and P is any proper R-scheme, then the functor

$$Z \rightsquigarrow \{Z \bmod \mathfrak{m}_R^n\}$$

from closed subschemes of P to systems of compatible closed subschemes of the infinitesimal fibers of P over $\operatorname{Spec} R$ is a *bijection*. (Even for $P = \mathbf{P}^1_R$ this is not obvious, and it fails miserably if we consider the affine line instead of the projective line.) Thus, we get the existence and uniqueness of the desired X. ■

Remark 3.12. Although each infinitesimal fiber X_n of X over $\operatorname{Spec} R$ has moduli-theoretic meaning for points valued in arbitrary $R/\mathfrak{m}_R^n$-algebras, we do *not* claim that X has a convenient moduli-theoretic meaning for its points valued in arbitrary R-algebras. In particular, $X_E = X \otimes_R R_E$ has no easy interpretation.

Nonetheless, it is X_E which will be of most interest to us. Thus, to work with X_E we need a way to understand its properties by studying the X_n's. This problem will be taken up in the next section.

4. Properties and applications of the ordinary crystalline moduli scheme

The construction of the proper R-scheme $f : X \to \operatorname{Spec} R$ is indirect, as formal GAGA is very abstract, but we can artfully use the construction to infer global properties of X which will be especially useful for the study of X_E. Our analysis rests on the following hypothesis which is in force throughout this section (unless we say otherwise):

Assume that $\det \rho : G_K \to R^\times$ *is cyclotomic and that* ρ *is the universal framed cyclotomic-determinant deformation ring of its reduction* ρ_k*. If* ρ_k *has only scalar endomorphisms, we also allow that* (ρ, R) *is the universal deformation of* ρ_k *with cyclotomic determinant.*

Proposition 4.1. *The* Λ*-scheme* X *is regular and flat, and* $X \bmod \pi$ *is reduced.*

Proof. By Proposition 2.1, Lemma 2.2, and Proposition 2.3, it suffices to prove that $X(B) \to X(B/J)$ is surjective for every artin local finite Λ-algebra B. Choose such a B, and let k' be its residue field, so B is canonically an algebra over $\Lambda' = W(k') \otimes_{W(k)} \Lambda$. It is harmless to make the finite étale scalar extension by $\Lambda \to \Lambda'$ throughout (this is compatible with the formation of X) to reduce to the case $k' = k$.

The maximal ideal of B is nilpotent, say with vanishing nth power for some $n \geq 1$, so the map of interest on points of X coincides with the analogue for X_n. Thus, the task is to show that if $\overline{L}$ is an ordinary crystalline structure on $\rho_{B/J}$ then it lifts to one on ρ_B. Let $x_0 \in X$ be the closed k-point corresponding to the specialization $(\rho_k, \overline{L}_k)$ of $(\rho_{B/J}, \overline{L})$ over the residue field k of B. Our problem is equivalent to showing that any local Λ-algebra map $\mathscr{O}^\wedge_{X,x_0} \to B/J$ lifts to B. Thus, it is sufficient (and even necessary) to prove that $\mathscr{O}^\wedge_{X,x_0}$ is a formal power series ring over Λ. To do this, we need to give a *deformation-theoretic* interpretation of this completion.

Since x_0 is a closed point, $R \to \mathscr{O}^\wedge_{X,x_0}$ is a local map and its reduction modulo $\mathfrak{m}_R^n$ recovers $\mathscr{O}^\wedge_{X_n,x_0}$ due to the relationship between X and the X_n's. But X_n is an actual moduli scheme over the ring $R/\mathfrak{m}_R^n$ (unlike X over R). In view of the assumed universal property of (ρ, R), it follows that $\mathscr{O}^\wedge_{X_n,x_0}$ is the deformation ring for ordinary crystalline structures lifting $\overline{L}_k$ on cyclotomic-determinant deformations of ρ_k (possibly with framing) having coefficients in Λ-finite artin local rings whose nth power vanishes. Hence, $\mathscr{O}^\wedge_{X,x_0}$ is the analogous formal deformation ring for arbitrary Λ-finite artin local coefficients (without restriction on the nilpotence order of the maximal ideal).

Our problem is therefore to prove that there is no obstruction to infinitesimal deformation of $(\rho_k, \overline{L}_k)$ as ordinary crystalline structures with cyclotomic determinant. (There is no obstruction when we impose the additional data of a framing, as that amounts to simply lifting some bases through a surjection of finite free modules.) That is, given such a representation over B/J we wish to lift it to one over B. The given representation with ordinary crystalline structure over B/J has diagonal characters $\{\chi\eta^{-1}, \eta\}$ for some unramified continuous $\eta : G_K \to (B/J)^\times$. Since $G_K/I_K = G_k = \widehat{\mathbf{Z}}$, we can lift η to an unramified continuous $\widetilde{\eta} : G_K \to B^\times$ (choose an arbitrary lift of $\eta(\mathrm{Frob}_k) \in (B/J)^\times$). We claim that this lifts to an ordinary crystalline deformation to B with diagonal characters $\{\chi\widetilde{\eta}^{-1}, \widetilde{\eta}\}$.

Thinking in terms of "upper-right matrix entries", we have to prove the surjectivity of the natural map

$$\mathrm{H}^1_{\mathrm{crys}}(K, B(\chi\widetilde{\eta}^{-2})) \to \mathrm{H}^1_{\mathrm{crys}}(K, (B/J)(\chi\eta^{-2})).$$

For $M = B(\widetilde{\eta}^{-2})$ we have $M/JM = B(\eta^{-2})$, and these are unramified discrete G_K-modules killed by a power of p. By Lemma 3.10, the natural map

$$\mathrm{H}^1_{\mathrm{crys}}(K, M(\chi)) \to \mathrm{H}^1_{\mathrm{crys}}(K, (M/JM)(\chi))$$

is surjective, so we are done. ■

Proposition 4.2. *The natural map* $f_E : X_E \to \mathrm{Spec}(R_E)$ *is a closed immersion, and this regular closed subscheme meets* $\mathrm{MaxSpec}(R_E)$ *in precisely the set of closed points* $x \in \mathrm{MaxSpec}(R_E)$ *such that* $\rho_x : G_K \to \mathrm{GL}_2(E(x))$ *is ordinary crystalline in the sense of Definition* 3.1*. In particular, the locus of ordinary crystalline points in* $\mathrm{MaxSpec}(R_E)$ *is Zariski-closed.*

Proof. By Proposition 2.5, it suffices to prove that f_E is injective on C-valued points for all E-finite C. We may assume C is local, so its residue field E' is E-finite. By Hensel's Lemma and the separability of E'/E, the E-algebra structure on C uniquely extends to an E'-algebra structure lifting the residue field. Thus, $C = E' \oplus I$ for the nilradical I of C. In particular, if Λ' denotes the valuation ring of E' then $\Lambda' \oplus I$ is a local Λ'-subalgebra of C, though it is generally not Λ'-finite since the E'-vector space I is generally nonzero.

To prove that the map $X(C) \to (\mathrm{Spec}\, R)(C)$ between sets of C-valued points over Λ is injective, we can first replace $f : X \to \mathrm{Spec}\, R$ with its scalar extension by $\Lambda \to \Lambda'$. This scalar extension is compatible with the formation of X (as may be checked on infinitesimal fibers over $\mathrm{Spec}\, R$), so we may assume $E' = E$.

We fix a map $\overline{\phi} : \mathrm{Spec}\, C \to \mathrm{Spec}\, R$ over Λ and seek to prove that it has at most one lift to a map $\phi : \mathrm{Spec}\, C \to X$. Consider the Λ-algebra map

$$\Lambda[\![x_1, \dots, x_m]\!]/(f_1, \dots, f_s) = R \to C = E \oplus I$$

corresponding to $\overline{\phi}$. Passing to the quotient $C/I = E$, this map carries each x_j to (c_j, y_j) for some $y_j \in I$ and $c_j \in (\pi)$. Thus, we can make the formal change of parameters $x_j \mapsto x_j - c_j$ to get to the case when $x_j \mapsto y_j \in I$ for all j. Since $I^N = 0$ for some large N, any monomial in the x's with large enough degree maps to 0 in C. Hence, the image of R in C is contained in $\widetilde{C} := \Lambda \oplus J$ for a finite Λ-submodule $J \subset I$, and we can increase J so that $\widetilde{C}$ is a local finite flat Λ-subalgebra of C. Note that $\widetilde{C}_E = C$.

Now consider any ϕ lifting $\overline{\phi}$. The restriction of ϕ to the closed point of $\operatorname{Spec} C$ is a map $\phi_0 : \operatorname{Spec} E \to X$ over the specialization $\overline{\phi}_0 : \operatorname{Spec} E \to R$ over Λ. This latter specialization is a Λ-algebra map $R \to E$ and hence lands in Λ. This resulting Λ-valued point of $\operatorname{Spec} R$ *uniquely* lifts to a Λ-valued point of the R-proper X extending ϕ_0, due to the valuative criterion for properness. Consider an open affine $\operatorname{Spec} A$ in X around the image of the Λ-valued point of X extending ϕ_0. This open affine contains the image of ϕ since $E = C/I$ with I nilpotent. The resulting Λ-algebra map

$$A \to C = E \oplus I$$

with A a *finite type* R-algebra lands in $\Lambda \oplus I$ by the choice of A, and so lands in $\widetilde{C}$ upon taking J big enough.

We have now constructed an R-map $\operatorname{Spec} \widetilde{C} \to X$ that serves as an "integral model" (over Λ) for ϕ. The choice of J can always be increased even further, so to prove the uniqueness of ϕ (if one exists) it suffices to consider a pair of R-maps $\operatorname{Spec} \widetilde{C} \rightrightarrows X$ over a *common* local Λ-map $\operatorname{Spec} \widetilde{C} \to R$, and to show that the resulting "generic fiber" maps $\operatorname{Spec} C \rightrightarrows X$ coincide.

By locality of the Λ-map $R \to \widetilde{C}$ to the Λ-finite $\widetilde{C}$, a cofinal system of open ideals in $\widetilde{C}$ is given by the $\mathfrak{m}_R^n \widetilde{C}$. Using the formal GAGA construction of X from the X_n's, it follows that the pair of maps $\operatorname{Spec} \widetilde{C} \rightrightarrows X$ corresponds to a pair of ordinary crystalline structures on $\rho_{\widetilde{C}}$ (i.e., compatible such structures over each artinian quotient of $\widetilde{C}$). These coincide provided that the resulting pair of filtrations on ρ_C coincide, since a $\widetilde{C}$-line in $\widetilde{C}^2$ is uniquely determined by the associated C-line in C^2 (via saturation of Λ-finite submodules in finite-dimensional E-vector spaces, as $\widetilde{C}$ is finite flat over Λ and $\widetilde{C}_E = C$).

Thus, the injectivity of f_E on C-points is reduced to proving that if $G_K \to \mathrm{GL}_2(C)$ is a homomorphism admitting an upper triangular form

$$\begin{pmatrix} \chi\eta^{-1} & * \\ 0 & \eta \end{pmatrix}$$

relative to some C-basis with $\eta : G_K \to C^\times$ unramified then the $\chi\eta^{-1}$-line is uniquely determined. This C-line is precisely the locus of vectors on which I_K acts through χ. Indeed, to prove this it suffices to check that in the quotient C-line η the space of χ-isotypic vectors for the I_K-action vanishes. Since χ is valued in $E^\times$ and C has an E-linear filtration by ideals with successive codimension 1 over E, we just need to observe that $\chi \neq 1$ in $E^\times$ since $\operatorname{char}(E) = 0$. (In more sophisticated p-adic Hodge theory settings, the analogue of this step requires results such as Tate's isogeny theorem for p-divisible groups over Λ: uniqueness results for integral structures in case of generic characteristic 0.)

To identify the closed points of X_E within $\operatorname{MaxSpec}(R_E)$, we will use Example 3.9. Closed points of X_E are obtained from E'-valued points for finite extension fields E'/E; let Λ' be the valuation ring of E'. The preceding argument with the valuative criterion for properness shows that any E'-valued point of X_E uniquely extends to a Λ'-valued point of X over $\operatorname{Spec} R$, and such a point corresponds precisely to a filtration on $\rho_{E'}$ as in Definition 3.1, due to Example 3.9. This proves that the closed points of X_E are the ordinary crystalline points of $\operatorname{MaxSpec}(R_E)$. ■

Corollary 4.3. *Let $\rho_0 : G_K \to \mathrm{GL}_2(k)$ be an ordinary crystalline representation with cyclotomic determinant. Let $\rho : G_K \to \mathrm{GL}_2(R)$ be the universal framed deformation with cyclotomic determinant; if ρ_0 has only scalar endomorphisms we allow alternatively that (ρ, R) is the universal deformation of ρ_0 with cyclotomic determinant.*

The locus of ordinary crystalline points in $\mathrm{MaxSpec}(R_E)$ *is Zariski-closed, and if*

$$\mathrm{Spec}(R^{\mathrm{ord}})$$

denotes the Zariski closure in $\mathrm{Spec}(R)$ *of this locus in* $\mathrm{Spec}(R_E)$ *then R_E^{ord} is regular, and it is a domain except precisely when $(\rho_0)_{\overline{k}} = \psi_1 \oplus \psi_2$ with $\psi_1 \neq \psi_2$ and each ψ_i an unramified $\overline{k}^{\times}$-valued twist of $\omega := \chi \bmod p$.*

Note that the exceptional cases at the end of the corollary do not include the case when ρ_0 makes G_K act trivially. This is really useful: we will apply this corollary later for the universal framed deformation ring of a 2-dimensional *trivial* residual representation (after making a preliminary finite extension on K).

Proof. Apply the preceding theory to ρ, so we get the "moduli scheme" $f : X \to \mathrm{Spec}\, R$ that is regular and induces a closed immersion f_E over E whose image on closed points is the set of ordinary crystalline points of $\mathrm{MaxSpec}(R_E)$. This gives the Zariski-closedness and regularity claims, so the domain property amounts to the assertion that X_E is connected. We saw above that $X \bmod \pi$ is reduced, so by Lemma 2.2 the connectedness of X_E is equivalent to the connectedness of the proper special fiber $f_0 : X_0 \to \mathrm{Spec}\, k$ (since X_0 is certainly non-empty, due to its moduli-theoretic meaning and the fact that the ordinary crystalline hypothesis on ρ_0 provides a k-point of X_0).

Loosely speaking, X_0 is the moduli scheme of ordinary crystalline structures on ρ_k. That is, it parameterizes all G_K-stable lines in ρ_0 on which I_K acts by χ. By construction, the non-empty X_0 is a closed subscheme of $\mathbf{P}^1_k$, so it is connected except precisely when it is not the entire projective line nor is a single geometric point (as we know $X_0(k) \neq \emptyset$). The condition $X_0 = \mathbf{P}^1_k$ says that every line in $(\rho_0)_{\overline{k}}$ is G_K-stable and has I_K-action by ω. In other words, ρ_0 is a scalar representation via an unramified twist of ω. Thus, the disconnectedness case is when $(\rho_0)_{\overline{k}}$ has more than one – but only finitely many! – G_K-stable line with action by an unramified twist of ω. The number of such lines is therefore exactly two, by the Jordan-Hölder theorem. Such cases are precisely when $(\rho_0)_{\overline{k}}$ is a direct sum of distinct $\overline{k}^{\times}$-valued characters of G_K which are each an unramified twist of ω. ■

Remark 4.4. There is a variant of the preceding considerations which is useful in practice: require the determinant of ρ_0 and its deformations to be $\chi\psi$ for a fixed unramified (possibly nontrivial) continuous character $\psi : G_K \to \Lambda^{\times}$.

In such cases the conclusions of Corollary 4.3 hold, by essentially the same proof. The point is that to prove these claims we can first make a scalar extension from Λ to the valuation ring of a finite extension of E, so we can arrange that the unramified $\psi : G_K \to \Lambda^{\times}$ admits a square root. It is then harmless in the Galois deformation theory to twist everything by the reciprocal of this square root, so we are thereby reduced back to the case $\psi = 1$ which was treated above.

The final application we take up is the determination of the dimension of the regular Zariski closure X_E of the locus of ordinary crystalline points in $\mathrm{Spec}(R_E)$ for a framed deformation ring $R_{\rho_0}^{\square,\det=\chi\psi}$. In some cases this Zariski closure is disconnected, but we claim that its connected components are always of the same pure dimension:

Proposition 4.5. *The* E*-fiber of the ordinary crystalline framed deformation ring*

$$R_{\rho_0}^{\square,\mathrm{ord},\det=\chi\psi}$$

has dimension $3+[K:\mathbf{Q}_p]$ *at all closed points.*

Proof. As above, we may reduce to the case $\psi=1$ by making a suitable finite extension on E (which is harmless for our purposes). In view of the regularity, we just have to compute the dimension of the *tangent space* at each closed point in characteristic 0. This will be a Galois H^1 with coefficients in a p-adic field.

By the proof of Proposition 4.2 (relating C-valued points and $\widetilde{C}$-valued points) and the lecture in the fall on charateristic-0 deformation rings, if we identify a closed point $x\in X_E$ with an ordinary crystalline representation $\rho_x : G_K \to \mathrm{GL}_2(E(x))$ then the (regular) completed local ring of X_E at x is the deformation ring of ρ_x relative to the conditions of having determinant χ and being ordinary crystalline (in the sense of Definition 3.1, generalized in the evident manner to allow coefficients in any finite E-algebra, not just finite extension fields of E).

The method of the proof of the Corollary in §1 of Samit's lecture in the fall shows (OK, this should be revised for clarity!) carries over to characteristic-zero deformation theory, so the dimension of the tangent space to the cyclotomic-determinant framed deformation functor exceeds the dimension of the tangent space to the cyclotomic-determinant deformation functor by $\dim \mathrm{PGL}_2 + h^0(\mathrm{ad}^0(\rho_x)) = 3 + h^0(\mathrm{ad}^0(\rho_x))$. Thus, the problem is to prove that in the tangent space $\mathrm{H}^1(K, \mathrm{ad}^0(\rho_x))$ to the cyclotomic-determinant deformation ring of ρ_x, the space of first-order deformations which are ordinary crystalline has $E(x)$-dimension $[K:\mathbf{Q}_p]$ if the reducible ρ_x has only scalar endomorphisms and $1+[K:\mathbf{Q}_p]$ otherwise (the case when ρ_x is a direct sum of characters, necessarily distinct due to ramification considerations). The representation ρ_x has the form

$$\rho_x \simeq \begin{pmatrix} \chi\eta^{-1} & * \\ 0 & \eta \end{pmatrix}$$

for some unramified $\eta : G_K \to \mathscr{O}_{E(x)}^\times$, and upon restriction to I_K (which kills η and η^{-1}) the resulting class in $\mathrm{H}^1(I_K, \mathbf{Z}_p(1))$ arises from units in $\mathscr{O}_{K^{\mathrm{un}}}^\wedge$ via Kummer theory (i.e., it is killed by the natural map $\mathrm{H}^1(I_K, \mathbf{Z}_p(1)) \to \mathbf{Z}_p$ defined by the valuation map $K^{\mathrm{un}\times} \to \mathbf{Z}$ and Kummer theory). It is harmless to rename $E(x)$ as E, so this is now a very concrete problem in Galois cohomology and Kummer theory using the "explicit" upper-triangular description of ρ_x.

The only method we know to carry out the dimension calculation in the general case is to bring in deeper methods related to p-adic Hodge theory or p-divisible groups over very ramified p-adic discrete valuation rings. But we will only apply the Proposition in the special case that ρ_0 is the *trivial* 2-dimensional residual representation. So now we will give a proof only in this case. Note that the residual triviality forces the mod-p cyclotomic character of G_K to be trivial, so there is a distinguished ordinary crystalline lift with cyclotomic determinant:

$\rho_\chi := E(\chi) \oplus E$. We have seen that for our ρ_0, the ord-crystalline framed deformation ring R with determinant χ has the property that R_E is regular with *connected* spectrum. Provided that the $E(x)$-dimension of its tangent space at any closed point $x \in \mathrm{MaxSpec}(R_E)$ is *independent* of x, it would suffice to carry out the dimension computation at a single x. For example, we would be reduced to computing the E-dimension of the ord-crystalline subspace of

$$\mathrm{H}^1(K, \mathrm{ad}^0(\rho_\chi)) = \mathrm{Ext}^1_K(\rho_\chi, \rho_\chi)^{\det=\chi}.$$

The equidimensionality of the tangent spaces on $\mathrm{MaxSpec}(R_E)$ is a special case of:

Lemma 4.6. *Let R be a quotient of a formal power series ring over Λ, and assume that R_E is normal with connected spectrum. Then all maximal ideals of R_E have the same height.*

Proof. Replacing R with the quotient by its nilradical has no effect on R_E, so we can assume that R is reduced. Likewise we can assume it has vanishing π-power torsion, so R is Λ-flat. Hence, R is a domain (as R_E is a domain, due to regularity and connectedness of its spectrum). But R is excellent, so the normalization map $R \to R'$ is module-finite. The residue field may increase in the normalization process, so R' is a quotient of a formal power series ring over the valuation ring of some finite unramified extension E' of E. Then $R_E = R'_E = R'_{E'}$, so we can replace (R, E) with (R', E') to reduce to the case when R is a normal domain.

There are now two ways to proceed: commutative algebra, or rigid-analytic geometry. For the commutative algebra method, let $\mathfrak{p}$ be a maximal ideal of $R_E = R[1/p]$. The complete local noetherian domain R is catenary ([5, 31.6(iv)]; in general, the catenary property is also part of the definition of excellence), so $\dim R_P + \dim(R/P) = \dim(R)$ for any prime ideal P of R. Taking P corresponding to $\mathfrak{p}$, we get

$$\dim(R_E)_\mathfrak{p} = \dim(R_P) = \dim(R) - \dim(R/P),$$

so it suffices to prove that $\dim(R/P) = 1$ for all such P. The quotient R/P is a Λ-flat quotient of a formal power series ring over Λ such that its generic fiber ring is $R_E/\mathfrak{p}$, which is a field of finite degree over E. Hence, the *subring* R/P lies in the valuation ring of this finite extension of E, whence R/P is module-finite over Λ and so is of dimension 1. This completes the commutative algebra proof.

We merely sketch the rigid-analytic method, which provides nice geometric intuition (and can be made rigorous). By choosing a presentation

$$R \simeq \Lambda[\![t_1, \ldots, t_m]\!]/(f_1, \ldots, f_s),$$

it is natural to associate to R the rigid-analytic space M over E defined by $f_1 = \cdots = f_s = 0$ in the open unit m-disc over E. This construction is given in more intrinsic terms in [1, 7.1]. That exposition proves some very useful related facts: there is a natural bijective correspondence between $\mathrm{MaxSpec}(R_E)$ and the underlying set of M such that the completed local rings at corresponding points are naturally E-isomorphic [1, 7.1.9], and R_E is identified with the ring of bounded global analytic functions on M (here we use the normality of R) [1, 7.3.6]. Thus, the completed local rings on M are normal (as the excellent R_E is normal, by hypothesis, so its completed local rings are normal). But it is elementary to check that a noetherian local ring is a normal domain if its completion is, so the analytic local rings on M are normal and the affinoid opens in M have normal coordinate ring. Moreover, M is *connected* for the Tate topology since idempotents are bounded analytic functions and R_E is a domain (due to normality and connectedness of its spectrum).

Since the completed local rings of R_E at its maximal ideals coincide with the completed local rings on M, and completion preserves dimension for local noetherian rings, to prove that all maximal ideals of R_E have the same height it suffices to prove that all local rings on M have the same dimension. More generally, any normal rigid-analytic space has pointwise dimension that is locally constant for the Tate topology (and hence globally constant in the *connected* case): this comes down to the fact that an affinoid space associated to a domain has constant pointwise dimension, which is [2, Lemma 2.1.5].

As an alternative argument in the rigid-analytic case if we assume R_E is regular (as holds in the cases we need), regularity of M implies smoothness of M since $\mathrm{char}(E) = 0$, so the coherent sheaf $\Omega^1_{M/E}$ is locally free on M with rank $\dim_m(M)$ at any $m \in M$. But connectedness of M forces this rank to be globally constant, whence M has constant pointwise dimension as desired. ■

Returning to the proof of Proposition 4.5, we just have to prove that the ordinary crystalline subspace of $\mathrm{Ext}^1_K(\rho_\chi, \rho_\chi)^{\det=\chi}$ has E-dimension equal to $[K : \mathbf{Q}_p]$. Since $\rho_\chi = E(\chi) \oplus E$, we have an equality of E-vector spaces

$$\mathrm{Ext}^1_K(\rho_\chi, \rho_\chi) = \mathrm{Ext}^1_K(E(\chi), E(\chi)) \oplus \mathrm{Ext}^1_K(E(\chi), E) \oplus \mathrm{Ext}^1(E, E(\chi)) \oplus \mathrm{Ext}^1_K(E, E). \tag{4.1}$$

The condition that an $E[\epsilon]$-deformation of ρ_χ has cyclotomic determinant amounts to the condition that its Ext-class ξ on the left side of (4.1) has components in outer terms that are Cartier dual to each other (as one checks with a direct 4×4 matrix calculation). The ordinarity condition likewise amounts to the vanishing of the $\mathrm{Ext}^1_K(E(\chi), E)$ component. The crystalline condition then says that the component in $\mathrm{Ext}^1_K(E, E) = \mathrm{H}^1(K, E)$ is unramified (a 1-dimensional subspace) and the component in $\mathrm{Ext}^1_K(E, E(\chi)) = \mathrm{H}^1(K, E(1))$ is crystalline.

Since $\rho_\chi = E(\chi) \oplus E$ has non-scalar endomorphisms, our problem is to prove that the ordinary-crystalline Ext-space with cyclotomic determinant inside of the left side of (4.1) has E-dimension $1 + [K : \mathbf{Q}_p]$. We have already accounted for one dimension, and it remains to prove that $\mathrm{H}^1_{\mathrm{crys}}(K, E(1)) = \mathrm{H}^1_{\mathrm{crys}}(K, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} E$ has E-dimension $[K : \mathbf{Q}_p]$. But $\mathrm{H}^1_{\mathrm{crys}}(K, \mathbf{Z}_p(1))$ is the multiplicative p-adic completion of $\mathscr{O}_K^\times = k^\times \times (1 + \mathfrak{m}_K)$, which is $1 + \mathfrak{m}_K$. Via the logarithm, its $\mathbf{Z}_p$-rank as a multiplicative $\mathbf{Z}_p$-module is $[K : \mathbf{Q}_p]$. ■

References

[1] A.J. de Jong, "Crystalline Dieudonné theory via formal and rigid geometry", Publ. Math. IHES **82** (1995), pp. 5–96.

[2] B. Conrad, "Irreducible components of rigid spaces", Annales Fourier **49**(2), 1999, pp. 473–541.

[3] M. Kisin, "Modularity of 2-adic Barsotti-Tate representations", Inv. Math. **178**(3), 2009, pp. 587–634.

[4] H. Matsumura, *Commutative algebra* (2nd ed.), Benjamin, 1980.

[5] H. Matsumura, *Commutative ring theory*, Cambridge Advanced Studies in Mathematics **8**, Cambridge, 1986.