Lecture 11: Hecke characters and Galois characters

Andrew Snowden
January 28, 2010

1. Introduction

Let K be a number field, let $\mathbf{A}_K^\times$ be its idele group and let G_K be its absolute Galois group. Class field theory states that there is a natural map (up to a choice of normalization)

$$\mathbf{A}_K^\times/K^\times \to G_K^{\mathrm{ab}}$$

which identifies G_K^{ab} as the profinite completion of $\mathbf{A}_K^\times/K^\times$. Equivalently, class field theory can be stated as an isomorphism

$$\{\text{finite order characters of } \mathbf{A}_K^\times/K^\times\} = \{\text{finite order characters of } G_K\}.$$

Thus we have a description of the finite order characters of G_K.

A *p-adic character* of G_K is a continuous homomorphism $G_K \to \overline{\mathbf{Q}}_p^\times$; since G_K is compact any such character takes values in $\mathscr{O}_F^\times$ for some finite extension $F/\mathbf{Q}_p$. There are many interesting p-adic characters which are not of finite order: for instance, the cyclotomic character χ_p. Since $\mathscr{O}_F^\times$ is profinite, p-adic characters of G_K are limits of finite order characters, and so we can use class field theory to understand them. Define a *p-adic Hecke character* (of K) to be a continuous homomorphism $\mathbf{A}_K^\times/K^\times \to \overline{\mathbf{Q}}_p^\times$; again, the image is always contained in $\mathscr{O}_F^\times$ for some F finite over $\mathbf{Q}_p$. We then have an identification

$$\{p\text{-adic Hecke characters of } K\} = \{p\text{-adic characters of } G_K\}.$$

induced by class field theory.

We thus have an understanding of p-adic characters of the Galois group. However, this is not the end of the story: there are *compatible systems* of characters. Such a system consists of a p-adic character ψ_p of G_K for each prime p such that for each place v of K the quantity $\psi_p(\mathrm{Frob}_v)$ is independent of p in a suitable sense. We would like to understand the collection of compatible systems. The Langlands program suggests that compatible systems of characters should correspond to automorphic representations of $\mathrm{GL}_1(\mathbf{A}_K)$, so we now examine these objects.

What is an automorphic representation of $\mathrm{GL}_1(K)$? To begin with, it should be an irreducible subrepresentation of $\mathrm{GL}_1(\mathbf{A}_K)$ acting on the space of automorphic forms on $\mathrm{GL}_1(\mathbf{A}_K)$ by right translation. (Recall that an automorphic form on $\mathrm{GL}_1(K)$ is a function $K^\times\backslash\mathrm{GL}_1(K) \to \mathbf{C}$ satisfying certain natural conditions.) Since $\mathrm{GL}_1(\mathbf{A}_K)$ is commutative, such a representation must be one dimensional. It is thus spanned by some non-zero automorphic form f. Since $\mathbf{C}f$ is stable by right translation, we find $f(xg) = \lambda_g f(x)$ for all $x, g \in \mathrm{GL}_1(\mathbf{A}_K)$. Taking $x = 1$, we find $\lambda_g f(1) = f(g)$ and so $f(1)f(xg) = f(g)f(x)$ holds for all x and g. Since f is non-zero we find that $f(1)$ is non-zero; scale f so that $f(1) = 1$. We then find that f is a homomorphism, and since it is invariant under $K^\times$ it also satisfies $f(K^\times) = 1$. The properties of automorphic forms that we did not list amount to f being continuous. A continuous homomorphism $\mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$ is called a *Hecke character*. We have thus shown that every automorphic representation of $\mathrm{GL}_1(K)$ is spanned by a Hecke character. It is clear that the character is unique. It is also not difficult to show that every Hecke character spans an automorphic representation. We thus have an identification

$$\{\text{automorphic representations of } \mathrm{GL}_1(K)\} = \{\text{Hecke characters of } K\}.$$

Consider now the diagram

$$\begin{array}{ccc} \{\text{Hecke characters of } K\} & \dashleftarrow\dashrightarrow & \{\text{compatible systems of characters of } G_K\} \\ \downarrow & & \downarrow \\ \{p\text{-adic Hecke characters of } K\} & = & \{p\text{-adic characters of } G_K\} \end{array}$$

We have already explained the bottom map. The right map takes a compatible system of characters $\{\psi_p\}$ to its pth member ψ_p. The top arrow means "we expect a relationship." Given a top arrow, the left arrow is obtained by going around the diagram.

As we have introduced it, the left arrow might seem the most mysterious: it is given by combining three operations, one of which is itself somewhat unclear. However, it is actually quite accessible. We will explain this in the coming sections. Once one has this left arrow, it is not difficult to understand the top arrow. The story goes like this. There are certain special Hecke characters, the *algebraic* ones. Given an algebraic Hecke character f one can build a p-adic Hecke character f_p for any prime p. Each f_p is associated to some p-adic character ψ_p of G_K and these ψ_p form a compatible system. In fact, this is a bijection, that is, every compatible system arises from a unique algebraic Hecke character.

2. The case $K = \mathbf{Q}$

We begin by considering the case $K = \mathbf{Q}$. The general case does not differ much from this case except that it is more notationally complicated. We have

$$\mathbf{A}_{\mathbf{Q}}^{\times}/\mathbf{Q}^{\times} = \prod_p \mathbf{Z}_p^{\times} \times \mathbf{R}_+.$$

Here $\mathbf{R}_+$ denotes the group of positive real numbers under multiplication. Each $\mathbf{Z}_p^{\times}$ has its usual topology and the product has the product topology; it is profinite.

Let f be a Hecke character. The restriction η of f to $\prod \mathbf{Z}_p^{\times}$ is a finite order character, as is any continuous homomorphism from a profinite group to $\mathbf{C}^{\times}$. The restriction of f to $\mathbf{R}_+$ is of the form $x \mapsto x^a$ for some real number a. We call f *algebraic* if this number a is an integer. Let α_{∞} be the Hecke character which is trivial on $\prod \mathbf{Z}_p^{\times}$ and on $\mathbf{R}_+$ is given by the standard inclusion $\mathbf{R}_+ \to \mathbf{C}^{\times}$. Then an arbitrary Hecke character f is algebraic if and only if it is of the form $\eta\alpha_{\infty}^n$ for some finite order character η and integer n. The character η and the integer n are then uniquely determined.

Now let f be a p-adic Hecke character. The restriction of f to $\mathbf{R}_+$ is then trivial. The restriction of f to $\prod_{\ell \neq p} \mathbf{Z}_{\ell}^{\times}$ is of finite order. The restriction of f to $\mathbf{Z}_p^{\times}$ is a continuous homomorphism $\mathbf{Z}_p^{\times} \to \overline{\mathbf{Q}}_p^{\times}$. It is not difficult to classify all such maps, but we will not do this. We call f *algebraic* if this restriction is of the form $x \mapsto x^n$ on a compact open subset of $\mathbf{Z}_p^{\times}$. Let α_p be the p-adic Hecke character which is trivial on $\mathbf{R}_+$ and $\prod \mathbf{Z}_{\ell}^{\times}$ and on $\mathbf{Z}_p^{\times}$ is given by the standard inclusion $\mathbf{Z}_p^{\times} \to \overline{\mathbf{Q}}_p^{\times}$. Then an arbitrary p-adic Hecke character f is algebraic if and only if it is of the form $\eta\alpha_p^n$ for some finite order character η and integer n. Again, η and n are uniquely determined.

Let f be an algebraic Hecke character. We can then write $f = \eta\alpha_{\infty}^n$. Define a p-adic Hecke character f_p by $f_p = \eta\alpha_p^n$. (Here we are implicitly identifying the roots of unity in $\mathbf{C}$ and $\overline{\mathbf{Q}}_p$ so that we may regard η as being valued in either field.) Under class field theory, the p-adic Hecke character α_p corresponds to the cyclotomic character χ_p. Thus f_p corresponds to $\psi_p = \eta'\chi_p^n$, where η' is the finite order character of G_K corresponding to η. Since the χ_p form a compatible system, we thus see that the ψ_p do as well. Therefore, starting from a Hecke character f we can produce a system $\{f_p\}$ of p-adic Hecke characters and from this obtain a compatible system $\{\psi_p\}$ of one dimensional Galois representations.

3. The general case

Let K be an arbitrary number field. We will find it convenient to treat p-adic Hecke characters and normal Hecke characters (which we now call ∞-adic Hecke characters) simultaneously. Thus let p be a prime or ∞. Let C_p be $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$ correspondingly (one could use $\mathbf{C}_p$ in place of $\overline{\mathbf{Q}}_p$). A p-adic Hecke character is then just a continuous homomorphism

$$\mathbf{A}_K^{\times}/K^{\times} \to C_p^{\times}.$$

We fix an emedding $K \to C_p$ for each p. We explain how this large number of choices can be cut down at the end of the section.

Let f be a p-adic Hecke character. We regard f as a character of

$$\mathbf{A}_K^{\times} = (K \otimes \mathbf{Q}_p)^{\times} \times \prod_{\ell \neq p} (K \otimes \mathbf{Q}_{\ell})^{\times}$$

which is invariant under $K^{\times}$. (If $p = \infty$ then $\mathbf{Q}_p$ means $\mathbf{R}$.) Of course, f restricts to a finite order character on the second factor since ℓ-adic and p-adic topologies do not interact. On the first factor, however, f can be much more complicated. We say that f is *algebraic* if its restriction to the first factor is given by a rational function on an open subgroup, in the following sense. Regard $K \otimes \mathbf{Q}_p$ as an n-dimensional $\mathbf{Q}_p$ vector space, where $n = [K : \mathbf{Q}_p]$, and let $x_i : K \otimes \mathbf{Q}_p \to \mathbf{Q}_p$ be the n coordinates. Then we want f to be a rational function function of the x_i, with coefficients in $\overline{\mathbf{Q}}_p$, after it is restricted to some open subgroup of $(K \otimes \mathbf{Q}_p)^\times$. (Note that we do not say compact open here. If $K \otimes \mathbf{Q}_p = \mathbf{R}$ then we allow f to be the absolute value character. This is an algebraic function when restricted to the open subgroup $\mathbf{R}_+$.)

We now give a nicer reformulation of this algebraic condition. Define a *weight* to be a character of the torus $\mathrm{Res}^K_{\mathbf{Q}}(\mathbf{G}_m)$ over $\overline{\mathbf{Q}}$, that is, a weight is a homomorphism of algebraic groups

$$w : (\mathrm{Res}^K_{\mathbf{Q}}(\mathbf{G}_m))_{\overline{\mathbf{Q}}} \to (\mathbf{G}_m)_{\overline{\mathbf{Q}}}.$$

Here Res denotes restriction of scalars. A weight gives a homomorphism $K^\times \to \overline{\mathbf{Q}}^\times$ which we also call w. For f to be algebraic, it is equivalent to ask that there exists a weight w and an open subgroup U of $(K \otimes \mathbf{Q}_p)^\times$ such that $f(x) = w(x)$ for $x \in K^\times \cap U$. The weight w is then unique and called the weight of f.

Before continuing we give an example. Let $K = \mathbf{Q}(\sqrt{d})$ be an imaginary quadratic field. Let u be a root of d in K and let u' be a root of d in $\mathbf{C}$. We fix an embedding $K \to \mathbf{C}$ by $u \mapsto u'$. Let f be a (∞-adic) Hecke character. At infinity, f gives a map $(K \otimes \mathbf{R})^\times \to \mathbf{C}^\times$. Now, $(K \otimes \mathbf{R})^\times$ is isomorphic to $\mathbf{C}^\times$. Every homomorphism $\mathbf{C}^\times \to \mathbf{C}^\times$ is of the form

$$re^{i\theta} \mapsto r^a e^{in\theta} = e^{a \log r + in\theta}$$

where a is a complex number and n is an integer. We wish to rephrase this classification in terms of K. Every element of K can be written as $x + yu$ with x and y in $\mathbf{Q}$. In these coordinates, r is given by the positive square root of $x^2 + dy^2$ while $e^{i\theta}$ is $(x + du')/\sqrt{x^2 + dy^2}$. We thus find that every homomorphism $(K \otimes \mathbf{R})^\times \to \mathbf{C}^\times$ is of the form

$$x + yu \mapsto (x^2 + dy^2)^{a/2} \left(\frac{x + du'}{\sqrt{x^2 + dy^2}} \right)^n = (x^2 + dy^2)^{(a-n)/2}(x + du')^n$$

where a is a complex number and n is an integer. Of course, a and n are uniquely determined. We find that this is a rational function of x and y if and only if $a - n$ is an even integer, say $2m$. In this case, the above formula can be written as

$$x + yu \mapsto (x - du')^m (x + du')^{n+m}.$$

Thus if we identify $K \otimes \mathbf{R}$ with $\mathbf{C}$ via $u \mapsto u'$ then any algebraic character $(K \otimes \mathbf{R})^\times \to \mathbf{C}^\times$ is of the form $z \mapsto z^n \overline{z}^m$. Of course, n and m are uniquely determined, but if we use the embedding $u \mapsto -u'$ then n and m are switched.

We now examine the same example from the point of view of weights. Let T be the torus $\mathrm{Res}^K_{\mathbf{Q}}(\mathbf{G}_m)$. It can be thought of as the group of all matrices inside of $\mathrm{GL}_2(\mathbf{Q})$ of the form

$$\begin{pmatrix} x & y \\ dy & x \end{pmatrix}$$

The group of characters $T_{\overline{\mathbf{Q}}} \to (\mathbf{G}_m)_{\overline{\mathbf{Q}}}$ is a free abelian group of rank two, generated by the two characters

$$\begin{pmatrix} x & y \\ dy & x \end{pmatrix} \mapsto x + uy, \qquad \begin{pmatrix} x & y \\ dy & x \end{pmatrix} \mapsto x - ux.$$

We thus see that the maps $K^\times \to \overline{\mathbf{Q}}^\times$ coming from weights are exactly the ones of the form

$$(x + uy) \mapsto (x + uy)^n (x - uy)^m.$$

(Here we identify $T(\mathbf{Q})$ with $K^\times$ by letting $x + uy$ correspond to the matrix whose top left entry is x and bottom right entry is y.) Thus the homomorphisms $(K \otimes \mathbf{R})^\times \to \mathbf{C}^\times$ which come from weights are exactly the ones $z \mapsto z^n \overline{z}^m$. This shows that our two characterizations of algebraic homomorphisms agree in this case.

We now return to the general setting. Let f be an algebraic ∞-adic Hecke character with weight w. For any prime p let $\alpha_{w,p}$ be the homomorphism $\mathbf{A}_K^\times \to C_p^\times$ which is trivial on the prime to p components of $\mathbf{A}_K^\times$ and given by $w(x)$ for $x \in (K \otimes \mathbf{Q}_p)^\times$. Since f is algebraic of weight w, the character $\eta = f\alpha_{w,\infty}^{-1}$ is locally constant on $\mathbf{A}_K^\times$, that is, its kernel is open. It is not difficult to see that there is a number field M such that η takes values in $M^\times$. Choose an embedding of M into $\overline{\mathbf{Q}}_p$. We now define f_p to be $\eta\alpha_{w,p}$. We have thus associated a family of algebraic p-adic Hecke characters $\{f_p\}$ to our initial Hecke character f. Letting ψ_p be the character of G_K associated to f_p we also get a system $\{\psi_p\}$ of Galois characters. In fact, this is a compatible system.

Throughout we have been assuming an embedding of K into C_p for each p. This was actually not used in our first characterization of algebraic. However, it was used to move between f and f_p. We now explain a nicer set-up that requires fewer choices. The idea is to work with a field of coefficients. Let M be a number field. For a place v of M define a v-adic Hecke character to be a continuous homomorphism $\mathbf{A}_K^\times/K^\times \to M_v^\times$. Then, in this setting, one need only choose an embedding $K \to M$. That is, after having picked an embedding one can take an algebraic v-adic Hecke character and form from it a system of algebraic Hecke characters indexed by the places of M.

4. Conclusion

We have a diagram

$$\begin{array}{ccc}
\{\text{algebraic Hecke characters}\} & = & \{\text{compatible systems of characters}\} \\
\| & & \| \\
\{\text{algebraic } p\text{-adic Hecke characters}\} & = & \{\text{certain } p\text{-adic characters of } G_K\} \\
\big\downarrow & & \big\downarrow \\
\{\text{all } p\text{-adic Hecke characters}\} & = & \{\text{all } p\text{-adic characters of } G_K\}
\end{array}$$

We now explain this. Equal signs mean isomorphism. The top left vertical arrow is our construction to go between algebraic Hecke characters and algebraic p-adic Hecke characters. It is clearly an isomorphism, since one can run the construction in reverse. The bottom left vertical arrow is just the inclusion of the algebraic characters into all characters. The bottom horizontal arrow has already been discussed. The "certain p-adic characters of G_K" are just those that come from algebraic Hecke characters. The middle horizontal and bottom right vertical arrow are evident. The top right vertical arrow takes a compatible system to its pth member. It is injective since a compatible system is determined by any of its members. We now come to interesting part. We have shown how to attach to an algebraic Hecke character a system of p-adic Hecke characters, and therefore a system of p-adic characters of G_K. As we stated, this is a compatible system. This gives a map from the top left to the top right. It is easily seen to be injective. A more difficult result is that it is surjective — every compatible system is associated to an algebraic Hecke character. A diagram chase now gives the surjectivity of the top right vertical arrow.

A natural question one may now ask is: which are the "certain" p-adic characters of G_K that arise from algebraic Hecke characters? One answer is provided by the diagram: they are exactly those that fit into a compatible system of characters. There is a better answer, though, one that is intrinsic to the character. Namely, a p-adic character of G_K comes from an algberaic Hecke character if and only if it is Hodge-Tate. This is a condition from p-adic Hodge theory.