Warning – these notes were written for AV's personal use and have not been checked in any way whatsoever, nor have they been edited for coherence beyond adding one or two sentences. AV makes absolutely no warranty of correctness and these should be used with extreme caution. The talk sketched a proof of Jacquet-Langlands between the quaternion algebra ramified at $\{\infty, 11\}$ and GL_2 at level 11.

1. The adelic quotient associated to a quaternion algebra

D the quaternion algebra ramified at $\{11, \infty\}$. This is represented by $x + yi + zj + wk$, where $i^2 = -11, j^2 = -11, k^2 = -1$; also $ij = 11k, jk = i, ki = j$.

We want to have some "concrete" understanding of the adelic quotient $\mathbf{Q}^\times \backslash (D \otimes \mathbb{A})^\times / \mathbb{A}^\times U$, where U is a maximal compact subgroup of $(D \otimes \mathbb{A})^\times$. It comes with Hecke operators T_p for each prime p.

The maps[1] $q \mapsto \mathrm{Cliff}^0(q), E \mapsto \mathrm{End}(E)$ give bijections:

- Maximal orders in D, i.e. quaternion rings[2] of discriminant -11.
- Isomorphism classes of supersingular elliptic curves over $\overline{\mathbf{F}_{11}}$;
- (Definite) ternary quadratic forms of $\frac{1}{2}$-discriminant -11.

This set is equipped with the structure of a $p+1$-valent directed graph from the Hecke operators; they are represented by matrices

$$A_2 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \; A_3 = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \; A_5 = \begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}, \; A_7 = \begin{pmatrix} 4 & 6 \\ 4 & 2 \end{pmatrix}.$$

Exercise. Why are these matrices not symmetric?

We describe each of the three realizations in turn.

1.1. *Maximal orders.*– Write $\mathfrak{o} = \mathbf{Z}[\frac{1+j}{2}]$. Then $\mathfrak{o}[k]$ is a maximal order.

$$\mathcal{O}_1 := \frac{x + yi + zj + wk}{2} \quad x \equiv z, y \equiv w(2).$$

The group of units (elements of norm 1) is simply $\{\pm 1, \pm k\}$.

Write $\omega = \frac{2+i+k}{4}$, a cube root of 1, and $\nu = \frac{1+i-j-k}{2}$, with norm 6. Then

$$\mathcal{O}_2 := \mathbf{Z} + \mathbf{Z}j + \mathbf{Z}\omega + Z\nu$$

is another maximal order. The group of units is $\{\pm 1, \pm\omega, \pm\omega^2\}$.

$\mathcal{O}_1$ and $\mathcal{O}_2$ intersect in an order that has index 2 in both. We can describe the passage from $\mathcal{O}_1$ to $\mathcal{O}_2$ as follows (with $q = 2\omega$):

$$\mathcal{O}_2 = \frac{q}{2} + \langle z \in \mathcal{O} : \mathrm{tr}(zq) \text{ even} \rangle.$$

The element q has the property that $\mathrm{tr}(q) = 0, \mathrm{N}(q) \equiv 0$ modulo 2.

More generally: Given any maximal order B, there exist $p+1$ neighbours B'. Namely, there exist exactly $p+1$ elements of B/pB of trace and norm 0. For each such x, let $x^* \in B$ be a lift that has norm divisible by p^2. Set $B' = \frac{x^*}{p} + (x^*)^\perp$.

[1]Given a ternary quadratic form q the *even Clifford algebra* gives a ternary quadratic form of the same discriminant. For instance

$$ax^2 + by^2 + cz^2 \mapsto \langle i, j, k | i^2 = -bc, j^2 = -ac, k^2 = -ab \rangle$$

Note that upon composition with "trace form" we get the map "multiplication by disc," at least over a field.

[2]A quaternion ring over R is free of rank 4 together with an involution $x \mapsto x^*$ so that the characteristic polynomial is as expected (roots x, x^* with multiplicty 2).

1.2. *Supersingular elliptic curves.*–

The p-neighbour operation is described thus: For each supersingular elliptic curve and prime $p \neq 11$, we can consider the $p+1$ different curves formed by quotienting by a subgroup of order p. This gives a $\delta \times \delta$ matrix whose columns add up to $p+1$, a "Brandt matrix."

There are two supersingular j-invariants in characteristic 11, namely, 1728 and 0. Over $\mathbf{C}$ these are represented by points i, ω. The j-invariant of all points $3i, i/3, (i+1)/3, (i+2)/3$ satisfy the equation $x^2 - 153542016x - 1790957481984 = 0$, which has roots $\{0, 1728\}$ in $\mathbf{F}_{11}$. On the other hand, $j(3\omega) = -12288000$, as are two of the other Hecke translates; the other Hecke translate $(\omega+1)/3$ has j-invariant zero.

There is a 2-isogeny from $y^2 = x^3 - x$ to $y^2 = x^3 + 1$ in characteristic 11.

$$x \mapsto \frac{6x^2 + 5x + 1}{x - 1}, y \mapsto y\frac{x^2 + 9x + 10}{(x-1)^2}.$$

1.3. *Quadratic forms.* – There are two quadratic forms, represented by $q_1 := x^2 + y^2 + 3z^2 - xz, x^2 + y^2 + xy - yz - zx + 4z^2$, and they have respectively 8 and 12 automorphisms. The *total mass* of the genus is 5/24.

Example of q_1. The associated quaternion algebra spanned by $1, i = e_2e_3, j = e_3e_1, k = e_1e_2$ has relations

$$i^2 = -3, j^2 = -j - 3, k^2 = -1, jk = i^*, ki = j^*, ij = 3k^*$$

For instance $e_3e_1e_3e_1 = e_3^2e_1^2 + 2e_3^2\langle e_1, e_3\rangle = -3 - e_3e_1$. Note that we can write this as $(x - z/2)^2 + y^2 + 11z^2/4$; in other terms, in terms of the basis $e_1' = e_1, e_2' = e_2, e_3' = 2e_3 + e_1$ it is $x^2 + y^2 + 11z^2$. Note that $k' := e_1'e_2' = k, j' := e_3'e_1' = 2j + 1, i' := e_2'e_3' = 2i - k$. These satisfy

$$j'^2 = -11, k'^2 = -1, i'^2 = -13 - 2(ik + ki) \cdots = -11, \dots$$

Note $ki = j^*$, so also $i^*k^* = j$, that is $ik = j$. So $ki + ik = j + j^* = -1$ and so on. In particular, B_q is isomorphic to the suborder of D spanned by $k, (j-1)/2, (i+k)/2$.

2. The cusp form of weight 11

There is precisely weight 2 one cusp form of weight 11, viz.

$$q\prod(1-q^n)^2\prod(1-q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 \dots$$

(Note that it is congruent to Δ modulo 11.) This corresponds to the elliptic curve

$$y^2 + y = x^3 - x^2,$$

which is in fact $X_1(11)$ *not* $X_0(11)$. We denote by $N(p)$ the number of points on this curve modulo p. So $N(2,3,5,7) = 5, 5, 5, 10\dots$ whereas $a_p = -2, -1, 1, -2$.

Exercise. Why are all the numbers are all divisible by 5?

The trace formula. – By the Lefschetz formula, the number of fixed points of T_p is $2(p + 1 - a_p)$. (So, 10, 10, 10, 20.)

The contribution of each cusp is 2.

The fixed points of T_p on $X_0(11)$ are parameterized by pairs (E, Λ) together with a cyclic p-isogeny $\psi : E \to E$ that fixes Λ. In other terms, E has CM by some order $\mathfrak{o}$ that has an element of norm p. Once we fix $\mathfrak{o}$ and an element $t \in \mathfrak{o}$ of norm p, the remaining question is whether t acting on $\mathfrak{o}/11$ fixes a line; if we suppose that the $p \neq 11$ and that 11 is unramified in $\mathfrak{o}$, this will be so just when $\mathfrak{o}$ is split at 11; if so, there are two such points.

The orders of small discriminant split at 11 are $\mathbf{Z}[\sqrt{-2}]$ and $\mathbf{Z}[\sqrt{-7}]$. These orders are both of class number 1.

In all cases each solution (up to sign) contributes $+2$ if unramified at 11, $+1$ else.

Example. There is an element of norm 2 in $\mathbf{Z}[\sqrt{-2}]$, namely $\pm\sqrt{-2}$; also $\pm\frac{1+\sqrt{-7}}{2}$.

Example. Norm 3: $\pm 1 \pm \sqrt{-2}$, $\frac{1\pm\sqrt{-11}}{2}$.

Example. Norm 5. We need to solve $20 = x^2+dy^2$; the solutions are $\frac{\pm 1\pm\sqrt{-19}}{2}$, $\frac{\pm 3\pm\sqrt{-11}}{2}$.

Example. Norm 7. We need to solve $28 = x^2 + dy^2$. The solutions are $\pm 1 \pm \sqrt{-6}$, $\frac{\pm 3\pm\sqrt{-19}}{2}$, $\sqrt{-7}$. But $1 + \sqrt{-6}$ contributes $+4$ (class number two) and $\sqrt{-7}$ also does (two orders). Total $8 + 4 + 4 = 16$.

Example. Norm 13. $\frac{\pm 1\pm\sqrt{-51}}{2}$, same for 43. Class numbers are 2 and 1. Total $+12$. Finally $\sqrt{-13}$ gives $+4$ – class number two. total 20. Correct ($a_{13} = 4$).

3. The trace of Brandt matrices

The trace of T_p on the split side is a summation

$$-\sum_{\mathfrak{o},\lambda} h(\mathfrak{o})(1 + \left(\frac{-11}{d}\right)).$$

I'll sketch why the trace of T_p on the quaternionic side is

$$p + 1 + \sum_{\mathfrak{o},\lambda} h(\mathfrak{o})(1 - \left(\frac{-11}{d}\right)). \tag{1}$$

Why are these equal? We need to check

$$\sum_{\mathfrak{o},\lambda} h(\mathfrak{o}) = p + 1,$$

and there are two ways to proceed:

(1) Compute the trace of T_p on *forms of weight* 2*, level* 1.
(2) Use the fact that $(\sum q^{n^2})^3 \sum q^{n^2} = (\sum q^{n^2})^4$.

Sketch of proof of (1)*:* I'll explain it in terms of CM elliptic curves: Suppose $\mathfrak{o}$ is inert or ramified at 11 and $\lambda \in \mathfrak{o}$ has norm p. (Let H be the Hilbert class field, and choose a prime above p). For each ideal class J of $\mathfrak{o}$, the elliptic curve E_J, when reduced mod p, comes equipped with a p-isogeny $E_J \to E_J$, i.e., a loop in the adjacency graph.

Example. Let us reconsider norm 3 from this perspective. It is the norm of $\sqrt{-3}$, $\frac{1\pm\sqrt{-11}}{2}$ in inert orders.

First, $\sqrt{-3}$. It gives rise to the CM elliptic curve with j-invariant 0, $y^2 = x^3 - 1$. However, there are two CM-maps over $\overline{\mathbb{F}_{11}}$, namely $x \mapsto \zeta x$ and $x \mapsto -\zeta x$.

On the other hand, $\sqrt{-11}$ gives rise a CM elliptic curve of j-invariant -32768. Explicitly, with $a = 4 \times 24 \times 539$ and $b = 16 * 539^2$, it is $y^2 = x^3 - ax - b$, a curve of j-invariant -2^{15}. This curve is the minimal model (!!) and it has conductor $2^4 3^2 7^2 11^2$. On the other hand, over $\mathbf{Q}(\sqrt{-11})$ it becomes the curve

$y^2 = 4x^3 - 24x - \sqrt{539}$, which *does* have good reduction at the prime above 11; it has j invariant $1728 \in \overline{\mathbf{F}_{11}}$.

Remark. You can construct the curve with conductor 121 by twisting. It is

$$y^2 = x^3 - 9504x + 365904.$$

Alternate. If you are proficient with ternary quadratic forms [...]

The file contained a section concerning computations at level 121 (esp. local representations at 11). These have been commented out.