Automorphy, Potentially Automorphy and Langlands Base Change

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Automorphic Galois Representations

Notation

Let F be a number field (not necessarily totally real) and $n \in \mathbb{N}$. Let S_f, S_∞ denote the set of all finite and infinite places of F respectively. For $v \in S_f \cup S_\infty$ we denote the completion by F_v and for $v \in S_f$ we denote by $\mathcal{O}_v$ the ring of integers. Let $\mathbb{A}_F$ denote the adeles over F and $\mathbb{A}_F^\infty$ their finite component. Fix algebraic closures $\bar{F}$ and $\bar{F}_v$ and let $G_F := Gal(\bar{F}/F)$ and $G_{F_v} := Gal(\bar{F}_v, F_v)$. We fix embeddings $\bar{F} \subset \bar{F}_v$, which induces embeddings $G_{F_v} \subset G_F$. For $S \subset S_f$ a finite subset let $G_{F,S}$ denote the Galois group of the maximal extension of F (contained in $\bar{F}$) unramified outside S. For $v \in S_f \setminus S$ let $frob_v \in G_{F,S}$ denote the frobenius element at v.

Automorphic Representations of $GL_{n/F}$

Let π be an automorphic representation of $GL_{n/F}$. Rather than defining such a representation let us recall its predominant features:

1. π is a complex vector space with a tensor decomposition

$$\pi = \pi_f \bigotimes (\otimes_{v \in S_\infty} \pi_{\infty,v})$$

   where π_f is an irreducible, smooth, admissible representation of $GL_n(\mathbb{A}_F^\infty)$ and $\pi_{\infty,v}$ is an irreducible, admissible Harish-Chandra module associated to the Lie group $GL_n(F_v)$.

2. We have the restricted tensor decomposition

$$\pi_f = \otimes'_{v \in S_f} \pi_{f,v}$$

   where $\pi_{f,v}$ is an irreducible, smooth, admissible representation of $GL_n(F_v)$ $(v \in S_f)$. Moreover, for almost all finite places $\pi_{f,v}$ is unramified, i.e. posesses a $GL_n(\mathcal{O}_{F_v})$ fixed vector. Let $S_\pi \subset S_f$ denote the set of finite places of F for which $\pi_{f,v}$ is ramified.

Recall that for any $v \in S_f$ the Satake Isomorphism induces a natural bijection:

$$\{ \text{ Semisimple conjugacy classes of } GL_n(\mathbb{C}) \ \}$$

$$\updownarrow$$

$$\{ \text{ Isomorphism classes of unramified, irreducible, smooth admissible complex representations of } GL_n(F_v). \ \}$$

Hence for every $v \notin S_\pi$ we get the semisimple conjugacy class $\Upsilon_v \subset GL_n(\mathbb{C})$. By Strong Multiplicity One the set $\{\Upsilon_v\}_{v \notin S_\pi}$ determines π up to isomorphism.

Fix $k \in \mathbb{Z}$. In the case where $n = 2$ and F is totally real if we demand that the Harish-Chandra modules at ∞ are all holomorphic discrete series of weight k, then π is associated to a cuspidal Hilbert modular newform g of parallel weight $k \geq 2$. If $v \notin S_\pi$ and $\mathbb{T}_v$ is the usual Hecke operator at v then if $\mathbb{T}_v g = a_v g$ we have the equality

$$a_v = trace(\Upsilon_v).$$

Now fix a prime $p \in \mathbb{N}$ and an algebraic closure $\bar{\mathbb{Q}}_p$. Also fix an isomorphism $\varsigma : \bar{\mathbb{Q}}_p \cong \mathbb{C}$. Let ρ be a continuous (for the p-adic topology) representation

$$\rho : G_F \rightarrow GL_n(\bar{\mathbb{Q}}_p),$$

which is unramified outside the finite set $S_\rho \subset S_f$. Hence ρ factors through G_{F,S_ρ}. For $v \in S_f \setminus S_\rho$ let $\phi_v \subset GL_n(\mathbb{C})$ denote the semisimple conjugacy class associated to $\rho(Frob_v)$. Here was are implicitly using the isomorphism ς. The information $\{\phi_v\}_{v \notin S_\rho}$ determines ρ up to isomorphism by Tchebotarev density.

Definition. *ρ is automorphic if there exists π, an automorphic representation of $GL_{n/F}$ and a finite subset $S \subset S_f$ such that*

1. *$S_\pi \subset S$ and $S_\rho \subset S$.*
2. *$\{\phi_v\}_{v \notin S} = \{\Upsilon_v\}_{v \notin S}$.*

We say that ρ is potentially automorphic if there exists a finite extension E/F, contained in $\bar{F}$, such that $\rho|_{G_E}$ is automorphic.

Global Langlands Reciprocity Conjecture. *If ρ is semisimple, deRham (at all places over p) and unramified outside of a finite set of primes then ρ is automorphic.*

Of course conjecturally all local data should match under the Local Langlands correspondence.

In the Hilbert case this implies that the traces of Frobenius away from S_ρ arise as eigenvalues of Hecke operators on of the space of Hilbert modular forms.

Now let F be totally real and D a quaterion algebra over F. If ρ is 2-dimensional then we say it is automorphic for D if we can find π_D, an automorphic representation of $D_{/F}$, such that ρ is associated to π_D^{JL}, its Jacquet-Langlands transfer. Informally this is saying that the traces of Frobenius away from S_ρ arise as eigenvalues of Hecke operators of the space of automorphic forms associated to D. Of course there is a converse to this - to any cuspidal Hilbert eigenform π of parallel weight $k \geq 2$ there is an associated 2-dimensional p-adic representation of G_F.

1 Langlands Base Change

Let E/F be a finite extension (contained in $\bar{F}$). We have the natural inclusion $G_E \subset G_F$. If ρ is a representation as in the Global Reciprocity Conjecture then $\rho|_{G_E}$ is also of this form. If we where to believe the reciprocity conjecture then if π is associated to ρ there should be an automorphic representation of $GL_{n/E}$, denoted π', associated to $\rho|_{G_E}$.

This "base change" transfer is also predicted by Langlands principle of Functoriality. More precisely, let $G = Res_{E/F}(GL_{n/E})$. The L-group of this groups is

$$^LG = (\prod_{G_E \backslash G_F} GL_n(\mathbb{C})) \rtimes G_F$$

where the G_F acts by permutations. Thus there is a natural L-homomorphism

$$^LGL_{n/F} = GL_n(\mathbb{C}) \times G_F \to^L G$$

which is the diagonal embedding. Functoriality in this case would transfer an automorphic representation of $GL_{n/F}$ to an automorphic representation of G. However, observing that $G(\mathbb{A}_F) = GL_n(\mathbb{A}_E)$, we see that an automorphic representation of this latter group is simply an automorphic representation of $GL_{n/E}$.

We have the following:

1. For E/F a solvable extension functoriality has been established in this case by Langlands ($n = 2$) and by Aurthur and Clozel ($n > 2$). They use the trace formula to establish functoriality for cyclic extensions of prime degree, then the result follows by induction. We denote the functorial transfer by $BC_{E/F}$ (BC for base change).

2. If E/F is solvable and π is an automorphic representation of $GL_{n/F}$ associated to a Galois representation ρ then $\rho|_{G_E}$ is automorphic with associated representation $BC_{E/F}(\pi)$.

In the Hilbert case we have the converse:

Theorem. *Let E/F be a solvable extension of totally real number fields. Let ρ be a 2-dimensional p-adic representation of G_F. Suppose that $\rho|_{G_E}$ is automorphic with associated representation π', a cuspidal automorphic representation of $GL_{2/E}$ all of who infinite components are holomorphic discrete series of weight $k \geq 2$ (i.e. coming from a Hilbert eigenform over E of parallel weight k). Then there is a cuspidal automorphic representation of $GL_{2/F}$ all of whose infinite components are discrete series of weight k such that $\pi' = BC_{E/F}(\pi)$ (i.e. coming from a Hilbert eigenform over F of parallel weight k).*

The proof of this result relies on a determining the image of the base change map and then determining the fibres. Because π' is associated to a representation of G_E which extends to G_F it is invariant under $Gal(E/F)$. This ensures that it is the base change of some automorphic representation of $GL_{n/F}$. The trick then is to find a lift with the desired properties.

Corollary. *If ρ is a two dimensional p-adic representation of G_F what becomes automorphic (of parallel weight $k \geq 2$) over a solvable totally real extension then ρ is already automorphic (of parallel weight k).*

Hence constructing solvable extensions will be important to us. In particular it will be important to construct solvable global extensions with prescribed local behaviour. In this regard we have the following theorem (an extension of the classical Grunwald-Wang theorem):

Theorem. *Let F be a totally real number field and $\{v_1, \cdots, v_n\} \subset S_f$. Let K_i/F_{v_i} be a finite Galois extension for each i. Then there exists a totally real solvable extension E/F including K_i/F_{v_i} as the local extension at v_i for each i, such that E/F can be chosen to be unramified at any auxilliary finite set of places.*

For a proof of this see Brians number theory handout.