Calculating deformation rings

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1 Introduction

We are interested in computing local deformation rings away from p. That is, if L is a finite extension of $\mathbf{Q}_\ell$ and V is a 2-dimensional representation of G_L over $\mathbf{F}$, where $\mathbf{F}$ is a finite extension of $\mathbf{F}_p$, $\ell \neq p$, we wish to study the deformation rings $R_V^\square$ and $R_V^{\psi,\square}$. Here $\psi : G_L \to \mathcal{O}^\times$ is a continuous unramified character, $\mathcal{O}$ is the ring of integers of a finite extension E of $\mathbf{Q}_p$ which has residue field $\mathbf{F}$, and $R_{V_\mathbf{F}}^{\psi,\square}$ is the quotient of $R_{V_\mathbf{F}}^\square$ corresponding to deformations with determinant $\psi\chi$, where $\chi : G_L \to \mathbf{Z}_p^\times$ is the cyclotomic character.

Note that $R_{V_\mathbf{F}}^{\psi,\square}$ exists: There is a natural determinant map from the universal 2-dimensional (framed) representation to the universal 1-dimensional (framed) representation, and we take the fiber over the closed point corresponding to $\chi\psi$.

We define the following two deformation problems:

- $D_V^{ur,\psi,\square}$ is the deformation functor which spits out unramified framed deformations with determinant $\psi\chi$
- $L_V^{\chi,\square}$ is the deformation functor which spits out pairs (V_A, L_A) of framed deformations with determinant χ, together with a G_L-stable A-line with G_L acting via χ on L_A. That is, L_A is a projective rank 1 A-module such that V_A/L_A is a projective A-module with trivial G_L-action.

Most of this talk will be about the structure of the ring representing the second functor.

2 Lies I will tell, and auxiliary categories of rings

The minor lie I will tell is that I will entirely suppress the language of categories fibered in groupoids, and pretend we are working with functors. This will allow me to avoid 2-categorical language. But to make what I say literally true, one has to handle non-trivial isomorphisms of deformations via the language of groupoids.

The more major lie I will tell is that after I finish this section, I will try to avoid talking about the various categories of algebras that are involved.

The basic set-up is representing certain deformations of a fixed residual representation (in characteristic p). The deformations are a priori to finite local artinian rings with fixed residue field. But we want to be able to take generic fibers of our representing objects in a sensible way, so we need techniques for passing to characteristic 0 points.

To do this, we need a variety of confusing auxiliary categories of algebras. To demonstrate, let $E/\mathbf{Q}_p$ be a finite extension with residue field containing $\mathbf{F}$, let $\mathcal{O} \subset \mathcal{O}_E$ be a discrete valuation ring finite over $W(\mathbf{F})$, and let D be a deformation functor on the category $\mathfrak{AA}_{\mathcal{O}}$ of finite local artinian $\mathcal{O}$-algebras with residue field $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}$, and let $E/\mathbf{Q}_p$ be a finite extension with residue field containing $\mathbf{F}$. We will be interested in the category $\mathfrak{AR}_E$ of finite local $W(\mathbf{F})[1/p]$-algebras with residue field E. We also introduce the following categories:

- $\widehat{\mathfrak{AR}}_{\mathcal{O}}$: $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ is the category of complete local noetherian $\mathcal{O}$-algebras with residue field $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}$.
- $\widehat{\mathfrak{AR}}_{\mathcal{O},(\mathcal{O}_E)}$: $\widehat{\mathfrak{AR}}_{\mathcal{O},(\mathcal{O}_E)}$ is the category of $\mathcal{O}$-algebras A in $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ equipped with a map of $\mathcal{O}$-algebras $A \to \mathcal{O}_E$.
- Int_B: Given $B \in \mathfrak{AR}_E$, Int_B is the category of finite $\mathcal{O}_E$-subalgebras $A \subset B$ such that $A \otimes_{\mathcal{O}_E} E = B$.

Note that Int_B is a subcategory of $\widehat{\mathfrak{AR}}_{\mathcal{O},(\mathcal{O}_E)}$ (A obviously has a map to E, and by finiteness or the same sort of arguments as in Brian's talk, it actually lands in $\mathcal{O}_E$), and there is a natural functor $\widehat{\mathfrak{AR}}_{\mathcal{O},(\mathcal{O}_E)} \to \widehat{\mathfrak{AR}}_{\mathcal{O}}$.

Also note that we can canonically extend D to a groupoid on $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, by setting $D(\varprojlim R/\mathfrak{m}_R^{n+1}) = \varprojlim D(R/\mathfrak{m}_R^{n+1})$.

Now fix some $\xi \in D(\mathcal{O}_E)$, which makes sense by the preceding comment. We define a groupoid $D_{(\xi)}$ on $\widehat{\mathfrak{AR}}_{\mathcal{O},(\mathcal{O}_E)}$ by setting $D_{(\xi)}$ to be the fiber over ξ. More precisely, $D_{(\xi)}(A)$ consists of objects of $D(A)$ together with morphisms (in D) covering the given map $A \to \mathcal{O}_E$.

Finally, we can extend $D_{(\xi)}$ to $\mathfrak{AR}_E$. We note that $B \in \mathfrak{AR}_E$ can be exhausted by objects in Int_B, so we set $D_{(\xi)}(B) = \varinjlim_{A \in \mathrm{Int}\, B} D_{(\xi)}(A)$.

Now Kisin proves two crucial lemmas about these groupoids (which he calls a lemma and a proposition). The first tells us how to get universal deformation rings for the groupoids on $\mathfrak{AR}_E$, and the second relates those groupoids to the ones we would naively expect (for some deformation problems we already care about):

Lemma 2.1. *If D is pro-represented by a complete local $\mathcal{O}$-algebra R, then $D_{(\xi)}$ is pro-represented (on $\mathfrak{AR}_E$) by the complete local $\mathcal{O}[1/p]$-algebra $\hat{R}_\xi$ obtained by completing $R \otimes_{\mathcal{O}} E$ along the kernel I_ξ of the map $R \otimes_{\mathcal{O}} E \to E$ induced by ξ.*

Lemma 2.2. *Fix a residual representation V over $\mathbf{F}$, and carry out the above program for D_V and $D_V^\square$. Then there are natural isomorphisms of groupoids*

$$D_{V,(\xi)} \tilde{\to} D_{V_\xi} \text{ and } D^\square_{V,(\xi)} \tilde{\to} D^\square_{V_\xi}$$

3 Main result

The main result we will prove is the following:

Theorem 3.1. *Let V be any 2-dimensional representation of G_L (over $\mathbf{F}$). Fix a continuous unramified character $\psi : G_L \to \mathcal{O}^\times$ and consider $R_V^{\psi,\square}$, the quotient of $R_V^\square$ corresponding to deformations of V with determinant $\psi\chi$. Then* $\mathrm{Spec}\, R_V^{\psi,\square}[1/p]$ *is 3-dimensional, and it is the scheme-theoretic union of formally smooth components.*

There are several claims implicit in this theorem, namely the existence, smoothness, connectedness, and dimension of $R_V^{\mathrm{ur},\psi,\square}$ and $R_V^{\chi\gamma,\gamma,\square}$, as well as the connectedness of $R_V^{\psi,\square}$. We assume these for the moment and go on with the proof.

Proof. Let E'/E be a finite extension, let $x : R_V^{\psi,\square}[1/p] \to E'$ be a point of $\operatorname{Spec} R_V^{\psi,\square}[1/p]$ with residue field E' (so that it is actually an E'-point), and let V_x be the induced representation with coefficients in E'. We know (from Brian's talk on characteristic 0 points of deformation rings) that the completion of $R_V^{\psi,\square}[1/p]$ at the maximal ideal $\mathfrak{m}_x = \ker x$ represents deformations of V_x. The tangent space at x is $H^1(G_L, \operatorname{ad}^0 V_x)$. Obstructions to deforming representations live in H^2 groups, so $R_V^{\psi,\square}[1/p]$ at x will be formally smooth at any point x where $H^2(G_L, \operatorname{ad}^0 V_x)$ vanishes.

Given any framed deformation problem $D^\square$ (with coefficients in some unspecified field H), there is a natural morphism $D^\square \to D$ to the unframed problem given by "forgetting the basis". This morphism is formally smooth in the sense that artinian points of D can be lifted.

Furthermore, the fibers of the morphism of tangent spaces $D^\square(H[\varepsilon]) \to D(H[\varepsilon])$ are principal homogeneous spaces under $\operatorname{ad}/\operatorname{ad}^{G_L}$. Specifically, given a residual representation V_H and a choice of (unframed) deformation $V_{H[\varepsilon]}$, $\ker(\mathrm{GL}_2(H[\varepsilon]) \to \mathrm{GL}_2(H)) = 1 + \varepsilon M_2(H[\varepsilon]) \cong \operatorname{End}_H V_H$ acts (via conjugation) on the fiber over $V_{H[\varepsilon]}$. Then it is easy to check that $1 + \varepsilon M$ acts trivially on the fiber if and only if M is in $\operatorname{ad}^0 V_H$.

Counting dimensions,

$$\dim_F D^\square(\mathbf{F}[\varepsilon]) = \dim_F D(\mathbf{F}[\varepsilon]) + \dim_F \operatorname{ad} - \dim_F H^0(G_L, \operatorname{ad}) \tag{3.1}$$

Using this formula, we see that the tangent space to $\operatorname{Spec} R_V^{\psi,\square}[1/p]$ at x has E'-dimension

$$\begin{aligned}
&\dim_{E'} H^1(G_L, \operatorname{ad}^0 V_x) + \dim_{E'} \operatorname{ad} V_x - \dim_{E'} H^0(G_L, \operatorname{ad} V_x) \\
&\quad= \dim_{E'} H^1(G_L, \operatorname{ad}^0 V_x) + \dim_{E'} \operatorname{ad} V_x - (\dim_{E'} H^0(G_L, \operatorname{ad}^0 V_x) - 1) \\
&\quad= -\big(\dim_{E'} H^2(G_L, \operatorname{ad}^0 V_x) - \dim_{E'} H^1(G_L, \operatorname{ad}^0 V_x) + \dim_{E'} H^0(G_L, \operatorname{ad}^0 V_x)\big) \\
&\qquad+ \dim_{E'} H^2(G_L, \operatorname{ad}^0 V_x) + \dim_{E'} \operatorname{ad} V_x - 1 \\
&\quad= \dim_{E'} H^2(G_L, \operatorname{ad}^0 V_x) + 3
\end{aligned}$$

the last step following by the Euler characteristic formula for p-adic coefficients. Thus, if $H^2(G_L, \operatorname{ad}^0 V_x) = 0$, x will be a formally smooth point of $\operatorname{Spec} R_V^{\psi,\square}[1/p]$ with a 3-dimensional tangent space.

Now suppose $H^2(G_L, \operatorname{ad}^0 V_x) \neq 0$. By the p-adic version of Tate local duality, $\dim_{E'} H^2(G_L, \operatorname{ad}^0 V_x) = \dim_{E'} H^0(G_L, (\operatorname{ad}^0 V_x)^*)$, which is $\dim_{E'} H^0(G_L, \operatorname{ad}^0 V_x(1))$

(because ad^0 is self-dual). Now we have the split exact sequence of G_L-modules

$$0 \to \mathrm{ad}^0 V_x(1) \to \mathrm{ad}\, V_x(1) \to E'(1) \to 0$$

which gives us an exact sequence in cohomology:

$$0 \to H^0(G_L, \mathrm{ad}^0 V_x(1)) \to H^0(G_L, \mathrm{ad}\, V_x(1)) \to H^0(G_L, E'(1))$$

But $H^0(G_L, E'(1)) = 0$ so

$$H^0(G_L, \mathrm{ad}^0 V_x(1) = H^0(G_L, \mathrm{ad}\, V_x(1)) = H^0(G_L, \mathrm{Hom}(V_x, V_x(1)))$$

In particular, if $H^2(G_L, \mathrm{ad}^0 V_x) \neq 0$, there is a non-zero homomorphism (of G_L-modules) $V_x \to V_x(1)$. It has 1-dimensional (G_L-stable) image and kernel, so there is some character γ such that $0 \to \gamma \to V_x \to \gamma(1) \to 0$ is exact. But such extensions are classified by $H^1(G_L, E'(-1))$, which is 0: the Euler characteristic formula says that $\dim_{E'} H^0(G_L, E'(-1)) - \dim_{E'} H^1(G_L, E'(-1)) + \dim_{E'} H^2(G_L, E'(-1)) = 0$, but $H^0(G_L, E'(-1))$ is clearly zero, and $H^2(G_L, E'(-1))$ is dual to $H^0(G_L, E'(2))$, which is zero, so $H^1(G_L, E'(-1))$ is zero as well. So this extension splits.

We have shown that if $H^2(G_L, \mathrm{ad}^0 V_x) \neq 0$, then $V_x = \gamma \oplus \gamma\chi$ for some character $\gamma : G_L \to E'^{\times}$. If γ is unramified, then this implies that x is in the image of both $R_V^{\mathrm{ur},\gamma^2,\square}$ and $R_V^{\chi\gamma,\gamma,\square}$.

So the only singular points of $\mathrm{Spec}\, R_V^{\psi,\square}[1/p]$ lie in the intersection of two formally smooth components.

$\square$

The definition of formal smoothness requires us to be able to lift through *any* square-zero thickening, but we only looked at what happens at artinian points of $\mathrm{Spec}\, R_V^{\psi,\square}[1/p]$; the commutative algebra necessary to justify this is discussed in Brian's notes on $\ell = p$.

4 Unramified deformations

We've seen previously that for the unframed case, the tangent space at x for unramified deformations with fixed determinant is $H^1(G_L/I_L, (\mathrm{ad}^0 V_x)^{I_L})$, and the obstruction space should be $H^2(G_L/I_L, (\operatorname{ad}^0 V_x)^{I_L}) = 0$. We have the exact sequence

$$0 \to (\operatorname{ad}^0 V_x)^{G_L} \to (\operatorname{ad}^0 V_x)^{I_L} \stackrel{\text{Frob}-\text{id}}{\longrightarrow} (\operatorname{ad}^0 V_x)^{I_L} \to (\operatorname{ad}^0 V_x)^{I_L}/(\text{Frob}-\text{id})(\operatorname{ad}^0 V_x)^{I_L} \to 0$$

This implies that $\dim_{E'} H^0(G_L, \operatorname{ad}^0 V_x) = \dim_{E'} H^1(G_L/I_L, (\operatorname{ad}^0 V_x)^{I_L})$. And since the tangent space for the framed case has dimension $\dim_{E'} H^1(G_L/I_L, (\operatorname{ad}^0 V_x)^{I_L}) + \dim_{E'} \operatorname{ad}^0 V_x - \dim_{E'} H^0(G_L, \operatorname{ad}^0 V_x)$ by the discussion in the previous section, this implies that the tangent space of $R_V^{ur,\psi,\square}$ has dimension $\dim_{E'} \operatorname{ad}^0 V_x = 3$.

So granting existence, $R_V^{ur,\psi,\square}$ is formally smooth and 3-dimensional.

5 $R^{\chi\gamma,\gamma,\square}$

We begin this section with a general lemma.

Lemma 5.1. *Let $\mathcal{O}$ be a local $W(k)$-algebra with residue field k, with K the fraction field of $W(k)$, and let X be a proper residually reduced $\mathcal{O}$-scheme. Then the components of the fiber of X over the closed point of $\mathcal{O}$ are in bijection with the components of $X[1/p]$.*

Proof. Consider a connected component of $X[1/p] = X \otimes_{W(k)} K$ and let e be the idempotent which is 1 on this component and 0 on the others. Then if ϖ is a uniformizer of $W(k)$, there is some n such that $\varpi^n e$ extends to a global section of X. But $(\varpi^n e)^2 = \varpi^n(\varpi^n e)$, so if $n > 0$, as a function on the special fiber $X \otimes_{\mathcal{O}} k$, $\varpi^n e$ is nilpotent. This contradicts our reducedness hypothesis, so $n = 0$ and e is already a global section of X.

So we know that the components of $X \otimes_{W(k)} K$ are in bijection with the components of X itself. But if $X^\wedge$ is the completion of X along its special fiber, the components of the special fiber $X \otimes_{\mathcal{O}} k$ are in bijection with the components of $X^\wedge$ (because they have the same underlying topological space), and formal GAGA implies that the components of $X^\wedge$ are in bijection with the components of X (X is proper over $\mathcal{O}$, so we can apply formal GAGA to see that the global idempotent functions on X and $X^\wedge$ are in bijection). $\square$

5.1 Representability

Proposition 5.2. *The morphism* $|L_V^{\chi,\square}| \to |D_V^{\chi,\square}|$ *is represented by a projective morphism* $\Theta_V : \mathcal{L}_V^{\chi,\square} \to R_V^{\chi,\square}$.

Proof. Given an A-point of $R_V^{\chi,\square}$, the A-points of $\mathcal{L}_V^{\chi,\square}$ should be certain line bundles on $\operatorname{Spec} A$, so we will cut $\mathcal{L}_V^{\chi,\square}$ out of $\mathbf{P}^1_{R_V^{\chi,\square}}$.

Consider $\mathbf{P}$, the projectivization of the universal rank 2 $R_V^{\chi,\square}$-module. That is, if V_R is the universal rank 2 $R_V^{\chi,\square}$-module (equipped with a representation of G_L), then $\mathbf{P} := \operatorname{Proj} \operatorname{Sym} V_R \cong \operatorname{Proj} R_V^{\chi,\square}[x_0, x_1]$.

If A is an an $R_V^{\chi,\square}$-algebra with residue field $\mathbf{F}$, a morphism $\operatorname{Spec} A \to \mathbf{P}$ (over $R_V^{\chi,\square}$) is the same as a surjection (of sheaves) $A^2 \to \mathcal{L} \to 0$.

Given a morphism $f : \operatorname{Spec} A \to \mathbf{P}$, there is a natural G_L-action on the quotient $\mathcal{L}$ if and only if $g^* f = f$ for all $g \in G_L$. The g^*-fixed locus of $\mathbf{P}$ is H_g defined by the Cartesian square

$$\begin{array}{ccc} H_g & \longrightarrow & \mathbf{P} \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id}, g^*)} \\ \mathbf{P} & \xrightarrow{\Delta} & \mathbf{P} \times_{R_V^{\chi,\square}} \mathbf{P} \end{array}$$

Since $\mathbf{P}$ is separated, H_g is a closed subscheme of $\mathbf{P}$. Thus, the intersection $H := \cap_{g \in G} H_g$ is a closed subscheme of $\mathbf{P}$ parametrizing G_L-equivariant quotients $A^2 \to \mathcal{L} \to 0$.

Now if A is a complete local $W(\mathbf{F})$-algebra, there is a natural map from H to the universal deformation of the residually trivial 1-dimensional representation, given (in the language of the functor of points) by sending $A^2 \to \mathcal{L} \to 0$ to $\mathcal{L}$. Then we can take the fiber over the (closed) point corresponding to the trivial representation to get a closed subscheme of $\mathbf{P}$ representing $L_V^{\chi,\square}$ on $\mathfrak{AR}_{W(\mathbf{F})}$.

Now take limits to get representability of $L_V^{\chi,\square}$ on $\mathfrak{Aug}_{W(\mathbf{F})}$. □

5.2 Smoothness and connectedness

Next we want to study smoothness and connectedness.

Proposition 5.3. *$\mathcal{L}_V^{\chi,\square}$ is formally smooth over $W(\mathbf{F})$. Furthermore, the $W(\mathbf{F})[1/p]$-scheme $\mathcal{L}_V^{\chi,\square} \otimes_{W(\mathbf{F})} W(\mathbf{F})[1/p]$ is connected.*

Proof. First, we will show that for any finite group M of p-power order, the natural map $H^1(G_L, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} M \to H^1(G_L, \mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} M)$ is an isomorphism. It suffices to consider the case $M = \mathbf{Z}/p^n\mathbf{Z}$. In that case, we have the exact sequence

$$0 \to \mathbf{Z}_p(1) \xrightarrow{\cdot p^n} \mathbf{Z}_p(1) \to M \to 0$$

Then the long exact sequence in group cohomology shows that

$$0 \to H^1(G_L, \mathbf{Z}_p(1))/p^n H^1(G_L, \mathbf{Z}_p(1)) \to H^1(G_L, M) \to H^2(G_L, \mathbf{Z}_p(1))[p^n]$$

is exact. The middle arrow is the natural map we started with, so we wish to show that $H^2(G_L, \mathbf{Z}_p(1))[p^n]$ is 0. But by Tate local duality (as in Simon's talk), $H^2(G_L, \mathbf{Z}_p(1))$ is Pontryagin dual to $\mathbf{Q}_p/\mathbf{Z}_p$, so has no p^n-torsion.

Thus, for any artinian algebra A, the composition

$$\mathrm{Ext}^1_{\mathbf{Z}_p[G_L]}(\mathbf{Z}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} A \to H^1(G_L, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} A \to H^1(G_L, \mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} A) \to \mathrm{Ext}^1_{\mathbf{Z}_p[G_L]}(A, A(1))$$

is an isomorphism.

To prove smoothness, it suffices to show that for any surjection of artinian rings $A \to A'$, the map $|L_V^{\chi,\square}|(A) \to |L_V^{\chi,\square}|(A')$ is a surjection. Now consider a pair $(V_{A'}, L_{A'})$ in $|L_V^{\chi,\square}|(A')$. It corresponds to an element of $\mathrm{Ext}^1_{\mathbf{Z}_p[G_L]}(A, A(1))$, so by the isomorphism we just proved, it corresponds to an element of $\mathrm{Ext}^1_{\mathbf{Z}_p[G_L]}(\mathbf{Z}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} A'$. But such an element clearly lifts to an element of $\mathrm{Ext}^1_{\mathbf{Z}_p[G_L]}(\mathbf{Z}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} A$, which is to say, an element of $|L_V^{\chi,\square}|(A)$.

Now we wish to prove connectedness after inverting p, and for this we use the lemma on connected components. Specifically, since $\mathcal{L}_V^{\chi,\square}$ is smooth, its special fiber $\mathcal{L}_V^{\chi,\square} \otimes_{W(\mathbf{F})} \mathbf{F}$ is reduced, so to show $\mathcal{L}_V^{\chi,\square}[1/p]$ is connected, it suffices to show that the special fiber $\mathcal{L} \otimes_{R_V^{\chi,\square}} \mathbf{F}$ is connected.

But the special fiber is simply the fiber over the residual representation. If $\mathbf{F} \cong \mathbf{F}(1)$ and the representation is split (i.e., the residual representation is trivial), any line in $\mathbf{F}^2$ is G_L-stable with G_L-acting by $\chi = \mathrm{id}$, so the fiber is a full $\mathbf{P}^1_{\mathbf{F}}$. Otherwise, there is at most one G_L-line with G_L acting via χ, and this is true for any A-point of the fiber, so it is either empty or it consists of a single reduced point. So the special fiber is connected. $\square$

The next proposition will show that $\mathcal{L}_V^{\chi,\square}[1/p] \to \operatorname{Spec} R_V^{\chi,\square}[1/p]$ is a monomorphism. More precisely, it shows that this morphism is injective on artinian points, but, as before, Brian's notes on $\ell = p$ explain why this is sufficient to let us conclude that it is actually a monomorphism.

Proposition 5.4. *Let $E/\mathbf{Q}_p$ be a finite extension, and let ξ refer to both an $\mathcal{O}_E$-valued point of $R_V^{\chi,\square}$ and an $\mathcal{O}_E$-valued point in the fiber of $\mathcal{L}_V^{\chi,\square}$ above it. Then the morphism of groupoids (functors) on $\mathfrak{AR}_E$ $L_{V_\xi}^{\chi,\square} \to D_{V_\xi}^{\chi,\square}$ is fully faithful. If the representation over E V_ξ corresponding to ξ is indecomposable, then this is an equivalence.*

Proof. Let B be an object of $\mathfrak{AR}_E$, and let V_B be an object of $D_{V_\xi}^{\chi,\square}(B)$. To prove the first assertion, we need to show that V_B admits at most one G_L-stable B-line $L_B \subset V_B$ such that G_L acts trivially on V_B/L_B. But $\operatorname{Hom}_{B[G_L]}(B(1), V_B/L_B) = \{0\}$ because the G_L-action on the target is trivial, so $\operatorname{Hom}_{B[G_L]}(B(1), V_B) = \operatorname{Hom}_{B[G_L]}(B(1), L_B)$ and L_B is unique.

Now suppose V_ξ is indecomposable; we wish to show that V_B actually does admit a suitable B-line. We will do this by showing that V_B is isomorphic to the trivial deformation $V_\xi \otimes_E B$. Note that by Tate local duality

$$\dim_E H^1(G_L, \operatorname{ad}^0 V_\xi) = \dim_E H^0(G_L, \operatorname{ad}^0 V_\xi) + \dim_E H^0(G_L, \operatorname{ad}^0 V_\xi(1)) = 0$$

the last equality following from indecomposability of V_ξ. The result then follows by induction on the length of B, since this calculation holds for any indecomposable extension of $A(1)$ by A. $\square$

But since we have a proper monomorphism of schemes $\mathcal{L}_V^{\chi,\square}[1/p] \to \operatorname{Spec} R_V^{\chi,\square}[1/p]$, it is a closed immersion.

Now we can prove the following proposition and corollary.

Proposition 5.5. *Let* $\operatorname{Spec} R_V^{\chi,1,\square}$ *be the scheme-theoretic image of the morphism* $\mathcal{L}_V^{\chi,\square} \to \operatorname{Spec} R_V^{\chi,\square}$. *Then*

1. *$R_V^{\chi,1,\square}$ is a domain of dimension 4 and $R_V^{\chi,1,\square}$ is formally smooth over $W(\mathbf{F})$.*

2. *If $E/\mathbf{Q}_p$ is a finite extension, then a morphism $\xi : R_V^{\chi,\square} \to E$ factors through $R_V^{\chi,1,\square}$ if and only if the corresponding two-dimensional representation V_ξ is an extension of E by $E(1)$.*

Proof. Since $R_V^{\chi,1,\square}$ is smooth and connected, it is a domain. We will find its dimension via a tangent space calculation. Suppose V_ξ is indecomposable (which we may assume, since most points on $R_V^{\chi,1,\square}$ are indecomposable). Then the dimension of $R_V^{\chi,1,\square}[1/p]$ is

$$\begin{aligned}\dim_E |D_{V_\xi}^{\chi,\square}|(E[\varepsilon]) &= \dim_E |D_{V_\xi}^{\chi}|(E[\varepsilon]) + 4 - \dim_E(\operatorname{ad} V_\xi)^{G_L} \\ &= \dim_E H^1(G_L, \operatorname{ad}^0 V_\xi) + 3 = 3\end{aligned}$$

So $R_V^{\chi,1,\square}$ itself is 4-dimensional, and we have proven the first part.

The second part follows from the definition of $\mathcal{L}_V^{\chi,\square}$ and $R_V^{\chi,1,\square}$. $\square$

Corollary 5.6. *Let $\mathcal{O}$ be the ring of integers in a finite extension of $W(\mathbf{F})[1/p]$, and $\gamma : G_L \to \mathcal{O}^\times$ a continuous unramified character. Write $R_{V,\mathcal{O}}^{\square} = R_V^{\square} \otimes_{W(\mathbf{F})} \mathcal{O}$. Then there exists a quotient $R_{V,\mathcal{O}}^{\chi\gamma,\gamma,\square}$ such that*

- *$R_{V,\mathcal{O}}^{\chi\gamma,\gamma,\square}$ is a domain of dimension 4 and $R_{V,\mathcal{O}}^{\chi\gamma,\gamma,\square}[1/p]$ is formally smooth over $\mathcal{O}$.*
- *If $E/\mathcal{O}[1/p]$ is a finite extension, then a map $\xi : R_{V,\mathcal{O}}^{\square} \to E$ factors through $R_{V,\mathcal{O}}^{\chi\gamma,\gamma,\square}$ if and only if V_ξ is an extension of γ by $\gamma(1)$.*

Proof. This basically follows because universal deformation rings behave reasonably well with respect to twisting by fixed characters, at least once the question makes sense.

More precisely, we may replace $\mathbf{F}$ by the residue field of $\mathcal{O}$ (corresponding to tensoring $R_V^{\square}$ with $\mathcal{O}$). Then twisting by γ^{-1} induces an isomorphism $R_{V,\mathcal{O}}^{\square} \tilde{\to} R_{V\times\gamma^{-1},\mathcal{O}}^{\square}$ (because twisting the residual representation by γ^{-1} doesn't change this deformation problem (except to multiply the determinant by γ^2), and the quotient $R_{V,\mathcal{O}}^{\chi\gamma,\gamma,\square}$ corresponds to $R_{V\otimes\gamma^{-1}}^{\chi,1,\square} \otimes_{W(\mathbf{F})} \mathcal{O}$ under this isomorphism. $\square$