Lecture 4: Generic fibers of deformation rings

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1. Some observations

Fix $\overline{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(k)$ absolutely irreducible, and let $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be the universal deformation. We're interested in the map $R \to \mathbb{T}_{\mathfrak{m}}$ for some Hecke algebra defined in terms of $\overline{\rho}$. Note that the Hecke algebra is 1-dimensional, and even finite free over $\mathbb{Z}_p$. The universal deformation ring R, however, often has dimension > 1 and nonzero p-torsion. In other words, the surjection $R \twoheadrightarrow \mathbb{T}_{\mathfrak{m}}$ is not even close to being an isomorphism in general.

Example. Consider $X_0(49)$ which is an elliptic curve. [Cf. Nigel Boston's papers on explicit deformation rings for the details of this example.] Let $\overline{\rho}$ the representation from the 3-torsion of E, and let $S = \{3, 7, \infty\}$. Boston computed the universal deformation as

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2\left(\frac{\mathbb{Z}_3[\![x_1, x_2, x_3]\!]}{(1 + x_1)^3 - 1}\right).$$

Just by looking at the ring on the right side, it's clear that its dimension is at least 2. (This example doesn't illustrate the phenomenon of p-torsion, but oh well...)

Morally, the reason for the higher dimension of R is that we are not imposing any local conditions at all for the places in S.

A key observation is that even when we succeed in proving a modularity lifting theorem, we don't know until we're done that R is $\mathbb{Z}_p$-finite and flat. In other words, even when in fact R turns out to be nice, we have very little grasp of why it is nice without proving an $R = \mathbb{T}$ theorem.

However, this is really not so bad. For example, if we could show that $R[1/p] \cong \mathbb{T}_m[1/p]$, that's totally fine. After all, we're trying to study deformations of $\overline{\rho}$ over p-adic integer rings, which are p-torsion free and reduced, so we rig the Hecke algebra to have the same properties. In other words, **we only care about the "p-adic points" of R** so we can just as well study the structure of $R[1/p]$/nilpotents. And via Kisin's methods, it turns out that a thorough understanding of the "structure" of this ring is attainable in interesting cases and is exactly what is needed for modularity lifting theorems. Things we would like to know:

- Characterize in some moduli-theoretic manner the connected components of its spectrum (e.g., so we can detect when two p-adic points lie on the same component).
- Dimension.
- Singularities, i.e. the extent to which an appropriately defined notion of smoothness fails to hold.

For the last point, it is just as good in practice to pass to a formally smooth R-algebra (such as a power series ring over R). So we can consider the framed deformation ring.

Remark. A key point is that $R[1/p]$ is **very** far from being a local ring. For example, say $R = \mathbb{Z}_p[\![x]\!]$ (which is a rough prototype of the sort of ring that arises). Then

$$R[1/p] = \mathbb{Z}_p[\![x]\!][1/p] = \{f \in \mathbb{Q}_p[\![x]\!] \mid \text{denominators are } \textit{bounded} \text{ powers of } p\} \subsetneq \mathbb{Q}_p[\![x]\!].$$

This ring has lots of $\mathbb{Q}_p$-algebra maps $\mathbb{Z}_p[\![x]\!][1/p] \twoheadrightarrow \mathcal{O}_K[1/p]$ for finite extensions $K/\mathbb{Q}_p$, sending x into $\mathfrak{m}_K$. Hence it has lots of maximal ideals.

2. Digression on Jacobson rings

Definition. A **Jacobson ring** is a Noetherian ring A such that any $\mathfrak{p} \in \operatorname{Spec} A$ is the intersection of the maximal ideals containing $\mathfrak{p}$.

Clearly a quotient of a Jacobson ring is Jacobson. Less evident, but in the exercises of Atiyah-MacDonald, is that a finitely generated algebra over a Jacobson ring is Jacobson. Note that any field is Jacobson, as is any Dedekind domain with *infinitely many* primes (but not a dvr, nor a local ring which is not 0-dimensional!). In particular, a general localization of a Jacobson ring is certainly not Jacobson, though localization at a single element is (since it is a finitely generated algebra).

A consequence of the definition is that if $X_0 = \mathrm{MaxSpec}(A) \stackrel{j}{\hookrightarrow} \operatorname{Spec} A = X$, then j is a dense *quasihomeomorphism*, which means that $U_0 = X_0 \cap U \leftrightarrow U$ is a bijection between the collections of open sets in X_0 and X. Jacobson rings abstract the nice properties enjoyed by algebras of finite type over a field.

Claim. If R is a quotient of a formal power series ring over a complete dvr A with uniformizer π then $R[1/\pi]$ is Jacobson, and for all maximal ideals $\mathfrak{m} \subset R[1/\pi]$, the quotient $R[1/\pi]/\mathfrak{m}$ is finite over the fraction field $K = A[1/\pi]$ of A. Moreover, every K-algebra map from $R[1/\pi]$ to a finite extension K' of K carries R into the valuation ring A' of K', with the map $R \to A'$ actually a local map.

Note it is elementary that every K-algebra map from $R[1/\pi]$ to a finite extension K' of K has kernel that is maximal: the kernel P is at least a prime ideal, and $R[1/\pi]/P$ is an intermediate ring between the field K and the field K' of finite degree over K, so it is a domain of finite dimension over a field (namely K) and hence is itself a field. Hence, P is maximal.

Also, everything in the Claim can be deduced from facts in rigid geometry concerning K-affinoid algebras, by using the approach in deJong's IHES paper *Crystalline Dieudonné theory via formal and rigid geometry.* For convenience, we give a direct proof using commutative algebra, avoiding rigid geometry (but inspired by it for some of the arguments).

The proof of the Claim is somewhat long (and was omitted in the lecture).

Proof. To prove the claim, first note that if the claim holds for R then it holds for any quotient of R. Hence, it suffices to treat the case when $R = A[\![x_1, \dots, x_n]\!]$ is a formal power series ring over A. We first check the more concrete second part of the Claim: for finite K'/K, any K-algebra map $R[1/\pi] \to K'$ carries R into the valuation ring A' of K' with $R \to A'$ moreover a local map. In other words, we are studying A-algebra maps $R \to K'$. This can be uniquely "promoted" to an A'-algebra map

$$A' \otimes_A R \to K',$$

and we can pass the tensor product through the "formal power series" formation since A' is a finite free A-module. In other words, we can rename A' as A to reduce to the case $K' = K$. So we claim that any A-algebra map $R \to K$ must be "evaluation" at an n-tuple in the maximal ideal of A. If we can show it carries each x_i to some a_i in the maximal ideal of A then the map kills $x_i - a_i$ for all i. By completeness of R it would be legal to make a "change of variables" renaming $x_i - a_i$ as x_i to reduce to the case when the map kills all x_i's. Since the quotient of R by the ideal generated by the x_i's is identified with A, after inverting π we get K (as a K-algebra!), so we'd have proved what we want.

Let's now show that indeed each x_i is carried to some a_i in the maximal ideal of A. By composing the given A-algebra map $R \to K$ with the natural inclusion $A[\![x_i]\!] \to R$ we are reduced to the case $n = 1$. That is, we wish to prove that any A-algebra map $A[\![x]\!] \to K$ must carry x to an element a in the maximal ideal of A. This map must kill some nonzero $f \in A[\![x]\!]$, as $A[\![x]\!][1/\pi]$ has infinite K-dimension as a vector space, and we can write $f = \pi^e f_0$ for some $e \geq 0$ and some f_0 not divisible by π. Thus, f_0 also dies in K, so by renaming it as f we arrange that f has some coefficient not divisible by π. This coefficient must occur in positive degree, as otherwise f would be a unit, which is absurd (as it is in the kernel of a map to a field). Now by the formal Weierstrass Preparation Theorem (in one variable –see Lang's *Algebra*), if $d > 0$ is the least degree of a coefficient of f not divisible by π then f is a unit multiple of a "distinguished" polynomial: a monic polynomial in x of degree d over A with all lower-degree coefficients divisible by π. Scaling away the unit, we can assume that f is a monic polynomial of degree $d > 0$ with all lower-degree coefficients divisible by π. Hence, $A[\![x]\!]/(f) = A[x]/(f)$ by long-division of formal power series (thanks to completeness of A!). Our map of interest therefore "is" an A-algebra map

$$A[x]/(f) \to K$$

and so it carries x to an element a of K that is a root of f. Since f is monic over A, we see $a \in A$. Since f has all lower-degree coefficients in the maximal ideal, necessarily a is in the maximal ideal too. That completes the proof of the second part of the Claim.

Now it remains to show the first part of the Claim: R is Jacobson, and if M is a maximal ideal of $R[1/\pi]$ then $R[1/\pi]/M$ is of finite degree over $A[1/\pi] = K$. We argue by induction on the number n of variables (motivated by the method of proof of the analytic Weierstrass Preparation theorem over $\mathbb{C}$ or non-archimedean fields), the case $n = 0$ being trivial. Also, it is harmless (even for the Jacobson property) to make a finite extension on K if we wish. We will use this later, to deal with a technical problem when the residue field k is finite (which is of course the case of most interest to us).

Assume $n > 0$, and consider a nonzero $f \in R = A[\![x_1, \dots, x_n]\!]$ contained in some chosen nonzero prime or maximal ideal; clearly f can be scaled by π-powers so it is not divisible by π. We want to get to the situation in which f involves a monomial term that is just a power of a single variable. Pick a monomial of least total degree appearing in f with coefficient in $A^\times$. (Such a term exists, since f is not divisible by π.) This least total degree d must be positive (as otherwise $f(0) \in A^\times$, so $f \in R^\times$, a contradiction). By relabeling, we may suppose x_1 appears in this monomial. If $n = 1$, this term is an $A^\times$-multiple of a power of x_1, so we're happy. Now assume $n > 1$ and consider the *homogeneous* change of variables which replaces x_i with $x_i + c_i x_1$ for all $i > 1$ (and leaves x_1 alone), with $c_i \in A$ to be determined in a moment. Each degree-d monomial

$$a_I x_1^{i_1} \cdots x_n^{i_n}$$

in f (before the change of variable) with total degree d contributes

$$a_I c_2^{i_2} \cdots c_n^{i_n} x_1^d$$

to the x_1^d term after the change of variable (with $i_1 = d - (i_2 + \cdots + i_n)$). All other monomials can only contribute to x_1^d with coefficient in maximal ideal of A. Thus, these other terms can be ignored for the purpose of seeing if we get x_1^d to appear with an $A^\times$-coefficient after the change of variables.

To summarize (when $n > 1$), whatever c_i's we choose in A, we get after change of variable that x_1^d appears with coefficient $h(c)$ for some polynomial h in $n - 1$ variables over A that has *some* coefficient in $A^\times$ (since i_1 is determined by $i_2, \dots, i_n$). Thus, h has nonzero reduction as a polynomial over the residue field k of A, so as long as this reduction is nonzero at some point in k^{n-1} we can choose the c's to lift that into A^{n-1} to get the coefficient of x_1^d to be in $A^\times$. If k is infinite, no problem. If k is finite (case of most interest!), for some finite extension k' of k we can find the required point in k'^{n-1}, so go back and replace A with the corresponding unramified extension (and the chosen prime with each of the ones over it after scalar extension) to do the job.

The upshot is that after a suitable change of variables (and possible replacement of A with a finite extension in case k is finite), we can assume that f contains some x_1^d with an $A^\times$-coefficient. Thus, if we view f in

$$R = (A[\![x_2, \dots, x_n]\!])[\![x_1]\!]$$

then it satisfies the hypotheses of the general Weierstrass Preparation (with complete coefficient ring) as in Lang's *Algebra*. This implies that f is a unit multiple of a *monic* polynomial in x_1 whose lower-degree coefficients are in the maximal ideal of $R' = A[\![x_2, \dots, x_n]\!]$ (which means A if $n = 1$). We can therefore scale away the unit so that f is such a "distinguished" polynomial, and then do long division in $R'[\![x_1]\!]$ due to completeness of R' to infer that

$$R/(f) = R'[\![x_1]\!]/(f) = R'[x_1]/(f).$$

This is a finite free R'-module!

We may now draw two consequences. First, if P is a prime ideal of $R[1/\pi]$ containing f then $R[1/\pi]/P$ is module-finite over the ring $R'[1/\pi]$ which is Jacobson by induction, so $R[1/\pi]/P$ is Jacobson. Hence, P is the intersection of all maximals over it, whence we have proved that $R[1/\pi]$ is Jacobson. Second, for a maximal ideal M of $R[1/\pi]$ containing f, the ring map $R'[1/\pi] \to R[1/\pi]/M$ is module-finite so its prime ideal kernel is actually maximal. That is, we get a *maximal* ideal M' of $R'[1/\pi]$ such that $R'[1/\pi]/M' \to R[1/\pi]/M$ is of finite degree. By induction, $R'[1/\pi]/M'$ is of finite degree over K, so we are done. □

3. Visualizing $R[1/p]$

Let $R = A[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_m)$ and K be in the last subsection. Observe that $\mathrm{Hom}_{\mathrm{loc}.A-\mathrm{alg}}(R, A') = \mathrm{Hom}_{\mathrm{Frac}(A)-\mathrm{alg}}(R[1/\pi], A'[1/\pi] = \mathrm{Frac}(A'))$ for any finite dvr extension A' of A. This suggests the following geometric perspective on the ring $R[1/\pi]$: it corresponds to the locus of geometric points (x_i) with coordinates in $\overline{\mathrm{Frac}(A)}$ lying in the open polydisk $\{|x_1|, \dots, |x_n| < 1\}$ at which the convergent power series $f_1, \dots, f_m$ all vanish. To make this viewpoint precise, one must regard the spaces in question as **rigid analytic spaces**.

4. Final thought

We'll see that for Galois deformation rings R, the completions of $R[1/p]$ at maximal ideals are deformation rings for *characteristic zero* representations corresponding to the maximal ideals in question. This is very interesting, since R itself was entirely about deforming $\bmod p$ things!

5. Back to examples of explicit universal deformation rings

Caveat: These sorts of examples are kind of "useless". The reference for N. Boston's examples is *Inv. Math.* **103** (1991).

Example 1 [loc. cit., Prop. 8.1.] Let $E: y^2 = x(x^2-8x+8)$, an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-2})$. Let $\overline{\rho}$ be the representation on the 3-torsion:

$$G_{\mathbb{Q},\{2?,3,5,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3).$$

In general we know that there is some surjection $\mathbb{Z}_3[\![T_1,\dots,T_d]\!] \twoheadrightarrow R(\overline{\rho})$ where we know the smallest d is (by NAK) $d = \dim \mathfrak{m}_R/(\mathfrak{m}_R^2, 3)$, and $\mathfrak{m}_R/(\mathfrak{m}_R^2,3) = \mathrm{H}^1(G_{\mathbb{Q},\{2?,3,5,\infty\}}, \mathrm{Ad}(\overline{\rho}))$. Here the adjoint module is $\mathrm{Ad}(\overline{\rho}) = \mathrm{End}_{\mathbb{F}_3}(\overline{\rho})$ with $G_{\mathbb{Q},\{2?,3,5,\infty\}}$ acting by conjugation via $\overline{\rho}$. In this particular case one can compute that $d = 5$, so

$$R(\overline{\rho}) = \mathbb{Z}_3[\![T_1,\dots,T_5]\!]/I$$

where the ideal of relations has the form

$$I = \delta\cdot(f,g)$$

for

$$f = 8u^4 - 8u^2 + 1, g = 8e^3 - 4u, \qquad u = (1 + T_4T_5)^{1/2}$$

and δ (which may involve all the T_is) is obtained by choosing a certain presentation of a pro-3 group (coming from a wild inertia group, perhaps for the splitting field of $\overline{\rho}$?), and setting $\delta = \det(\rho^{\text{univ}}(y) - 1)$ where y is a particular generator in said presentation. Consequently one can write down some "explicit" deformations of $\overline{\rho}$ by looking for solutions to the relations above in a $\mathbb{Z}_3$-algebra...

Example 2 [Boston-Ullom]. Let $E = X_0(49)$ and $\overline{\rho} = \overline{\rho}_{E,3}$ the representation on the 3-torsion:

$$G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3).$$

In this case the universal deformation ring is particularly simple:

$$R \cong \mathbb{Z}_3[\![T_1,\dots,T_4]\!]/((1+T_4)^3 - 1).$$

We have $(1+T_4)^3 - 1 = T_4(T_4^2 + 3T_4 + 3)$. The quadratic factor is irreducible over $\mathbb{Q}_3$, but not over $\mathbb{Q}_3(\sqrt{-3})$. So, loosely speaking, $\mathrm{Spec}\, R$ has two irreducible components but three "geometric" irreducible components: $T_4 = 0$ and T_4 equal to either of the conjugate roots of the quadratic factor. For example, to recover the 3-adic Tate module of E one considers the map $R \to \mathbb{Z}_3$ given by mapping all T_is to 0. This is a sort of "canonical" $\mathbb{Z}_3$-point of $\mathrm{Spec}\, R$. Since the quadratic factor of the relation is $\mathbb{Q}_3$-irreducible, so that quadratic field cannot be $\mathbb{Q}_3$-embedded into $\mathbb{Q}_3$, *every* $\mathbb{Z}_3$-point lies in the $T_4 = 0$ component.

The lesson to take from this seems to be that it can be hard to detect components, or more generally aspects of the geometry, of $\mathrm{Spec}\, R$, when only looking at p-adic points over a small field like $\mathbb{Q}_p$; we have to expect to work with points in many finite extensions in order to effectively probe the geometry. All this is by way of motivation for our interest in characteristic zero points of deformation rings, and (for example) our willingness to throw out all possible nastiness at p by studying $R[1/p]$ instead of R itself.

6. Back to Characteristic 0

Now let Λ be a p-adic dvr with fraction field K and residue field k. Let $R = \Lambda[\![X_n,\dots,X_n]\!]/I$ be the universal deformation ring of a residual representation $\overline{\rho}: \Gamma \to \mathrm{GL}_N(k)$, for a profinite group Γ satisfying the requisite p-finiteness conditions (e.g. G_K for local K or $G_{K,S}$ for a number field K).

Remark. We have seen above that for any maximal ideal $\mathfrak{m} \subset R[1/p]$, the residue field $R[1/p]/\mathfrak{m}$ is of finite degree over k. The intuition for this fact is that these closed points of $\operatorname{Spec} R[1/p]$ correspond to Galois orbits over K of $\overline{K}$-solutions to $I = 0$ in the open unit n-polydisk. (The case $n = 1$ is a consequence of the Weierstrass Preparation Lemma. One *can* relate the geometry of $\operatorname{Spec} R[1/p]$ to the geometry of the aforementioned "rigid analytic space" $I = 0$. For example, if $R[1/p]$ is connected (no nontrivial idempotents) then $I = 0$ is connected in the sense of rigid geometry. One can also match up the dimensions of the components. The input for this equivalence is the (self-contained!) §7 of de Jong's IHES paper *Crystalline Dieudonné theory* ..., but we won't use it.

We also saw above that any K-algebra map $R[1/p] \to K'$ for a finite extension K'/K is actually given by sending all the X_is to elements $x_i \in \mathfrak{m}_{K'} \subset \mathcal{O}_{K'} \subset K'$. In other words, $R \subset R[1/p]$ actually lands in $\mathcal{O}_{K'}$!

Now fix a K-algebra map $x : R[1/p] \twoheadrightarrow K'$ into a finite extension of K. ("Contemplate a p-adic point of $\operatorname{Spec} R$".) Let

$$\rho_x : \Gamma \xrightarrow{\rho^{\text{univ}}} \mathrm{GL}_N(R) \to \mathrm{GL}_N(R[1/p]) \to \mathrm{GL}_N(K')$$

be the specialized representation. (In the Boston-Ullom example above, when we take $x : R[1/3] \to \mathbb{Q}_3$ to be the map sending all the T_is to zero, then ρ_x is the 3-adic Tate module of $X_0(49)$.)

Goal: Understand the dimension $\dim R[1/p]_{\mathfrak{m}_x} = \dim R[1/p]^\wedge_{\mathfrak{m}_x}$**.** (Here $(\cdot)^\wedge$ denotes completion.)

For instance, is this complete local ring regular? Perhaps even a power series ring over K'? If so, then its dimension is $\dim \mathfrak{m}_x/\mathfrak{m}_x^2$.

Theorem. *Let* $\rho_x^{\text{univ}} : \Gamma \to \mathrm{GL}_N(R[1/p]^\wedge_{\mathfrak{m}_x})$ *be induced from* ρ^{univ} *by the natural map* $R \to R[1/p]^\wedge_{\mathfrak{m}_x}$*. Then the diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho_x^{\text{univ}}} & \mathrm{GL}_N(R[1/p]^\wedge_{\mathfrak{m}_x}) \\ & \underset{\text{cont}}{\overset{\rho_x}{\searrow}} & \downarrow \\ & & \mathrm{GL}_N(K') \end{array}$$

commutes, and in fact ρ_x^{univ} *is the universal for continuous deformations of* ρ_x*.*

More precisely, if one considers the category $\widehat{\mathcal{C}}$ *of complete local noetherian* K'*-algebras with residue field* K'*, and the functor on the category* $\mathcal{C}$ *of artinian quotients of objects in* $\widehat{\mathcal{C}}$ *which picks out those deformations of* ρ_x *which are continuous for the* p*-adic topology on such artinian quotients, regarded as finite-dimensional* K'*-vector spaces, then* $R[1/p]^\wedge_{\mathfrak{m}_x}$ *is the representing object.*

Remark. If A is a complete local Noetherian F-algebra and the characteristic of F is zero, and $A/\mathfrak{m} = F'$ is a finite extension of F, then there exists a unique F-algebra lift $F' \hookrightarrow A$. Why? By completeness we have Hensel's lemma and by characteristic zero we have F'/F separable. So we can find solutions in A to the defining polynomial of F' over F.

Why do we care about the theorem?

(1) The deformation ring $R[1/p]^\wedge_{\mathfrak{m}_x}$ is isomorphic to $K'[\![T_1, \dots, T_n]\!]$ if and only if $R[1/p]^\wedge_{\mathfrak{m}_x}$ is regular (by the Cohen structure theorem), and the power series description is precisely the condition that the corresponding deformation functor for ρ_x is formally smooth (i.e., no obstruction to lifting artinian points in characteristic 0). This holds precisely when $\mathrm{H}^2(\Gamma, \mathrm{Ad}(\rho_x)) = 0$. So that is interesting: a computation in Galois cohomology in characteristic 0 can tell us information about the structure of $R[1/p]$ at closed points.

(2) $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \cong \mathrm{H}^1_{\text{cont}}(\Gamma, \mathrm{Ad}(\rho_x))$, by the continuity condition we imposed on the deformations in the theorem.

Combining (1) and (2), we can check regularity of $R[1/p]$ at a closed point and in such cases then even compute $\dim_x R[1/p]$ by doing computations in (continuous) Galois cohomology with p-adic coefficients!

7. Proof of theorem

Step 1: Reduce to the case $K' = K$**.** Here is the trick. Set $\Lambda' = \mathcal{O}_{K'}$. Note that $\Lambda' \otimes_\Lambda R$ is local because $(\Lambda' \otimes_\Lambda R)/\mathfrak{m}_R = \Lambda' \otimes_\Lambda k = k'$ is a field. The Λ'-algebra $\Lambda' \otimes_\Lambda R$ is the universal deformation ring of $\overline{\rho} \otimes_k k'$

(where k' is the residue field of K') when using Λ'-coefficients; this behavior of deformation ring with respect to finite extension of the coefficients will be proved in Samit's talk. Consider the diagram

$$\begin{array}{ccc} K'\otimes_K R[1/p] & \xrightarrow{x'} & K' \\ \| & & \\ (\Lambda'\otimes_\Lambda R)[1/p] & & \uparrow \\ \uparrow & & \\ \Lambda'\otimes_\Lambda R & \longrightarrow & \Lambda' \end{array}$$

Exercise: $(\Lambda'\otimes_\Lambda R)[1/p]^\wedge_{\mathfrak{m}_{x'}} \cong R[1/p]^\wedge_{\mathfrak{m}_x}$ as K'-algebras.

So we can rename Λ' as Λ, completing the reduction.

Step 2: Observe that since $\overline{\rho}$ is absolutely irreducible, so is ρ_x. Consequently any deformation of ρ_x has only scalar endomorphisms.

Step 3: Consider any deformation

$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta} & \mathrm{GL}_N(A) \\ & \searrow^{\rho_x} & \downarrow \\ & & \mathrm{GL}_N(K) \end{array}$$

where A is a finite local K-algebra with residue field K. We would like to show that there exists a unique K-map $R[1/p]^\wedge_{\mathfrak{m}_x} \to A$ which takes ρ_x^{univ} to θ, up to conjugation. Why is this sufficient? Because if so, then there would be lifts of ρ_x to $\mathrm{GL}_N(A)$, one coming from ρ_x^{univ} and the other being θ, which are $\mathrm{GL}_N(A)$-conjugate to one another by some matrix M. Upon reduction to $\mathrm{GL}_N(K)$, the matrix $\overline{M}$ would centralize ρ_x. So by Step 2, $\overline{M}$ must be a scalar endomorphism $c \in K^\times$. Consequently we can replace M by $c^{-1}M$ to conclude that the two lifts are conjugate to one another by a matrix which is residually trivial. The latter is precisely what we need to prove that ρ_x^{univ} is universal. (Note that if we used framed deformations throughout then this little step wouldn't be needed. It is important because in later applications we will certainly want to apply the Theorem to cases for which $\overline{\rho}$ is not absolutely irreducible. The reader can check that the proof of the Theorem works in the framed setting once the preceding little step is bypassed.)

The map we need is the same as making a local K-algebra map

$$R[1/p]_{\mathfrak{m}_x} \to A$$

with the same property with respect to θ, since A is a complete K-algebra. (Note that this "uncompletion" step is only possible since we already did Step 1! We originally completed $R[1/p]_{\mathfrak{m}_x}$, which is a K-algebra and generally not a K'-algebra.) The latter is the same as a K-algebra map $R[1/p] \to A$ such that $R[1/p] \to A \to K$ is the original point x, which takes ρ^{univ} to θ. ("It's all a game in trying to get back to R".) In other words, we wanted a dotted map in the diagram

$$\begin{array}{ccccc} R & \longrightarrow & R[1/p] & \overset{\exists!?}{\dashrightarrow} & A \\ \downarrow{\scriptstyle \exists!\widetilde{x}} & & & \searrow^{x} & \downarrow \\ \Lambda & & \longrightarrow & & K \end{array}$$

(The existence of $\widetilde{x}$ is by one of the propositions from §6.) But $R[1/p]$ is just a localization of R and A is a $\Lambda[1/p]$-algebra (it is a K-algebra!), so in fact the existence of a unique dotted map above is equivalent to the existence of a unique dotted map α in the diagram

$$\begin{array}{ccc} R & \overset{\exists!?}{\underset{\alpha}{\dashrightarrow}} & A \\ \downarrow{\scriptstyle \widetilde{x}} & & \downarrow \\ \Lambda & \longrightarrow & K \end{array}$$

such that α takes ρ^{univ} to θ. Now unfortunately A is not in the category $\widehat{\mathcal{C}}_\Lambda$ [typically it is something like $K[t]/(t^7)$], so θ is not quite a deformation of $\overline{\rho}$, so we cannot appeal directly to the universal property of (R, ρ^{univ}). Instead we need to mess around a bit.

Here's the point. $A = K \oplus \mathfrak{m}_A$ and $\mathfrak{m}_A$ is a finite-dimensional K-vector space which is nilpotent.

Claim. $\mathfrak{m}_A = \varinjlim I$ where the limit is taken over Λ-finite multiplicatively stable Λ-modules I.

(Idea of the proof: take products and products and more products. By nilpotence and finite-dimensionality of $\mathfrak{m}_A$ over K, you don't have to keep going forever. Then take the Λ-span of finite collections of such products to get the desired I's.)

Write Λ_I for $\Lambda \oplus I$.

Lemma/Exercise: Any Λ-algebra map $R \to A$ lands in some Λ_I. (Hint: choose I containing the images of all the X's.)

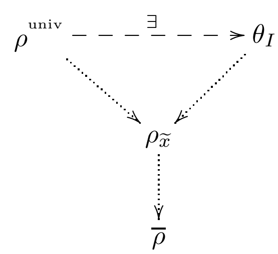
So it's enough to show two things.

(1) For some I we have a map $R \to \Lambda_I$ giving a deformation θ_I of the "integral lattice" version $\rho_{\widetilde{x}}$ of ρ_x. The image of Γ under ρ^{univ} is topologically finitely generated (since $\mathrm{GL}_N(R)$ is essentially pro-p and Γ satisfies the p-finiteness condition), so then there exists *some* I_0 such that θ factors through $\mathrm{GL}_N(\Lambda_{I_0})$, giving a map $\theta_{I_0} : \Gamma \to \mathrm{GL}_N(\Lambda_{I_0})$.

(2) The map from (1) is unique.

Indeed, by then comparing any two I and I' with a common one, we'd get the desired existence and *uniqueness* at the level of coefficients in A.

To prove (1), note that $\Lambda_I \in \widehat{\mathcal{C}}_\Lambda$ and θ_I deforms $\rho_{\widetilde{x}}$, and hence $\overline{\rho}$. Here is the picture:



The induced map $R \to \Lambda_I$ respects the map to Λ coming from the fact that $\rho_{\widetilde{x}}$ deforms $\overline{\rho}$, because if not, then we would have another map $R \to \Lambda_I \to \Lambda$, which contradicts the universal property of R.

To prove (2) just use the uniqueness from the universal property of (R, ρ^{univ}) for deforms on $\widehat{\mathcal{C}}_\Lambda$.