Lecture 5: Schlessinger's criterion and deformation conditions

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1. What does it take to be representable?

We have been discussing for several weeks deformation problems, and we have said that we would like our deformation functors to be representable so we can study their ring-theoretic properties. We have stated that the framed deformation functor is always representable and that the unrestricted deformation functor is under certain hypotheses, but we have yet to prove either assertion.

There is a general theory of functors on C_Λ the category of Artin local Λ-algebras. My goal in this section is to, in as concrete terms as possible, describe what it takes for such a functor to be representable and how we might verify these properties. I then verify these properties for $D_{\bar{\rho}}$ with $\mathrm{End}_k(\bar{\rho}) = k$.

Along the way, I will point out some subleties of the relationship between C_Λ and $\hat{C}_\Lambda$.

Let k be a finite field. Recall that C_Λ is the category of local Artin Λ-algebras with residue field k, where Λ is any complete noetherian ring with residue field k. One can just think of $\Lambda = W(k)$ in which case every local Artin ring with residue field k admits a unique Λ-algebra structure. Denote by $\hat{C}_\Lambda$ the category of complete local Noetherian rings with residue field k.

We are interested in functors $F : C_\Lambda \to \mathrm{Set}$. We say F is representable if there exists $R \in \hat{C}_\Lambda$ such that F is naturally isomorphic to $\mathrm{Hom}_\Lambda(R, \cdot)$. (Technically you might call this pro-representable but it won't cause any confusion to just say "representable").

Elementary Properties of Representable Functors On C_Λ

- I If F is representable, $F(k) = \mathrm{Hom}_\Lambda(R, k) =$ single point. We assume from now on that $F(k)$ is the one point set.
- II If F is representable, $F(k[\epsilon]) = \mathrm{Hom}_k(m_R/m_R^2 + m_\Lambda, k) = t_F$ is a finite dimensional vector space over k.
- III If F is representable, then F commutes with fiber products, i.e. if $A \to C$ and $B \to C$ are two maps in C_Λ then the natural map

$$F(A \times_C B) \to F(A) \times_{F(C)} F(B) \tag{1}$$

  is a bijection.

Exercise 1.1. If you have not seen it before, you should verify that fiber products exist in C_Λ induced by set-theoretic fiber products. This is not true in $\hat{C}_\Lambda$ (See Conrad's example in Mazur's article **??**).

Exercise 1.2. Show that there is a natural multiplicative map $k \to End_\Lambda(k[\epsilon])$ given by $a \mapsto \alpha_a$ where $\alpha_a(x + y\epsilon) = x + ay\epsilon$.

Remark 1.3 (Tangent Space). Without knowing F is representable, its worth noting what is required for the tangent space to make sense. Let $t_F := F(k[\epsilon])$. The natural map $k \to End_\Lambda(k[\epsilon])$ induces scalar multiplication on t_F. We also know $k[\epsilon]$ is a group object in C_Λ compatible with scaling. Functoriality gives a map

$$F(k[\epsilon] \times_k k[\epsilon]) \to t_F.$$

If we can identify the LHS with $t_F \times t_F$, then we are set. The LHS does admit a natural map to $F(k[\epsilon]) \times_{F(k)} F(k[\epsilon]) = t_F \times t_F$. If this map is bijective (a special case of (III)), then t_F has a vector space structure.

Exercise: Check for F representable that the vector space structure on t_F given functorially as above is the same as the natural vector space structure on $\mathrm{Hom}_k(m_R/m_R^2 + m_\Lambda, k)$.

It turns out that property (III) along with the $\dim_k t_F < \infty$ is necessary and sufficient for F to be representable. However, I will not prove this because it is far too general to be useful. It could be quite difficult to check (III) for every possible pair of morphisms. Luckily, we don't have to! And this leads us to Schlessinger's criterion after a brief definition.

Definition 1.4. A map $A \to B$ in C_Λ is *small* if its surjective and its kernel is principal and annihilated by m_A.

Theorem 1.5 (Schlessinger's Criterion)**.** *Let F be a functor from C_Λ to Sets such that $F(k)$ is a single point. For any two morphism $A \to C$ and $B \to C$ consider the morphism* (1).

Then if F has the following properties:

- H1 (1) *is a surjection whenever $B \to C$ is small.*
- H2 (1) *is a bijection when $C = k$ and $B = k[\epsilon]$.*
- H3 t_F *is finite dimensional*
- H4 (1) *is a bijection whenever $A \to C$ and $B \to C$ are equal and small.*

then F is representable.

I will return to the proof at the end given sufficient time, but one essentially makes clever use of the structure of Artin local rings working at each nilpotent level to build the representing ring as an inverse limit (see Schlessinger [**?**]. We denote the criterion by (SC).

Schlessinger's criterion is just one of many ways to show the deformation functor is representable. It has the advantage that it is concrete and allows one to really exploit the fact that we are working over Artin rings.

Proposition 1.6. *Assume that $End_G(\bar{\rho}) = k$ and G satisfies the p-finiteness condition. Then the deformation functor $D_{\bar{\rho}}$ is representable.*

Proof. I leave it to the reader to verify the following useful fact $\mathrm{Gl}_n(A \times_C B) \cong Gl_n(A) \times_{\mathrm{Gl}_n(C)} \mathrm{Gl}_n(B)$ as groups. This says that given any two lifts ρ_A and ρ_B which agree when pushed-forward to C come from a lift to the fiber product. The only difficulty then in verifying H1 through H4 will be the ambiguity coming from conjugation.

In what follows ρ_A and ρ_B will always lifts of $\bar{\rho}$. We assume we have maps $A \to C$ and $B \to C$ satisfying the hypotheses in (SC). Further, we denote by $\tilde{\rho}_A$ and $\tilde{\rho}_B$ the respective push-forwards of ρ_A and ρ_B under the given maps.

(H1) We are given ρ_A and ρ_B such that $\tilde{\rho}_A = M\tilde{\rho}_B M^{-1}$ for some $M \in \mathrm{Gl}_n(C)$. Since $B \to C$ is surjective, we can lift M to $\mathrm{Gl}_n(B)$. Replacing ρ_B by $M\rho_B M^{-1}$ in the same deformation class yields compatible lifts which can then be lifted to $A \times_C B$.

(H2) Here we start with ρ and ρ' over the fiber product such that ρ_A and ρ_B are conjugate to ρ'_A and ρ'_B respectively. Choose conjugators M_A and M_B. Note that if $\tilde{M}_A$ and $\tilde{M}_B$ were equal, we could lift them to $\mathrm{Gl}_n(A \times_C B)$ and we would be done. This is true in general with no hypotheses on A, B, and C.

We are free to multiply M_B on the right by any matrix N such $N\rho_B N^{-1} = \rho_B$. Let $\mathrm{Stab}(\rho_B)$ be the set of such N. Further note that $\tilde{M}_B^{-1} * \tilde{M}_A$ is in $\mathrm{Stab}(\tilde{\rho}_B)$. If we can lift this to $\mathrm{Stab}(\rho_B)$ then we are done. Hence a sufficient condition for the desired map to be injective is that:

$$\mathrm{Stab}(\rho_B) \to \mathrm{Stab}(\tilde{\rho}_B) \tag{2}$$

is surjective.

This is clear since for $C = k$, $\mathrm{Stab}(\tilde{\rho}_B) = k^*$. I leave it as a exercise to show that for $B = k[\epsilon]$ and $C = k$, the equation (2) holds without any hypotheses on $\bar{\rho}$.

(H3) Follows from p-finiteness of Galois groups, given that (H2) implies the existence of the tangent space.

(H4) I leave it to the reader to verify that surjectivity of (2) follows from the following lemma:

If $\mathrm{End}_G(\bar{\rho}) = k$ and ρ_A is any lift of $\bar{\rho}$, then

$$\mathrm{End}_G(\rho_A) = A.$$

Set $L = \mathrm{End}_G(\rho_A)$ and note that L is an A-submodule of $\mathrm{End}(\rho_A)$ which contains the scalar matrices $A * I$. Further, we have that $L \times_A A/m_A = \mathrm{End}_G(\bar{\rho})$. By Nakayama, L is generated over k by any lift of I. Thus, $L = A$. □

If $\bar{\rho}$ is absolutely irreducible then it will satisfy the above hypothesis by Schur's lemma. However, there is important other case where $\bar{\rho}$ is not irreducible but still satisfies $\mathrm{End}_G(\bar{\rho}) = k$.

Proposition 1.7. *Let k be any field, and let V be any representation of G with a G-stable filtration $V_1 \subset V_2 \subset \ldots \subset V_n = V$ such that:*

- I *V_{i+1}/V_i is one-dimensional with G acting by χ_i.*
- II *The χ_i are distinct.*
- III *The extension $V_i/V_{i-1} \to V_{i+1}/V_{i-1} \to V_{i+1}/V_i$ is non-split for all i.*

Then $\mathrm{End}_G(V) = k$.

Proof. Before you read this proof, I recommend doing the 2 by 2 case by hand which I may or may not have gotten to in the lecture.

Let $M \in \mathrm{End}_G(V)$ i.e. M commutes with the G-action. We want to show that M is a scalar. We first note that V_1 is the unique 1-dimensional subspace on which G acts via χ_1. For if V_1' were another, we could build a Jordan-Holder series $V_1 \subset V_1 \cup V_1' \subset \ldots$ and thus χ_1 would appear at least twice in Jordan-Holder decomposition which can't happen since χ_i are distinct.

It follows then that M preserves V_1 and by induction the whole flag. Let M act on V_1 by multiplication by a. We claim that $M = aI$. Consider $M - aI : V \to V$ also in $\mathrm{End}_G(V)$. Since $M - aI|_{V_1} = 0$, it factors as a morphism

$$T : V/V_1 \to V.$$

By induction, the induced map $V/V_1 \to V/V_1$ which is G-invariant is multiplication by a scalar c. If $c \neq 0$, then $T|_{V_2}$ would give a splitting of the extension where $i = 1$ and so we can assume $c = 0$.

Thus, T is actually a G-invariant map

$$V/V_1 \to V_1.$$

If $T = 0$ we are done, else let V_i be the first subspace on which it is non-trivial. Then $T : V_i/V_{i-1} \to V_1$ is an isomorphism as G-modules, contradiction. □

Remark 1.8. Schlessinger's criterion is a statement purely about a functor on C_Λ. However, once we know F is representable, its quite natural to talk about its points valued in complete local Noetherian rings for we have $\mathrm{Hom}_\Lambda(R, A) = \mathrm{Hom}_\Lambda(R, \lim A/{m_A}^n) = \lim F(A/{m_A}^n)$. In fact, that's really what we were interested in all along, for example, $\mathbb{Z}_p$-deformations, not representations on Artin local rings. So we must ask ourselves, are the points of our universal deformation ring valued in $\hat{C}_\Lambda$ what we want them to be?

Let A be a complete local Noetherian ring. Its clear the any deformation to A yields a map $R_{\bar{\rho}}^{\mathrm{univ}} \to A$ (here's where you use that your representation is continuous). However, the other direction requires an argument. Denote A/m_A^n by A_n. We are given deformations $\rho_n \in D_{\bar{\rho}}(A_n)$ such that $\rho_n \otimes A_{n-1}$ is equivalent to ρ_{n-1}. If ρ_n formed a compatible system of lifts, we would be fine, but we have conjugations interfering at each level. In this case, it can be resolved quite easily. Assume we have compatibility up to ρ_n. Given that $M(\rho_{n+1} \otimes A_n)M^{-1} = \rho_n$ change ρ_{n+1} by any lift of M to $\mathrm{Gl}_N(A_n)$ and proceed by induction.

We will return to this point again later where the argument will require some extra input.

Remark 1.9. (Framed Deformations) The fact mentioned earlier that $\mathrm{Gl}_n(A \times_C B) \cong Gl_n(A) \times_{\mathrm{Gl}_n(C)} \mathrm{Gl}_n(B)$ implies that the framed deformation functor $D_{\bar{\rho}}^{\square}$ commutes with all fiber products and thus is representable. However, (SC) is probably way to fancy a way to prove existence for framed deformations. For the record, I give a proof that $R_{\bar{\rho}}^{\square}$ exists.

Proof. Since $\mathrm{Gl}_N(k)$ is finite, $\bar{\rho}$ is trivial on some finite index subgroup H of G. For any lift, we have that

$$\rho_A(H) \subset \ker(\mathrm{Gl}_N(A) \to \mathrm{Gl}(k))$$

which is a p-group for any $A \in C_\Lambda$. Thus, $\rho_A|_H$ factors through maximal pro-p quotient of H which by p-finiteness is topologically finitely generated. Pick generators $g_1, \ldots, g_j$. Also, pick coset representatives for $g_{j+1}, \ldots, g_m$ for G/H.

Any lift ρ_A is determined by where the $\{g_i\}$ are sent. Consider the power series ring $R = \Lambda[[X_{i,j}^l]]$ where $1 \leq i, j \leq N$ and $1 \leq l \leq m$. I claim that we can construct the universal framed deformation ring as a quotient of R such that the universal framed deformation ρ^{univ} is given by g_l goes to the matrix $(X_{i,j}^l)$. Let S be the set of relations in G amongst the g_l. For any relation, we can consider the corresponding relation on matrices under the map g_l goes to $(X_{i,j}^l)$. We form the ideal I in R generated by these relations. Then $D_{\bar{\rho}}^{\square}$ is represented by R/I. □

Before we move on to deformation conditions, I would like to recall several different interpretation of the tangent space which will be useful in the future. We would like to give a concrete interpretation of the abstract $t_{\bar{\rho}} := D_{\bar{\rho}}(k[\epsilon])$. Let (V, τ) be a deformation to $k[\epsilon]$ where τ is an isomorphism $V/\epsilon \to \bar{\rho}$. Consider the following exact sequence of k-vector spaces:

$$0 \to \epsilon V \to V \to V/\epsilon V \to 0.$$

Simply because G commutes with the action of $k[\epsilon]$, this is an extension of G-modules. Further one can identify via τ the terms on both ends with $\bar{\rho}$.

Hence we get a map $t_{\bar{\rho}} \to \text{Ext}(\bar{\rho}, \bar{\rho})$. It is an exercise to show the map is bijective. By general nonsense, one can identify this Ext-group with $H^1(G, \text{ad}(\bar{\rho}))$. Note that $\text{ad}(\bar{\rho})$ is just $\text{End}(\bar{\rho})$ where G acts via conjugation (ad stands for adjoint).

I will give you the map $\text{Ext}(\bar{\rho}, \bar{\rho})$ to $H^1(G, \text{ad}(\bar{\rho}))$, but I leave it to you to check that the vector space structures on $t_{\bar{\rho}}$ and $H^1(G, \text{ad}(\bar{\rho}))$ agree.

Given an extension $0 \to V' \to V \to V'' \to 0$ choose a splitting $\phi : V'' \to V$ just as vector spaces. The map $g \mapsto g\phi g^{-1} - \phi$ is a co-cycle with values in $\text{ad}(\bar{\rho})$.

In the next section, as we impose various deformation conditions, we will eventually want to keep track of the effect on the tangent space.

2. Deformation Conditions

As we have mentioned already several times in this seminar, whether in the local or global situation, the unrestricted universal deformation ring if it exists will be far too "big" to be useful. Hence we will want to impose some conditions on what kinds of deformations we allow. Deformation conditions can come in different varieties. Often we have global representations on which we impose local conditions at finite set of primes. At these local places, we might impose matrix conditions for example, fixed determinant, ordinary, etc. We could also impose conditions coming from geometry or p-adic Hodge theory: flat, crystalline, semi-stable. I will discuss some of these in more detail later.

In Mazur's article [**?**], he defines the notion of deformation condition quite generally such that everything we will talk about probably fits into that framework. However, for our purposes and for the purpose of intuition, the definitions are unilluminating. Instead, I will give two different perspective from which one could derive all the definitions.

Functorial Perspective If $D_{\bar{\rho}}$ is the deformation functor, a deformation condition should define a subfunctor $D'_{\bar{\rho}}$ of $D_{\bar{\rho}}$. Further, if $D_{\bar{\rho}}$ is representable, then $D'_{\bar{\rho}}$ should be as well. One could use the term relatively representable as Mok did, but its not necessary.

The first statement is usually immediate for any deformation condition. The second one is not. This is a place you might use Schlessinger's criterion, maybe you know already that $D_{\bar{\rho}}$ satisfies Schlessinger then you just have to show the $D'_{\bar{\rho}}$ does too. We will see an example of this soon.

Deformation Space Perspective If $D_{\bar{\rho}}$ is represented by $R_{\bar{\rho}}$, then we can talk about Spec $R_{\bar{\rho}}$ as the space of all deformations of $\bar{\rho}$. Personally, I find this picture quite compelling.

A brief aside. Say $\bar{\rho}$ is modular, then Akshay explained that we get a surjective map $R_{\bar{\rho}} \to T$, a Hecke ring. In geometric language, Spec T is a closed subspace of the deformation space which includes the closed point corresponding to $\bar{\rho}$. Imagine this as the locus of "modular" deformations. Given a representation coming from an elliptic curve, etale cohomology, or somewhere else, whose reduction is $\bar{\rho}$, its natural to ask does it land in that locus. Our goal then, as I understand it, is to impose enough purely representation theoretic condition to cut out the "modular" locus. Then whatever representation we started with will presumably have those properties and hence will be modular.

From this perspective then a deformation condition is just a closed condition on the space of all deformations. More concretely, there exist an ideal I such that for any $f : R \to A$, $f \circ \rho_{\text{univ}}$ satisfies the deformation condition if and only if f factors through R/I.

Remark 2.1. We can connect the two perspectives as follows: let D' be subfunctor of D and assume they are both representable by R and R', then we get a natural map $R \to R'$. I claim this map is surjective. It suffices to check that the map on cotangent spaces is surjective. But the map on cotangent spaces is dual to the map on tangent spaces which is injective because $D'(k[\epsilon]) \subset D(k[\epsilon])$.

Remark 2.2 (Relative Perspective). There is relative perspective which doesn't require $D_{\bar{\rho}}$ to be representable. Given any deformation ρ to A, we can ask if the subset of maps Spec $B \to$ Spec A such that the pullback of ρ has a given condition is represented by a closed subset of Spec A? If this holds for all A and if the univeral deformation ring exists then we can apply it to $(R_{\bar{\rho}}, \rho^{\text{univ}})$. This is the perspective Kisin often takes.

2.1. **Determinant Condition.** Let G be any local or global Galois group.

Definition 2.3. Let $A \in \hat{C}_\Lambda$, and let $\delta : G \to \Lambda^*$ be a character. We say a representation ρ on a free rank n A-module has *determinant* δ if $\wedge^n : G \to A^*$ factors through δ.

Consider the functor of deformations with fixed determinant δ (assume that $\bar{\rho}$ has determinant δ). I claim this is a deformation condition.

Here the second perspective is most natural. Let $\delta^{\text{univ}} : G \to R_{\bar{\rho}}^*$ be the the determinant of the universal deformation of $\bar{\rho}$ assuming it exists. Then let I be the ideal generated by $\delta^{\text{univ}}(g) - i(\delta(g))$ where $i : \Lambda^* \to R_{\bar{\rho}}^*$ is inclusion coming from algebra structure. Then, $R_{\bar{\rho}}/I$ represents deformations with determinant δ. Its usually denoted by $R_{\bar{\rho}}^{\delta}$.

I haven't worked it out, but I suspect it would quite a bit more tedious to show for example that the determinant condition defines a subfunctor which satisfies Schlessinger's criterion or that it is relatively representable.

Note that even if $D_{\bar{\rho}}$ is not representable, the same proof goes through for any (A, ρ) to show relative closedness as in Remark 2.2.

2.2. **Unramified Condition.** Let K be a global field and let S be a finite set of primes. We denote by $G_{K,S}$ the maximal Galois group unramified outside S. Take $\bar{\rho}$ to be a residual representation of $G_{K,S}$ which happens also to be unramified at some $\nu \in S$.

Definition 2.4. Let ρ_A be any deformation of $\bar{\rho}$. We say that ρ_A is unramified at ν if $\rho_A|_{G_{K_\nu}}$ is unramified for any choice of decomposition group G_{K_ν}.

In showing this is a deformation condition, I will illustrate the relative perpective. Again, let ρ_A be any deformation of $\bar{\rho}$. Consider any map $f : A \to B$. The push-forward $f_*(\rho_A)$ will be unramified an ν iff its trivial on the inertia group I_{K_ν} (for some choice of inertia).

Let J be the ideal in A generated by the entries of $\{I - \rho_A(g)\}$ for all $g \in I_{K_\nu}$. Then, one can verify that A/J represents the unramified at ν condition. This is the relative condition; if $R_{\bar{\rho}}$ exists, we can apply the same argument to construct the universal deformation ring unramified at ν.

2.3. **Ordinary Deformations.** We will go into extensive detail in this section as the notion of ordinary will play a prominent role in what is to come. There seem to be several definitions of ordinary floating around. I chose one that is both concrete and sufficiently general for now.

Definition 2.5. Let $G = G_K$ be a local Galois group where the residue characteristic is p. Let $\psi : G \to \mathbb{Z}_p^*$ be the p-adic cyclotomic character. An n-dimensional representation ρ of G is *ordinary* if

$$\rho|_{I_K} \sim \begin{pmatrix} \psi^{e_1} & \star & \star & \star \\ 0 & \psi^{e_2} & \star & \star \\ 0 & 0 & \ddots & \star \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $e_1 > e_2 > \ldots > e_{n-1} > 0$. Implicitly we are including $\mathbb{Z}_p \to \Lambda$ so that the definition makes sense over any Λ-algebra and hence on our category $\hat{C}_\Lambda$.

Before we continue, let me say where this condition is coming from.

Example 2.6. If E is an elliptic curve over K a local field of residue characteristic p which has good ordinary reduction at p, then the representation of G_K on $T_p(E)$ is ordinary. In particular, it has the form

$$g \mapsto \begin{pmatrix} \psi(g)\chi_1(g) & \star \\ 0 & \chi_2(g) \end{pmatrix}$$

where χ_1 and χ_2 are unramified characters on G_K.

There is an corresponding notion of what it means for a modular form to be ordinary, but I won't get into it here.

Remark 2.7. Note that though ψ^e is non-trivial on I_K for any $e \neq 0$ (in fact its infinitely ramified), its possible for ψ^e to be residually trivial. The residually trivial case will be of interest to us later on, but for now, we assume that all ψ^{e_i} are residually distinct and non-trivial. Its not hard to work out exactly when this happens based on K, p, e_i.

Next, we would like to show that if we conjugate ρ such that $\rho|_{I_K}$ is upper triangular, then ρ will be upper triangular. This follows from the following useful lemma.

Lemma 2.8. *Let $\rho : I_K \to \mathrm{Gl}_N(A)$ be a representation, $A \in C_\Lambda$ landing in the upper triangular matrices with residually distinct characters along the diagonal. If $M\rho M^{-1}$ is also upper triangular with same characters occurring in the same order, then M is upper triangular.*

Proof. We prove it in two steps. First we show that ρ preserves a unique flag. We know this fact residually using a Jordan-Holder component argument as in Proposition 1.7. Let $M \cong A^N$ with G acting through ρ. Let $L_1 \subset M$ be the line corresponding to e_1 on which ρ acts by χ_1. We want to show that given any $m \in M$ on which G acts via χ_1, $m \in L_1$. From there, it is a simple induction on N.

Consider the quotient $N = M/L_1$. I claim N contains no non-zero v on which G acts via χ_1. Assume there existed such an v. Filter N by $m_A^n N$, and let n_0 be the smallest n such that $v \notin m_A^n N$. Clearly G acts on the image of v in $m_A^{n_0-1}N/m_A^{n_0}N$ via the character χ_1. However, its not hard to see that for any n,

$$m_A^n N/m_A^{n+1}N \cong m_A^n/m_A^{n+1} \otimes_k (\bar{\rho}/(L_1 \otimes k))$$

as $k[G]$-modules. The RHS breaks up as the direct sum of copies of $\bar{\rho}$ quotiented by the χ_1 subspace and hence χ_1 doesn't appear anywhere in semi-simplification (using residual distinctness). Hence the flag is unique.

To say M is upper triangular is equivalent to saying M preserves the flag which we have now shown to be unique. Again, by induction on N, it will suffice to show the M preserves L_1. Let $e_1 \in L_1$. Our hypotheses imply that

$$M\rho M^{-1}e_1 = \rho e_1 = \chi_1 e_1.$$

Multiplying by M^{-1} and using that M^{-1} commutes with χ_1, we get

$$\rho(M^{-1}e_1) = \chi(M^{-1}e_1).$$

By uniqueness then, $M^{-1}e_1 \in L_1$ and hence M^{-1} preserves the flag so M does as well. □

Corollary 1. *If ρ is ordinary, then it lands in a Borel subgroup, i.e. is upper-triangular with respect to some basis.*

Proof. By assumption, we can conjugate $\rho|_I$ to be upper-triangular so it suffices to show that $\rho(g)$ is, where g is some Frobenius element. Since ψ^e are invariant under conjugation by g, we see that $\rho(g)$ satisfies the hypotheses on M in the previous lemma and hence is upper-triangular. □

Corollary 2. *The ordinary deformation functor satisfies (SC) and so is representable under the assumption that the residual representation is non-split in the sense of Proposition 1.7.*

Proof. By Proposition 1.7, the residual deformation satisifies the necessary conditions for the universal deformation functor to exist. We denote the ordinary deformation functor by $D_{\bar{\rho}}^{\mathrm{ord}}$; it is clearly a subfunctor of $D_{\bar{\rho}}$. As a subfunctor, injectivity of the map (1) is automatic in H1, H2, and H4. Hence it suffices to check that (1) is surjective under the hypotheses of H1, namely when $B \to C$ is small.

We are free to choose ordinary lifts ρ_A and ρ_B

$$\tilde{\rho}_A = M\tilde{\rho}_B M^{-1}.$$

Since $\tilde{\rho}_A$ and $\tilde{\rho}_B$ are both ordinary M satisfies the hypotheses of the previous lemma and so is upper-triangular. We can choose a lift M' of M to $\mathrm{Gl}_n(B)$ which is upper triangular. Changing ρ_B by M' maintains its ordinary form. Hence, we have ρ_A and ρ_B agreeing after push-forward and both have ordinary form and so their fiber product will also be ordinary. □

Exercise 2.9. (Continuity) Let A be complete local Noetherian ring and set $A_n = A/m_A^n$. Given a compatible system of ordinary deformations ρ_n, show that there exist an ordinary deformation ρ_A such that $\rho_A \otimes A_n \equiv \rho_n$. Hint: See Remark 1.8.

As a final comment, it is possible to interpret ordinarity as a closed condition at least under the assumptions of residually distinct and nonsplit, but I did not have to write it up. Hopefully, I will have a chance to present it in seminar. Otherwise, feel free to ask me about it afterwards.