Math 396. Universal bundles and normal bundles

1. Overview

We begin with a striking example.

Example 1.1. Let M be a smooth 2-manifold. Do there exist finitely many smooth vector fields $\vec{v}_1, \dots, \vec{v}_r$ on M that span all tangent spaces (i.e., $\{\vec{v}_i(m)\}$ spans $\mathrm{T}_m(M)$ for all $m \in M$)? Equivalently, is there a C^∞ bundle surjection $M \times \mathbf{R}^r \to TM$ for some $r \geq 1$? (The constant sections $\underline{e}_i \in (M \times \mathbf{R}^r)(M)$ maps to the $\vec{v}_i \in (TM)(M) = \mathrm{Vec}_M(M)$.) In the case of compact M, it is easy to give an affirmative answer, as follows. Let $\{U_i\}$ and $\{U_i'\}$ be coverings by coordinate charts such that U_i has compact closure contained in U_i' and the U_i's cover M. By compactness of M, we may take these collections to be finite. Using coordinates $\{x_{i1}, \dots, x_{i,n_i}\}$ on U_i', the vector fields $\partial_{x_{ij}}$ on U_i' span all tangent spaces, and we can find a smooth bump function ϕ_i equal to 1 on U_i and supported in U_i'. Hence, each $\phi_i \partial_{x_{ij}}$ makes sense as a global smooth vector field and for each fixed i with j varying from 1 to n_i we get a spanning set for the tangent spaces over U_i. Thus, since the U_i's cover M, the entire collection gives an affirmative answer to the original question.

However, this proof uses no serious geometry and cannot give an explicit r. It also says nothing in case M is non-compact. Is compactness just an artifact of the method of proof? In fact, it can be proved that with no compactness assumption on the 2-manifold M, there exist 7 smooth vector fields that do the job: we can always take $r = 7$ for 2-manifolds. How can one prove such things? The key is to apply all of the techniques of our subject (especially the Whitney embedding theorem) to the tangent bundle TM, viewed as a manifold in its own right, and to make a serious study of the significance of Grassman manifolds. In this handout, we will develop the necessary techniques to prove the above result (as well a higher-dimensional version, and other generalizations). This illustrates the power of the theory of vector bundles to answer concrete questions whose formulation does not involve vector bundles.

We now turn to a general problem with vector bundles, the solution of which is vital to the application described above. For a positive integer c and a vector space V of dimension $d+1$ over $\mathbf{R}$, with $1 \leq c \leq d$, recall that the underlying set of the compact smooth Grassmann manifold $\mathbf{G}_c(V)$ is a "parameter space" for the set of codimension-c subspaces of V. Let $c' = d + 1 - c$, so the points of $\mathbf{G}_c(V)$ parameterize the set of c'-dimensional subspaces of V. In the case $c = 1$ (i.e., $c' = d$), this is the *projective space* $\mathbf{P}(V)$ whose points parameterize the set of hyperplanes in V. But are these *ad hoc* constructions? That is, how do we know there isn't some other reasonable way to put a smooth manifold structure on the set of subspaces of V of a fixed codimension? Put in more conceptual terms, is there a *universal property* satisfied by the Grassmannian $\mathbf{G}_c(V)$, which is to say a way to think about $\mathbf{G}_c(V)$ *independent* of its explicit construction?

What constitutes a "good" method of parameterizing a set by the set of points of a manifold anyway? The right answer to such questions was given by Grothendieck in the late 1950's, revolutionizing large chunks of mathematics in the process, and we will adapt his answer to our present circumstances. The first indication that there may be a deeper way to understand the true significance of $\mathbf{G}_c(V)$ is to reflect on the fact that to each point $x \in \mathbf{G}_c(V)$ we naturally associate a subspace $W_x \subseteq V$ of codimension c (i.e., of dimension $c' = d + 1 - c$), and as x varies this subspace W_x ranges through all codimension-c subspaces of V without repetition. In other words, we have a "family" of subspaces $\{W_x\}_{x \in \mathbf{G}_c(V)}$ of a fixed vector space V. This sounds suspicously like the data of a subbundle of rank c' inside of the trivial bundle $\mathbf{G}_c(V) \times V \to \mathbf{G}_c(V)$. Our aim in this handout is to make precise sense of this observation.

First, we shall construct a C^∞ subbundle $W \subseteq \mathbf{G}_c(V) \times V$ over $\mathbf{G}_c(V)$ with rank c' such that its x-fiber is exactly the c'-dimensional subspace $W_x \subseteq V$ that "corresponds" to the point x. Then we do something much more profound: we shall prove that for $0 \le p \le \infty$ this is the *universal* "C^p-varying family" of c'-dimensional subspaces of V. "Universal" means this: for $0 \le p \le \infty$, if M is *any* C^p premanifold with corners and $E \subseteq M \times V$ is *any* C^p subbundle of constant rank c' over M inside of the trivial bundle $M \times V \to M$ (so the fibers $E(m)$ for $m \in M$ are viewed as a C^p varying family of c'-dimensional subspaces of V, parameterized by the points of M), then there is a *unique* C^p map $f : M \to \mathbf{G}_c(V)$ such that under the canonical equality $f^*(\mathbf{G}_c(V) \times V) = M \times V$ of vector bundles over M the C^p subbundle f^*M is *equal* to $E \subseteq M \times V$. This represents a *vast* strengthening of the set-theoretic statement that the set of points of the Grassmannian parameterizes the set of codimension-c subspaces of V. Indeed, this earlier feature of the Grassmannian is exactly the special case of the above universal property upon taking M to be a 1-point space!

Example 1.2. In the case $c = 1$, we are studying *hyperplane bundles* (i.e., subbundles whose fibers are hyperplanes). Take $V = \mathbf{R}^{d+1}$ and $c = 1$. In this case, the above discussion says that over $\mathbf{P}(\mathbf{R}^{d+1}) = \mathbf{P}^d(\mathbf{R})$ there is a C^∞ hyperplane bundle $H \subseteq \mathbf{P}(\mathbf{R}^{d+1}) \times \mathbf{R}^{d+1}$ whose fiber over $x = [a_0, \dots, a_d]$ is the hyperplane $H_x = \{\sum a_j t_j = 0\}$ in $\mathbf{R}^{d+1}$. Moreover, if M is *any* C^p premanifold with corners and $E \subseteq M \times \mathbf{R}^{d+1}$ is *any* C^p hyperplane bundle then there is a unique C^p map $f : M \to \mathbf{P}^d(\mathbf{R})$ such that under the equality $f^*(\mathbf{P}^d(\mathbf{R}) \times \mathbf{R}^{d+1}) = M \times \mathbf{R}^{d+1}$ we have $f^*(E) = H$.

Explicitly, if $E(m) \subseteq \mathbf{R}^{d+1}$ is the hyperplane $\sum a_j t_j = 0$ then $f(m) = [a_0, \dots, a_n] \in \mathbf{P}^d(\mathbf{R})$. Thus, $\mathbf{P}^d(\mathbf{R})$ *equipped with its universal hyperplane bundle* $H \subseteq \mathbf{P}^d(\mathbf{R}) \times \mathbf{R}^{d+1}$ is the universal smoothly varying family of hyperplanes in $\mathbf{R}^{d+1}$. In particular, we should *not* think of $\mathbf{P}^d(\mathbf{R})$ as a "bare" compact manifold, but rather as equipped with the data of the hyperplane bundle H. It is this *extra structure* over $\mathbf{P}^d(\mathbf{R})$ that explains its real importance in differential geometry.

Example 1.3. The preceding example (taking $V = \mathbf{R}^{d+1}$) carries over *verbatim* with any $1 \le c \le d$, except we replace "hyperplane bundle" with "codimension-c subbundle" and $\mathbf{P}^d(\mathbf{R})$ with $\mathbf{G}_c(d) = \mathbf{G}_c(\mathbf{R}^{d+1})$ throughout. Working with $V = \mathbf{R}^{d+1}$ makes the situation feel more concrete, but since varying subspaces have no canonical basis it tends to keep the structure clearer if we avoid a choice of basis of V in the statements of results (though using a basis in the middle of a proof is a reasonable thing to do). If we use $V = \mathbf{R}^{d+1}$ throughout, it may become too tempting to try to describe everything in terms of the standard coordinates, thereby leading to lots of big matrices.

Is there a down-to-earth illustration of why such a universal property gives something interesting? As we will see later in this handout, in conjunction with the Whitney embedding theorem it yields the following remarkable consequence: for any $r \ge 1$ and $n \ge 1$, *any* smooth vector bundle with constant rank r over *any* smooth (perhaps *non-compact*) manifold with constant dimension n is C^∞-isomorphic to a pullback (by many smooth maps) of the universal subbundle over the compact manifold $\mathbf{G}_{2n+r+1}(\mathbf{R}^{2n+2r+1})$. Roughly speaking, this says that in the general theory of vector bundles over manifolds (especially classification problems), the universal subbundles over the compact Grassmannians play a very distinguished role. This is the starting point of the theory of characteristic classes, which constitutes the fundamental topological technique for studying vector bundles on manifolds. (See the book "Characteristic classes" by Milnor and Stasheff for more on this story; it assumes some knowledge of algebraic topology.)

Example 1.4. Let M be a smooth manifold with constant dimension n, and $E = TM$. This bundle has constant rank n, and so its dual bundle $E^\vee$ also has rank n. The preceding paragraph applied to $E^\vee$ asserts that $E^\vee$ can be realized as a C^∞ subbundle of the trivial bundle $M \times \mathbf{R}^{4n+1}$ over M. Dualizing this subbundle inclusion and using double duality, we express $E \simeq E^{\vee\vee}$ as a bundle quotient of $(M \times \mathbf{R}^{4n+1})^\vee \simeq M \times (\mathbf{R}^{4n+1})^\vee \simeq M \times \mathbf{R}^{4n+1}$. Thus, the images $s_1, \dots, s_{4n+1} \in E(M) = \mathrm{Vec}_M(M)$ of the frame of constant sections $\underline{e}_1, \dots, \underline{e}_{4n+1}$ of $M \times \mathbf{R}^{4n+1} \to M$ have the property that $\{s_i(m)\}$ spans $E(m) = \mathrm{T}_m(M)$ for all $m \in M$. That is, we have built $4n+1$ smooth vector fields on M that span all tangent spaces of M. For $n = 2$, we have $4n + 1 = 7$.

Remark 1.5. It must be emphasized that by far the deepest input in the preceding results is the Whitney embedding theorem in the non-compact case (it is applied to the total space of vector bundles, which are never compact except possibly when bundle has rank 0). The course text proves Whitney's theorem only in the compact case, so you will have to look elsewhere to see how to handle the general case.

After we prove the universal property of Grassmannians, we will require one further ingredient before we can explain the preceding remarkable assertion relating general vector bundles to those over Grassmannians. It will be necessary to make a digression to study the concept of *normal bundles* along submanifolds (roughly, the space of directions pointing "away" from a submanifold, taken modulo directions along the manifold). Normal bundles are certain quotient bundles that are an important tool in their own right, as will be seen later in the course, but for our present purposes they play a crucial role in proving the above ubiquitous nature of the universal subbundles over Grassmannians in the general theory of vector bundles over arbitrary manifolds. The geometric significance of normal bundles emerges when one studies the geometry of how submanifolds of a fixed ambient manifold interact with each other.

2. Universal bundles over Grassmannians

We now undertake two goals: to "glue" the c'-dimensional subspaces $W_x \subseteq V$ for $x \in \mathbf{G}_c(V)$ into a rank-c' subbundle W of the trivial bundle $\mathbf{G}_c(V) \times V \to \mathbf{G}_c(V)$, and to prove that the resulting data is the *universal* pair (E, M) consisting of a rank-c' subbundle $E \subseteq M \times V$ over a smooth manifold M (i.e., all such pairs are obtained from the one over the Grassmannian $\mathbf{G}_c(V)$ via pullback along a *uniquely determined* smooth map to the Grassmannian). We begin with the construction over the Grassmannian:

Theorem 2.1. *Fix $1 \leq c \leq d$ and let $c' = d + 1 - c = \dim V - c$. Let $G = \mathbf{G}_c(V)$, and for each $x \in G$ let $W_x \subseteq V$ be the corresponding c'-dimesional subspace of V. There is a unique rank-c' subbundle $W \subseteq G \times V$ over G whose x-fiber is $W_x \subseteq V$ for every $x \in G$.*

Proof. The uniqueness is clear: if W and W' are two such subbundles then $W(x) = W'(x)$ inside of V for all x and so as embedded smooth submanifolds of $G \times V$ we have $W = W'$ set-theoretically and thus as subbundles (see Lemma 2.1 in the handout on subbundles and quotient bundles). Our problem is therefore one of existence. By Lemma 2.1 in the handout on subbundles and quotient bundles, if W is to exist then it must be a smooth closed submanifold of $G \times V$ and its underlying set has to be the union of the subsets $W_x \subseteq (G \times V)(x) = V$ for all $x \in G$. We call this subset W, so $W \subseteq G \times V$ has x-fiber equal to the c'-dimensional subspace of $(G \times V)(x) = V$ for all $x \in G$. Hence, by Theorem 2.5 in the handout on subbundles and quotient bundles it is necessary and sufficient to prove that the subset $W \subseteq G \times V$ is a closed C^∞ submanifold.

This problem is local over G (why?), so it suffices to find an open covering of G by open subsets U such that the part of W lying over U (i.e., the union of the W_x's for $x \in U$) is a closed C^∞ submanifold in the open locus $U \times V$ in $G \times V$ over U. Let $\{v_0, \dots, v_d\}$ be an ordered basis of V, and for each ordered c-tuple $I = \{i_1 < \cdots < i_c\}$ in $\{0, \dots, d\}$ let $U_I \subseteq G$ be the corresponding standard open subset that is (as smooth manifold) a Euclidean space $\mathbf{R}^{I \times I'}$ with $I' = \{0, \dots, d\} - I$. Recall that the points of U_I are exactly the c'-dimensional subspaces $W_x \subseteq V$ such that $v_{i_1}, \dots, v_{i_c}$ represent a basis of the c-dimensional quotient V/W_x. Explicitly, the coordinates of $x \in U_I$ are the unique tuple $(a_{ii'}(x))_{(i,i')\in I\times I'} \in \mathbf{R}^{I\times I'}$ such that W_x has as basis the c' vectors

$$(1)\qquad v_{i'} - \sum_{i\in I} a_{ii'}(x)v_i = v_{i'} - \sum_{j=1}^{c} a_{i_j,i'}(x)v_{i_j}$$

for $i' \in I'$. As we vary x through U_I, the point $(a_{ii'}(x)) \in \mathbf{R}^{I\times I'}$ varies.

What is the subset $W_I = \cup_{x\in U_I} W_x$ inside of $U_I \times V = \mathbf{R}^{I\times I'} \times V \simeq \mathbf{R}^{I\times I'} \times \mathbf{R}^{d+1}$ (the final step using the chosen ordered basis $\{v_0, \dots, v_d\}$ of V)? In other words, given a point

$$((a_{ii'}), (b_0, \dots, b_d)) \in \mathbf{R}^{I\times I'} \times \mathbf{R}^{d+1},$$

what is the condition that $\sum b_j v_j \in V$ lies in W_x? That is, under what condition on the b_j's (in terms of the $a_{ii'}$'s) does it happen that $\sum b_j v_j$ is in the span of the linearly independent vectors in (1)? Rather than write out the equations, we use the explicit fibral basis as a guide for what to do: consider the mapping

$$U_I \times \mathbf{R}^{I'} \to U_I \times V$$

defined by

$$\begin{aligned} ((a_{ii'}), (b_{i'})) \quad &\mapsto \quad ((a_{ii'}), \sum_{i'\in I'} b_{i'}(v_{i'} - \sum_{i\in I} a_{ii'}v_i)) \\ &= \quad ((a_{ii'}), \sum_{i'\in I'} b_{i'}v_{i'} - \sum_{i\in I}(\sum_{i'\in I'} a_{ii'}b_{i'})v_i). \end{aligned}$$

This is visibly a smooth mapping of trivial bundles over U_I, and on fibers over $x \in U_I$ it is a linear map $\mathbf{R}^{I'} \to V$ that is an injection *onto* the subspace with basis (1); that is, the fibral image is the subspace $W_x \subseteq V$!

Aha, so we have built a subbundle $U_I \times \mathbf{R}^{I'}$ inside of $U_I \times V$ whose x-fiber is $W_x \subseteq V$ for all $x \in U_I$. In particular, its image is a closed smooth submanifold (see Lemma 2.1 in the handout on subbundles and quotient bundles), and this image is exactly the subset W_I that we needed to prove is a closed smooth submanifold of $U_I \times V$. ∎

Now we consider general pairs (E, M) with M a C^p premanifold with corners and E a rank-c' C^p subbundle of $M \times V$. The preceding theorem gives a specific such pair $(W, \mathbf{G}_c(V))$ over $M = \mathbf{G}_c(V)$ (viewed in the C^p sense). This is a very special construction because it is the universal one; that is, all other pairs are obtained from it by "unique pullback":

Theorem 2.2. *Let M be a C^p premanifold with corners, $0 \le p \le \infty$, and let $E \subseteq M \times V$ be a C^p subbundle of rank $c' = \dim V - c$. There is a unique C^p mapping $f : M \to \mathbf{G}_c(V)$ such that inside of $f^*(\mathbf{G}_c(V) \times V) = M \times V$ we have $f^*(W) = E$, with $W \subseteq \mathbf{G}_c(V) \times V$ the rank-c' subbundle constructed above.*

Before we prove Theorem 2.2, we emphasize that this theorem gives a property of $(W, \mathbf{G}_c(V))$ (viewed in the C^p sense) with intrinsic meaning *regardless* of the method used to construct the pair in the first place. That is, if tomorrow somebody shows you some other pair (W', G') that "C^p parameterizes" families of subspaces of dimension c' inside of V in a *universal manner* (which is to say, satisfying the same property as indicated for $(W, \mathbf{G}_c(V))$ in the above theorem), their output will have to be *uniquely C^p isomorphic* to $(W, \mathbf{G}_c(V))$. That is, there is a unique C^p isomorphism $f : G' \simeq \mathscr{G}_c(V)$ such that $f^*(W) = W'$ inside of $G' \times V$. We are therefore justified in calling W the *universal subbundle* over $\mathbf{G}_c(V)$; it is definitely to always be thought of as a subbundle of $\mathbf{G}_c(V) \times V$ (rather than as just a "bare" vector bundle over $\mathbf{G}_c(V)$), since the subbundle aspect is what makes it interesting.

Proof. Let us first show that the map f has only one possibility in the set-theoretic sense. If there is to be a C^p map $f : M \to \mathbf{G}_c(V)$ such that $f^*(W) = E$ inside of $f^*(\mathbf{G}_c(V) \times V) = M \times V$, then on fibers over $m \in M$ we must have $W_{f(m)} = E(m)$ as c'-dimensional subspaces of V. That is, $f(m) \in \mathbf{G}_c(V)$ *must* be the *unique* point for which the corresponding c'-dimensional subspace $W_{f(m)} \subseteq V$ is exactly $E(m)$. Note that here we are using the old "set-theoretic parameterization" property of the points of $\mathbf{G}_c(V)$, namely that the points of the Grassmannian are in bijection with the set of c'-dimensional subspaces of V (and the bundle W encodes this via its fibers over the Grassmannian).

Hence, we now proceed in reverse: we *define* the map of sets $f : M \to \mathbf{G}_c(V)$ by requiring that $f(m) \in \mathbf{G}_c(V)$ is the unique point such that the c'-dimensional subspace $W_{f(m)} \subseteq V$ is $E(m) \subseteq V$. The problem is to prove that f is a C^p mapping. Note that this problem is *local on* M, since we have already constructed the unique possible f globally! Hence, it suffices to study the situation over opens that cover M.

Since there is a covering of M by opens over which E is trivial, by working on such opens we reduce the apparently overwhelming generality of our task to the special case when the subbundle $E \to M$ inside of $M \times V$ is trivial as a vector bundle. It will be necessary to make some further refinements on M, and the reader should observe how the argument below gradually "rediscovers" the manifold structure on the Grassmannian.

Pick an ordered basis $\{v_0, \dots, v_d\}$ of V and let $\{s_1, \dots, s_{c'}\}$ be a trivializing frame for E. The inclusion $E \hookrightarrow M \times V$ carries s_j to some C^p section of $M \times V \to M$, say

$$s_j \mapsto \sum_i h_{ij} \underline{v}_i$$

with $\underline{v}_i : M \to M \times V$ the constant section $m \mapsto (m, v_i)$ and $h_{ij} \in C^\infty(M)$. In other words, for all $m \in M$ the subspace $E(m) \subseteq V$ has as basis the vectors

$$s_j(m) = \sum_i h_{ij}(m) \cdot \underline{v}_i(m) = \sum_i h_{ij}(m) \cdot v_i \in V.$$

The linear independence of the $s_j(m)$'s implies that the matrix $(h_{ij}(m))$ with $d + 1 = \dim V$ rows and c' columns has linearly independent columns. That is, the column rank is c'. Hence, the row rank is also c', so there exist c' linearly independent rows. Upon picking such rows, we get a $c' \times c'$ submatrix with independent rows and hence it is invertible. That is, for each $m \in M$ there is some non-vanishing determinant $\det(h_{j'j}(m))_{j' \in J', 1 \leq j \leq c'}$ with $J' \subseteq \{0, \dots, d\}$ a subset of size c'.

For each subset $J' \subseteq \{0, \dots, d\}$ with size c', let $M_{J'} \subseteq M$ be the *open* subset over which the determinant $\det(h_{j'j})_{j' \in J', 1 \leq j \leq c'}$ is non-vanishing. The preceding argument shows that as J' varies over its finitely many possibilities, the associated open sets $M_{J'}$ (some of which may well be empty) *cover* M.

Remark 2.3. We digress to note that we are beginning to see shadows of the construction of the Grassmannian. With respect to the chosen ordered basis of V the subset $M_{J'}$ is the set of $m \in M$ such that in the c-dimensional quotient $V/E(m)$ the c vectors v_j for $j \in J = \{0, \dots, d\} - J'$ represent a basis (as the problem of expressing the $v_{j'}$'s for $j' \in J'$ as linear combinations of these v_j's modulo the span $E(m)$ of the independent $s_1(m), \dots, s_{c'}(m)$ is *exactly* the problem of inverting the $c' \times c'$ submatrix of coefficients for $s_1(m), \dots, s_{c'}(m)$ along the $v_{j'}$'s for $j' \in J'$). Thus, $M_{J'} = f^{-1}(U_J)$ where $U_J \simeq \mathbf{R}^{J \times J'} \subseteq \mathbf{G}_c(V)$ is the standard Euclidean chart associated to J.

Since our main problem is local on M, we therefore lose no generality in working on the $M_{J'}$'s separately. That is, we may assume $M = M_{J'}$ for some fixed subset $J' \subseteq \{0, \dots, d\}$ of size c'. Letting J be the complement of J' with size c, Remark 2.3 shows that condition $M = M_{J'}$ is exactly the condition that the set-theoretic map $f : M \to \mathbf{G}_c(V)$ whose smoothness we wish to prove has image contained in the open subset U_J. It is therefore equivalent to prove that the set-theoretic map $f : M \to U_J$ is C^p. Aha, but $U_J = \mathbf{R}^{J \times J'}$ has global C^p (even C^∞) coordinates! Hence, to check that the map f to U_J is C^p all we have to do is to check that the component functions of f are C^p. We will not compute these functions explicitly, as an inverse matrix intervenes, but we will describe them in terms of some matrix operations from which the desired C^p property will drop out (essentially because of the universal Cramer formula for how to invert an invertible matrix).

Consider the problem of solving the system of linear equations

$$v_{j'} = \sum_{j \in J} b_{jj'} v_j + \sum_{k=1}^{c'} \beta_{kj'} s_k(m) \tag{2}$$

for all $j' \in J'$ with $m \in M$ fixed. Since $W_{f(m)} = E(m)$ is the subspace with basis given by the $s_k(m)$'s, the point $(b_{jj'}) \in \mathbf{R}^{J \times J'}$ is *exactly* the tuple of coordinates of $f(m) \in U_J$. Thus, we know that the $b_{jj'}$'s exist and are uniquely determined (so the $\beta_{kj'}$'s are as well, since the $s_k(m)$'s are a basis of $E(m)$). The numbers $b_{jj'} = b_{jj'}(m) \in \mathbf{R}$ as functions of $m \in M$ are therefore the component functions of the set-theoretic map $f : M \to U_J = \mathbf{R}^{J \times J'}$. Hence, our problem is precisely to prove that $m \mapsto b_{jj'}(m)$ is C^p for each $(j, j') \in J \times J'$.

Look at the coefficient of v_j on each side of (2). Since $j' \in J'$ and $j \in J$ with $J \cap J' = \emptyset$, there is no v_j on the left side. Hence, when we expand the $s_k(m)$'s in terms of the basis $\{v_0, \dots, v_d\}$ of V then the total coefficient of v_j on the right side of (2) must *vanish*. That is,

$$b_{jj'} = -\sum_{k=1}^{c'} \beta_{kj'} h_{jk}$$

as functions on M. The h's are C^p functions, so our problem is to show $\beta_{kj'}$ is C^p for all $1 \le k \le c'$ and each $j' \in J'$. Now we fix $i \in J'$ and compare coefficients of v_i on both sides of (2): the coefficient on the left is $\delta_{ij'}$, and since J is disjoint from J' the coefficient on the right is $\sum_{k=1}^{c'} \beta_{kj'} h_{ik}$. In other words, if we let i, j' vary through the set of indices J' of size c' and we let k vary from 1 to c', then we get the matrix equation $(h_{ik})(\beta_{kj'}) = \mathrm{id}_{c' \times c'}$. Since $M = M_{J'}$, the $c' \times c'$ submatrix (h_{ik}) built from the rows with $i \in J'$ is *invertible*, and so our matrix equation can indeed be uniquely solved for the $\beta_{kj'}$'s (with $j' \in J'$) in terms of the h_{ik}'s (with $i \in J'$) via Cramer's formula for inverting a matrix. In particular, we get a formula for $\beta_{kj'}$ as a rational function in the h_{ik}'s (for $1 \le k \le c'$ and $i \in J'$) with non-vanishing denominator. Hence, the C^p property of the h's gives the same for the β's as functions on M. ■

Let us summarize what we have proved: for any C^p premanifold with corners M $(0 \le p \le \infty)$ and any C^p subbundle $E \subseteq M \times V$ with rank $c' = d + 1 - c = \dim V - c$, the set-theoretic map $f : M \to \mathbf{G}_c(V)$ sending $m \in M$ to the point $f(m) \in \mathbf{G}_c(V)$ that "classifies" the codimension-c subspace $E(m) \subseteq (M \times V)(m) = V$ is a C^p *mapping*, and moreover $f^*(W) = E$ inside of $M \times V$ (as this may be checked on fibers over M, where it follows from the definition of f). The real content is that f is a C^p mapping and not merely set-theoretic.

3. Normal bundles

Our main goal is to show how to use Whitney's embedding theorem for smooth manifolds to prove the following remarkable result (also see Corollary 3.12 for a wonderful variant):

Theorem 3.1. *If X is a C^∞ manifold with constant dimension $n \ge 1$ and $E \to X$ is a C^∞ vector bundle with constant rank $r \ge 1$, then there exists a smooth mapping $f : X \to \mathbf{G}_{2n+r+1}(\mathbf{R}^{2n+2r+1})$*

such that the pullback $f^*(W)$ of the universal rank-r subbundle over the Grassmannian is isomorphic to E as a C^∞ vector bundle over X.

We emphasis that f in this theorem is far from unique. The uniqueness aspect of the universal mapping property of Grassmannians is not inconsistent with this, as E is not canonically presented as a subbundle of anything over X. The crux of the proof will be to show the fact (not at all obvious!) that the rank-r bundle E is *necessarily* a C^∞ subbundle of $X \times \mathbf{R}^{2n+2r+1}$. The proof will exhibit E as such a subbundle in zillions of different ways, and so that is why we will have very little control over what the map f actually is (since it will be determined by the mechanism by which we exhibit E as a subbundle of $X \times \mathbf{R}^{2n+2r+1}$). With some stronger techniques in topology, one can show that all choices of f may be related to each other in a manageable way upon passing to a so-called "infinite Grassmannian" (that we will not discuss here). In this sense, the lack of control over f turns out to not be a hindrance in applications of Theorem 3.1.

The proof of Theorem 3.1 requires a new and very important concept, that of normal bundles. The notion of normal bundle arises whenever one has a submanifold. An important example for our purposes is this: the zero section $0 : X \to E$ identifies X is a closed smooth submanifold of E. Note how here we are really viewing E as a smooth manifold in its own right; that is, the geometry of the total space of E is what matters (and not just its "vertical structure" as a vector bundle over X).

Definition 3.2. Let $i : Y \hookrightarrow X$ be a C^p embedding of C^p manifolds with $p \geq 1$, so $\mathrm{d}i : TY \to TX$ induces a C^{p-1} subbundle $TY \to i^*(TX)$ over Y. The C^{p-1} quotient bundle $\mathrm{N}_{Y/X} = (i^*(TX))/TY$ over Y is the *normal bundle* to Y in X; its y-fiber is $\mathrm{T}_{i(y)}(X)/\mathrm{T}_y(Y)$, the "space of directions in X at y taken modulo the directions along Y."

Note that the normal bundle lives on the submanifold.

Remark 3.3. Roughly speaking, the normal bundle encodes the global twistedness of Y inside of X: it tells how to move "away from Y" within the space X, at least infinitesimally near Y (through tangential information). Note that if $U \subseteq X$ is an open set containing Y, then $\mathrm{N}_{Y/U} = \mathrm{N}_{Y/X}$.

Before we study normal bundles in general, let's look at some interesting examples with hypersurfaces and transverse intersections in an inner product space (such as $\mathbf{R}^n$ with its standard inner product, the case $n = 3$ being a perfectly interesting one). First, we explain how to relate normal bundles to "orthogonal complements".

Example 3.4. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space with dimension $n > 1$. For each $x \in V$ there is a canonical linear isomorphism $j_x : V \simeq \mathrm{T}_x(V)$ given by $v \mapsto D_{x,v}$, and so via j_x we may transfer the inner product on V to an inner product $\langle \cdot, \cdot \rangle_x$ on $\mathrm{T}_x(V)$ (that is, for $\vec{v}, \vec{w} \in \mathrm{T}_x(V)$ we define $\langle \vec{v}, \vec{w} \rangle_x = \langle j_x^{-1}(\vec{v}), j_x^{-1}(\vec{w}) \rangle$). Concretely, if $\{v_i\}$ is an *orthonormal* basis of V and $\{t_i\}$ is the dual system of linear coordinates then $\partial_{t_j}|_x = D_{x,v_i} = j_x(v_i)$, so $\{\partial_{t_j}|_x\}$ is an orthonormal basis of $\mathrm{T}_x(V)$ with respect to $\langle \cdot, \cdot \rangle_x$.

We let $i : M \hookrightarrow V$ be an embedded C^p submanifold of V, $1 \leq p \leq \infty$, so TM is a C^{p-1} subbundle of the C^{p-1} pullback bundle $i^*(TV)$, with quotient equal to $\mathrm{N}_{M/V}$ *by definition.* But there is a more appealing way to visualize this normal bundle: it is an orthogonal complement to TM in $i^*(TV)$. More specifically, for each $m \in M$ let $T_m(M)^\perp$ be the orthogonal complement to $\mathrm{T}_m(M)$ in $\mathrm{T}_{i(m)}(V)$ with respect to $\langle \cdot, \cdot \rangle_{i(m)}$. I claim that these $T_m(M)^\perp$'s fit together into the fibers of a C^{p-1} subbundle $(TM)^\perp$ in $i^*(TV)$, so the natural bundle map $TM \oplus (TM)^\perp \to i^*(TV)$ is an isomorphism (as it is on fibers) and the composite $(TM)^\perp \to i^*(TV) \to \mathrm{N}_{M/V}$ is an isomorphism (as again we may check on fibers: $\mathrm{T}_m(M)^\perp \to \mathrm{T}_{i(m)}(V)/\mathrm{T}_m(M)$ is an isomorphism, as always for orthogonal complements to a subspace of an inner produce space). That is, the bundle $(TM)^\perp$ of "directions orthogonal to M in V along M" is identified with the normal bundle. This is the reason for the name "normal bundle". In the course text, $(TM)^{\perp}$ is taken as the *definition* of the normal bundle. There is no doubt that the identification of the normal bundle in our sense with an orthogonal bundle is one of the fundamental reasons for interest in normal bundles in differential geometry, but it is regretable to impose the data of inner products in the definition of normal bundles because we have seen that the notion of normal bundle makes sense *without* any inner products on tangent spaces. Indeed, the normal bundle is intrinsic to the geometry of how one manifold sits in another. The inner products are *extra structure* on the ambient manifold, and if we change this extra structure then the normal bundle really does not change (a fact that is hard to "see" if one does not give the general definition as we have done).

It remains to show that the orthogonal complements $T_m(M)^{\perp}$ really do form the fibers of a subbundle of $i^*(TV)$ over M. Recall that $TV \to V$ is naturally trivialized (via "constant vector fields") as $TV \simeq V \times V$; on fibers over $x \in V$, it is the map $\mathrm{T}_x(V) \simeq V$ inverse to $j_x : V \simeq \mathrm{T}_x(V)$. Thus, we get $i^*(TV) \simeq M \times V$ (on fibers over $m \in M$, this is the natural identification $\mathrm{T}_{i(m)}(V) \simeq V$). Hence, upon viewing M as a C^{p-1} manifold and renaming $p-1 \geq 0$ as p, our problem is to prove:

Lemma 3.5. *Choose $p \geq 0$ and a C^p submanifold M in V, as well as a C^p subbundle E in $M \times V$. The orthogonal complements $E(m)^{\perp} \subseteq V$ with respect to the inner product on V form the fibers of a C^p subbundle $E^{\perp}$ of $M \times V$.*

As we shall see, this lemma ultimately comes down to a "universal procedure" in linear algebra: the Gramm-Schmidt process.

Proof. We may assume M is connected, or more specifically that E has constant rank r. Let $N = \dim V$, and we may assume $1 \leq r < N$ (since the cases $r = 0$ and $N = r$ are trivial). The dimension of $E(m)^{\perp}$ is equal to the positive constant $N - r$ for all $m \in M$. By the criterion in Theorem 2.5 in the handout on subbundles and quotient bundles, it therefore suffices to prove that the union $E^{\perp}$ of the $E(m)^{\perp}$'s is a closed C^p submanifold of $M \times V$. This problem is local over M, so we may assume E is trivial, say with trivializing sections $s_1, \dots, s_r \in E(M)$. Since E is a subbundle of $M \times V$, by working locally over M we may even suppose that the collection $\{s_j\}$ extends to a trivializing frame $\{s_1, \dots, s_N\}$ of the ambient bundle $M \times V$. (See Corollary 2.4 in the handout on subbundles and quotient bundles.) For each $m \in M$, how do we compute $E(m)^{\perp}$ in terms of the $s_1(m), \dots, s_N(m)$?

For $r < j \leq N$ we let $s'_j(m)$ be the orthogonal projection of $s_j(m)$ onto $E(m)^{\perp}$. Fix an orthonormal basis $\{e_i\}$ of V. By smoothness of the s_j's, for $1 \leq j \leq N$ we have $s_j = \sum h_{ij} \underline{e}_i$ with $h_{ij} \in C^{\infty}(M)$ and $\underline{e}_i \in (M \times V)(X)$ the constant section on the basis vector e_i of V. Since $\{\underline{e}_i(m) = e_i\}$ is an orthonormal basis of V for all $m \in M$ and $\{s_1(m), \dots, s_r(m)\}$ is a basis of $E(m)$, there is a "universal formula" for the basis $s'_{r+1}(m), \dots, s'_N(m)$ of $E(m)^{\perp}$, as follows. We first apply the "universal procedure" of Gramm-Schmidt to the ordered basis $\{s_1(m), \dots, s_r(m)\}$ of $E(m)$ to get an orthonormal basis $\{s'_1(m), \dots, s'_r(m)\}$ for $E(m) \subseteq V$; the coefficients of $s'_1, \dots, s'_r$ with respect to the fiberwise-orthonormal frame $\underline{e}_1, \dots, \underline{e}_N$ are certain universal "smooth" expressions in the C^p functions h_{ij} (involving square roots of positive quantities, and so forth), so $s'_1, \dots, s'_r$ are C^p sections of $M \times V \to M$. Now for $r < j \leq N$ we define

$$s'_j(m) = s_j(m) - \sum_{i=1}^{r} \langle s'_i(m), s_j(m) \rangle s'_i(m),$$

and so clearly $s'_{r+1}, \dots, s'_N$ are all C^p sections as well. These are fiberwise independent, so they define a C^p subbundle of $M \times V$, and this is exactly $E^{\perp}$ as a subset of $M \times V$. Hence, $E^{\perp}$ is a closed C^p submanifold, as required to infer from these local considerations that the original global $E^{\perp}$ is a C^p subbundle of $M \times V$. ■

Example 3.6. Let us now apply Example 3.4 to the study of normal bundles along hypersurfaces in the inner product space V. Let $U \subseteq V$ be an open set, and $f : U \to \mathbf{R}$ a C^p function with $p \geq 1$. Assume for some $c \in \mathbf{R}$ that f has no critical points on $H = f^{-1}(c)$, so $i : H \hookrightarrow V$ is an embedded C^p submanifold (closed in U). In this case, I claim that the C^{p-1} normal bundle $\mathrm{N}_{H/V} = \mathrm{N}_{H/U}$ on H is trivial, essentially due to a "normal gradient vector field".

To make this precise, for each $x \in U$ let $(\nabla f)(x) \in \mathrm{T}_x(U)$ be the unique vector such that $\langle (\nabla f)(x), \cdot \rangle_x = (\mathrm{d}f)(x)$ as functionals on $\mathrm{T}_x(U)$, where $\langle \cdot, \cdot \rangle_x$ is the inner product on $\mathrm{T}_x(U)$ arising from the isomorphism $V \simeq \mathrm{T}_x(V) = \mathrm{T}_x(U)$. For all $h \in H$ the functional $\mathrm{d}f(i(h)))$ on $\mathrm{T}_{i(h)}(V) \simeq V$ is nonzero (no critical points on H) and has kernel that is the hyperplane $\mathrm{T}_{i(h)}(H)$, so $(\nabla f)(i(h)) \in \mathrm{T}_{i(h)}(H)$ is a nonzero vector in the line $\mathrm{T}_{i(h)}(H)^{\perp}$. That is, $h \mapsto (\nabla f)(i(h))$ is a non-vanishing set-theoretic section of the orthogonal line bundle $(TH)^{\perp}$ relative to the hyperplane bundle TH in $i^*(TV)$. As we saw in the discussion preceding Lemma 3.5, $(TH)^{\perp}$ is naturally isomorphic to $\mathrm{N}_{H/V}$.

I claim that the *gradient field* $\nabla f : x \mapsto (\nabla f)(x)$ is a C^{p-1} section of $TU = (TV)|_U$, so the resulting C^{p-1} pullback section $i^*(\nabla f)$ of $i^*(TV)$ over H lies in the line subbundle $(TH)^{\perp}$ and is nowhere-vanishing (as we see on fibers over H: $(\nabla f)(i(h)) \in \mathrm{T}_h(H)^{\perp}$ in $\mathrm{T}_{i(h)}(V)$ is nonzero for all $h \in H$). Hence, it *trivializes* the line bundle $(TH)^{\perp}$. We conclude that, granting the C^{p-1} property of the gradient field, for such a hypersurface H defined as a level set without critical points the normal bundle $\mathrm{N}_{H/V}$ on H is *trivial.*

To prove that ∇f is C^{p-1} on U, we just compute in local coordinates: if $\{e_i\}$ is an orthonormal basis of V and $\{t_i\}$ is the dual coordinate system on V then $(\nabla f)(x) = \sum (\partial_{t_j} f)(x) \partial_{t_j}|_x$ in $\mathrm{T}_x(U) = \mathrm{T}_x(V)$ for all $x \in U$ because forming the inner product of this sum against $\sum a_j \partial_{t_j}|_x$ gives the output

$$\sum (\partial_{t_j} f)(x) a_j = \sum (\mathrm{d}f)(x)(\partial_{t_j}|_x) \cdot a_j = (\mathrm{d}f)(x)(\sum_j a_j \partial_{t_j}|_x). \tag{3}$$

(Here we have used that $\{\partial_{t_j}|_x\}$ is an orthonormal basis of $\mathrm{T}_x(V)$ with respect to $\langle \cdot, \cdot \rangle_x$, as $\partial_{t_j}|_x = D_{x,e_j}$ with $\{e_j\}$ an orthonormal basis of V.) The equation (3) says that inner product against $\sum (\partial_{t_j} f)(x) \partial_{t_j}|_x$ in $T_x(V)$ via $\langle \cdot, \cdot \rangle_x$ is evaluation of the functional $(\mathrm{d}f)(x)$, as required in the definition of the gradient vector $(\nabla f)(x)$. From the explicit formula we have just derived for the gradient, its coefficient functions with respect to the tangent space basis of ∂_{t_j}'s is the collection of functions $\partial_{t_j} f$ that are C^{p-1}. (Warning: this formula for the gradient vector rests *crucially* on the fact that the t_j's are dual to an *orthonormal* basis of V, so that $\{\partial_{t_j}|_x\}$ is an orthonormal basis of $\mathrm{T}_x(V)$ with respect to $\langle \cdot, \cdot \rangle_x$ for all $x \in V$.)

Example 3.7. In the preceding example with gradients, we have used the crutch of f to exhibit a continuous (even C^{p-1}) non-vanishing normal vector field along H in V, and as a C^{p-1} section of $i^*(TV)$ over H it gives a fiberwise basis in $\mathrm{T}_{i(h)}(V)/\mathrm{T}_h(H) = \mathrm{N}_{H/V}(h)$. For example, consider the special case $H = S^n$ inside $V = \mathbf{R}^{n+1}$ (with the standard inner product), with $f = \sum x_j^2$ (so $H = f^{-1}(1)$). Pick $x = (b_0, \dots, b_n) \in S^n$, so since the standard coordinates $\{x_i\}$ on V are dual to the standard basis $\{e_i\}$ that is an orthonormal basis,

$$(\nabla f)(x) = \sum_i (\partial_{x_i} f)(x) e_i = \sum_i 2b_i e_i = (2b_0, \dots, 2b_n)$$

in $\mathbf{R}^{n+1} \simeq \mathrm{T}_x(\mathbf{R}^{n+1})$. Geometrically, $(\nabla f)(x)$ is the "outward" normal vector to S^n at x with length 2. (The outwardness corresponds to the fact that f increases as we move outward from the sphere: the gradient points in the direction of maximal local increase in f.)

Example 3.8. Triviality of the normal bundle to a hypersurface $i : H \hookrightarrow \mathbf{R}^{n+1}$ is *equivalent* to the existence of a nowhere-vanishing C^{p-1} normal vector field. Indeed, we have seen in Example 3.4 that $\mathrm{N}_{H/\mathbf{R}^{n+1}}$ is isomorphic to the orthogonal bundle $(TH)^{\perp}$ in $i^*(T(\mathbf{R}^{n+1}))$, and since triviality of a line bundle is equivalent to the existence of a nowhere-vanishing global section it follows that triviality of the normal bundle is equivalent to the existence of a nowhere-vanishing global section of $(TH)^{\perp} \to H$. But such a section is precisely the data of a C^{p-1}-varying non-vanishing normal field along H in $\mathbf{R}^{n+1}$. A short calculation shows that formation of the length of the vectors in the non-vanishing normal field is a positive C^{p-1} function on H, so dividing by this gives a C^{p-1} *unit normal* field along H. Note that the specification of a unit normal field gives a globally consistent sense of "direction" away from the hypersurface, and in particular suggests a notion of 2-sidedness (motion in the direction of the unit normal field or in the opposite direction).

But there are surfaces in $\mathbf{R}^3$ such as the Möbius strip that seem to be one-sided, and so should not admit C^{p-1} (or even continuous) unit normal fields. That is, we expect that surfaces such as the Möbius strip will give examples of embedded smooth surfaces in $\mathbf{R}^3$ for which the normal bundle should be *non-trivial* (even topologically). In particular, we arrive at the rather interesting conclusion that in no open subset of $\mathbf{R}^3$ containing a Möbius strip can the surface be given as the zero locus of a *single* smooth equation without critical points along the surface! Indeed, if such a function exists then its gradient along the hypersurface would (after division by its length) provide a continuous (even C^{p-1}) unit normal field, contrary to geometric intuition for this surface. This is very interesting: unlike linear algebra (where hyperplanes are always the zero locus of a single nonzero linear functional), when working with submanifolds of a manifold it is generally *false* that a hypersurface can be *globally* expressed as a the zero locus of a single (reasonable) function on the ambient manifold. We therefore see that the study of submanifolds through defining equations is not possible for *global* geometry.

Example 3.9. Let $U \subseteq \mathbf{R}^n$ be an open set and let $f_1, \dots, f_r : U \to \mathbf{R}$ be smooth functions such that f_j has no critical points along the level set $H_j = f_j^{-1}(c_j)$ for some $c_j \in f_j(U)$. Assume moreover that the H_j's are pairwise transverse submanifolds, so their intersection $Z = \cap H_j$ is a submanifold of constant codimension r. By the method as above, but now using our earlier work on transverse intersections of submanifolds, the non-vanishing gradients $(\nabla f_j)(z)$ gives a basis for the orthogonal space $\mathrm{T}_z(Z)^{\perp}$ in $\mathrm{T}_z(\mathbf{R}^n) = \mathbf{R}^n$ for all $z \in Z$. Hence, since $\mathrm{N}_{Z/\mathbf{R}^n}$ is isomorphic to the orthogonal bundle $(TZ)^{\perp}$ (Example 3.4), we conclude that it is a trivial bundle: the gradients of the f_j's at points of Z give a trivializing frame. But the Klein bottle K is "non-orientable", much like the Möbius strip, and so since K has a closed embedding into $\mathbf{R}^4$ it will follow from our later study of orientation on manifolds that this surface in $\mathbf{R}^4$ has non-trivial normal bundle. Hence, the smooth compact surface K in $\mathbf{R}^4$ *cannot* be expressed as a transverse intersection of critical-point-free level sets for a pair of smooth functions on an open domain in $\mathbf{R}^4$!

We now prove two general results concerning normal bundles, as preparation for the proof of Theorem 3.1.

Lemma 3.10. *If $j : Z \hookrightarrow Y$ and $i : Y \hookrightarrow X$ are two embeddings of C^p manifolds, $1 \leq p \leq \infty$, so $i \circ j : Z \hookrightarrow X$ is as well, $\mathrm{N}_{Z/Y}$ on Z is naturally realized as a C^{p-1} subbundle of $\mathrm{N}_{Z/X}$, with quotient $j^*(\mathrm{N}_{Y/X})$.*

On fibers over $z \in Z$ this just says that $\mathrm{T}_z(X)/\mathrm{T}_z(Z)$ contains $\mathrm{T}_z(Y)/\mathrm{T}_z(Z)$ with quotient $\mathrm{T}_z(X)/\mathrm{T}_z(Y)$. If we have a C^{p-1}-varying family of inner products on the tangent spaces, so quotients can be identified with orthogonal complements, then the lemma says that directions perpendicular to Z in X at z break up into two types: those perpendicular to the directions along the larger submanifold Y at z, and those that lie along the directions of Y at z but perpendicular to Z at z.

Proof. By definition, $\mathrm{N}_{Z/Y} = j^*(TY)/TZ$ and $\mathrm{N}_{Z/X} = (i \circ j)^*(TX)/TZ \simeq j^*(i^*(TX))/TZ$. Since i is an embedding, $\mathrm{d}i : TY \to TX$ over i induces a subbundle $TY \hookrightarrow i^*(TX)$ over Y (that on fibers over $y \in Y$ is just the inclusion $\mathrm{d}i(y) : \mathrm{T}_y(Y) \hookrightarrow \mathrm{T}_{i(y)}(X)$). Applying j, we realize $j^*(TY)$ as a subbundle of $j^*(i^*(TX))$ over Z. (On fibers over $z \in Z$, this is just the inclusion $\mathrm{d}i(j(z))$ of $\mathrm{T}_{j(z)}(Y)$ into $\mathrm{T}_{i(j(z))}(X)$.) By checking on fibers over each $z \in Z$, we see that the subbundle inclusion $j^*(TY) \to j^*(i^*(TX)) = (i \circ j)^*(TX)$ restricts to the identity on the subbundle TZ in each. (This amounts to the Chain Rule identity $\mathrm{d}(i \circ j)(z) = \mathrm{d}i(j(z)) \circ \mathrm{d}j(z)$ for all $z \in Z$.) Hence, there is an induced map on quotient bundles

$$\mathrm{N}_{Z/Y} = j^*(TY)/TZ \to (i \circ j)^*(TX)/TZ = \mathrm{N}_{Z/X}$$

that is an injection on fibers and so is a subbundle.

It remains to identify the quotient $\mathrm{N}_{Z/X}/\mathrm{N}_{Z/Y}$ with $j^*(\mathrm{N}_{Y/X})$ as vector bundles over Z. By definition, $\mathrm{N}_{Y/X} = i^*(TX)/TY$. Applying j^* to the bundle surjection $i^*(TX) \to \mathrm{N}_{Y/X}$ over Y yields a bundle surjection $(i \circ j)^*(TX) = j^*(i^*(TX)) \twoheadrightarrow j^*(\mathrm{N}_{Y/X})$ that kills the subbundle $j^*(TY)$. The induced bundle mapping

$$(i \circ j)^*(TX)/j^*(TY) \to j^*(\mathrm{N}_{Y/X})$$

is an isomorphism on fibers, and so is an isomorphism. (This argument really shows the more general fact that formation of pullback of bundles commutes with formation of quotients by subbundles.) Hence, it is equivalent to construct an isomorphism of bundles over Z,

$$(j^*(TY)/TZ)/((i \circ j)^*(TX)/TZ) \simeq j^*(TY)/(i \circ j)^*(TX).$$

Rather more generally, if $E \subseteq E' \subseteq E''$ is a pair of subbundles over Z, so E'/E is a subbundle of E''/E, then we claim that the quotient bundle of E''/E modulo E'/E is identified with E''/E'. To see this, consider the bundle surjection $E'' \to E''/E'$. This kills E', and so it kills E. Hence, it uniquely factors through a bundle mapping $E''/E \to E''/E'$ that is a fiberwise surjection. This new map kills $E'/E \subseteq E''/E$, so it induces a mapping of bundles $(E''/E)/(E'/E) \to E''/E'$ that is an isomorphism on fibers and hence an isomorphism. ■

Here is the key step where normal bundles work their magic:

Theorem 3.11. *Let E be a C^∞ vector bundle over a C^∞ manifold X. Using the zero section $0 : X \to E$ to realize X as a closed C^∞ submanifold of E, $\mathrm{N}_{X/E} \to X$ is* naturally *isomorphic to E as C^∞ vector bundles on X.*

The reader is urged to skip the proof (past the first or fourth paragraph) on an initial reading.

Proof. One approach is to build natural isomorphisms $\mathrm{T}_{0(x)}(E)/\mathrm{T}_x(X) \simeq E(x)$ for any $x \in X$, and to show that these vary nicely in x in the sense that they glue to define a bundle isomorphism $\mathrm{N}_{X/E} = 0^*(TE)/TX \simeq E$ over X. Intuitively, if we pick a local trivialization $E|_U = U \times \mathbf{R}^n$ then we get an isomorphism $\mathrm{T}_{0(x)}(E) = \mathrm{T}_{(x,0)}(U \times \mathbf{R}^n) = \mathrm{T}_x(U) \oplus \mathrm{T}_0(\mathbf{R}^n) = \mathrm{T}_x(X) \oplus \mathbf{R}^n$, so we get an isomorphism $\mathrm{T}_{0(x)}(E)/\mathrm{T}_x(X) \simeq \mathbf{R}^n = E(x)$. This short calculation is the *reason* we beleive the lemma to be true, and it has natural geometric appeal: after locally trivializing E over X, the "directions away from the zero-section in E" modulo those along the base directions are seen to be the "vertical" directions, which is to say along the fibers of $E \to X$. Unfortunately, this vague statement is local over X (as well as vague), and the whole point of vector bundles is to work *globally*: we need a bundle isomorphism $\mathrm{N}_{X/E} \simeq E$ over all of X, not unrelated isomorphisms over small opens in X. Keep in mind that *locally* over the base, any two vector bundles of the same rank are isomorphic (as both become trivial)! The real problem is to show that there is the "same global twisting", so we can build a global bundle isomorphism.

There were many implicit choices in the construction of the above isomorphisms on fibers, and so why are they independent of the choices? Moreover, once this point is settled then why do such fibral isomorphisms arise from a *global* bundle isomorphism (i.e., over the entire base space)? It is better to have intrinsic global definitions and to use local considerations only to verify properties (such as being an isomorphism). We therefore opt for an intrinsic and elegant global method that illustrates a variety of techniques we have at our disposal for working with vector bundles. The reader is invited to try to fill in the well-definedness and other details on the fibral approach suggested above. (At the very least, the need to do *some* work in the proof shows that the use of Theorem 3.11 in the proof of Theorem 3.1 constitutes genuine input.) It should also be noted that one could alternatively approach the problem of building the bundle isomorphism $\mathrm{N}_{X/E} \simeq E$ by using transition matrices. However, this is ugly because the true meaning of the global isomorphism can only be appreciated when it is built without the interference of matrices.

We wish to build a bundle isomorphism $0^*(TE)/TX \simeq E$ over X, so let us try to build a bundle mapping $0^*(TE) \to E$ and check that on x-fibers it is surjective with kernel $\mathrm{T}_x(X)$, so then the kernel subbundle contains (and hence equals) TX. This will give the result. How should we build a bundle mapping $0^*(TE) \to E$ over X? Since the universal property of pullback involves mappings *to* a pullback (rather than *from* a pullback), we are led to consider the dual problem: try to build a natural map of bundles $E^\vee \to 0^*(TE)^\vee = 0^*(T^*E)$ over X (here we have used that pullback and dual commute); we could then dualize this map and hope for the best. That is, we first seek to build a natural map of bundles $E^\vee \to T^*E$ over $0 : X \to E$.

Inspired by our work with $\mathscr{O}$-modules, for open $U \subseteq X$ can we build natural maps $f_U : E^\vee(U) \to (T^*E)(E|_U)$ respecting shrinking in U? If so, then since $0^{-1}(E|_U) = U$ we see that "pullback along the 0-section" gives a map $(T^*E)(E|_U) \to (0^*(T^*E))(U)$, and hence composing this with f_U would give maps $E^\vee(U) \to (0^*(T^*E))(U)$ compatibly with shrinking in U. Provided these latter maps are $\mathscr{O}(U)$-linear, it follows from the equivalence between bundles and $\mathscr{O}$-modules (surprise!) that such maps arise from a uniquely determined bundle map $E^\vee \to 0^*(T^*E)$ over X. The dual map $0^*(TE) \to E$ over X is then a viable candidate for our main goal.

The above considerations thereby lead us to ask the question: for open $U \subseteq X$ is there a natural map $f_U : E^\vee(U) \to \Omega^1_E(E|_U)$? Recall that $E^\vee(U) = \mathrm{Hom}_U(E|_U, U \times \mathbf{R})$. Hence, given a C^∞ bundle mapping $h : E|_U \to U \times \mathbf{R}$ over U we seek to build a C^∞ differential 1-form over the open submanifold $E|_U$ in E (i.e., an element in $\Omega^1_E(E|_U)$). Ah, but by viewing h as a smooth map between smooth manifolds (ignoring that it respects bundle structures), we may compose with the smooth projection $p_U : U \times \mathbf{R} \to \mathbf{R}$ to obtain a composite mapping $p_U \circ h : E|_U \to \mathbf{R}$ that is a *smooth function*. As with any smooth function on the manifold $E|_U$, we may form the differential $\mathrm{d}(p_U \circ h)$ that is a smooth 1-form over $E|_U$! Hence, $f_U : h \mapsto \mathrm{d}(p_U \circ h)$ is a mapping of sets $E^\vee(U) \to \Omega^1_E(E|_U) = (T^*E)(E|_U)$. This construction clearly respects shrinking on U and respects addition on both sides (since addition on the dual-bundle section h is expressed in terms of the mapping $h : E|_U \to U \times \mathbf{R}$ via addition in $\mathbf{R}$). We claim that these mappings f_U are the answer to our prayers. (The reader who has reached this point is urged to skip the rest on a first reading and to see how Theorem 3.11 is used in the proof of Theorem 3.1.)

The associated mappings $E^\vee(U) \to (0^*(T^*E))(U)$ are given by $h \mapsto 0^*(\mathrm{d}(p_U \circ h))$ and respect shrinking in U. Are these $\mathscr{O}(U)$-linear? The compatibility with addition has been explained, and so we have to choose $s \in \mathscr{O}(U)$ and prove $0^*(\mathrm{d}(p_U \circ (s \cdot h))) = s \cdot 0^*(\mathrm{d}(p_U \circ h))$. Passing to fibers at any $u \in U$, we want $\mathrm{d}(p_U \circ (s \cdot h))(0(u)) = s(u)\mathrm{d}(p_U \circ h)(0(u))$ in $T_{0(u)}(E)^\vee$, with $0(u)$ the origin in $E(u) \subseteq E$. Here, $h : E|_U \to U \times \mathbf{R}$ is a mapping bundles over U but it is viewed as a mapping of manifolds. By the definition of $p_U : U \times \mathbf{R} \to \mathbf{R}$ and the module structure on sections of bundles, $p_U \circ (s \cdot h) = (s \circ \pi_U) \cdot (p_U \circ h)$ as smooth functions on $E|_U$, with $\pi_U : E|_U \to U$ the structure map. Thus, by the Leibnitz Rule

$$\mathrm{d}(p_U \circ (s \cdot h))(0(u)) = \mathrm{d}((s \circ \pi_U) \cdot (p_U \circ h))(0(u)) = s(u)\mathrm{d}(p_U \circ h)(0(u)) + (p_U \circ h)(0(u))\mathrm{d}(s \circ \pi_U)(0(u))$$

since $(s \circ \pi_U)(0(u)) = s(u)$ (as $\pi_U(0(u)) = u$). But $h : E|_U \to U \times \mathbf{R}$ is a bundle mapping over U, so the function $p_U \circ h : E|_U \to \mathbf{R}$ carries the zero-section of $E|_U$ to the origin in $\mathbf{R}$ and hence it vanishes at all points $0(u) \in E|_U$. This gives the required identity.

We have now built a bundle mapping $E^\vee \to 0^*(T^*E) = 0^*((TE)^\vee) = 0^*(TE)^\vee$ over X, so dualizing (and using double duality) gives a C^∞ bundle mapping $\theta_E : 0^*(TE) \to E$ over X. It remains to show that this is a bundle surjection with kernel subbundle equal to TX (as then we get an induced isomorphism $\mathrm{N}_{X/E} \simeq E$ as C^∞ bundles over X). But the formation of our mapping over X clearly respects shrinking over X, and our remaining problems are of local nature over X, so it suffices to work locally over X. Thus, we may assume that E is a *trivial* smooth vector bundle. Rather than work with an isomorphism $E \simeq X \times \mathbf{R}^n$, it is more convenient to work with an equality $E = X \times \mathbf{R}^n$. Hence, we have to investigate how the formation of θ_E interacts with isomorphisms among bundles E over X.

If $\varphi : E' \to E$ is a bundle mapping over X (such as an isomorphism), then it is easy to check from the construction of θ_E and $\theta_{E'}$ that the diagram of bundle mappings

$$\begin{array}{ccc} 0'^*(TE') & \xrightarrow{\theta_{E'}} & E' \\ {\scriptstyle \mathrm{d}\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ 0^*(TE) & \xrightarrow[\theta_E]{} & E \end{array}$$

over X that is commutative (the left column is the map induced along zero-sections by the bundle mapping $\mathrm{d}\varphi : TE' \to TE$ over φ). In particular, if φ is an isomorphism then the columns are isomorphisms and so in such cases θ_E is an isomorphism if and only if $\theta_{E'}$ is an isomorphism. That is, the isomorphism problem for θ_E only depends on E up to bundle isomorphism over X. We are therefore reduced to the case $E = X \times \mathbf{R}^n$. Since we can work locally on X, we may also assume X admits global coordinates $\{x_1, \dots, x_m\}$.

Let $\pi_1 : E \to X$ and $\pi_2 : E \to \mathbf{R}^n$ be the projections, so $\xi_E : TE \simeq \pi_1^*(TX) \oplus \pi_2^*(T(\mathbf{R}^n))$ as bundles over E. The composites of $0 : X \to E$ with π_1 and π_2 are respectively the identity on X and the map $X \to \mathbf{R}^n$ that is the constant map to the origin. Hence, pulling back the mapping ξ_E along 0 gives an isomorphism $0^*(\xi_E) : 0^*(TE) \simeq TX \oplus (X \times \mathrm{T}_0(\mathbf{R}^n))$ as bundles over X. This decomposition recovers the subbundle inclusion $TX \hookrightarrow 0^*(TE)$ that is used in the definition of $\mathrm{N}_{X/E}$, so it suffices to prove that the composite bundle mapping,

$$TX \oplus (X \times \mathrm{T}_0(\mathbf{R}^n)) \simeq 0^*(TE) \overset{\theta_E}{\to} E = X \times \mathbf{R}^n,$$

kills TX and restricts to the canonical isomorphism $X \times \mathrm{T}_0(\mathbf{R}^n) \simeq X \times \mathbf{R}^n$ on the other factor.

Consider the constant sections $\underline{x}_i$ of $X \times (\mathbf{R}^n)^\vee = E^\vee$ that are the bundle mappings $p_i : E = X \times \mathbf{R}^n \to X \times \mathbf{R}$ given by the standard projections on $\mathbf{R}^n$; these are a global frame for $X \times (\mathbf{R}^n)^\vee = E^\vee$ as a bundle over X. Since $p_X \circ p_i : E = X \times \mathbf{R}^n \to \mathbf{R}$ is the smooth function given by projection to the ith factor of $\mathbf{R}^n$, its total differential as a fiberwise functional on tangent spaces

$$\mathrm{T}_{0(x)}(E) = \mathrm{T}_{(x,0)}(X \times \mathbf{R}^n) = \mathrm{T}_x(X) \oplus \mathrm{T}_0(\mathbf{R}^n)$$

vanishes under restriction to a functional on $\mathrm{T}_x(X)$ (giving the desired killing of TX) and restricts to the functional on $\mathrm{T}_0(\mathbf{R}^n)$ that is exactly the tangent mapping at origins for the ith standard projection $\mathbf{R}^n \to \mathbf{R}$. That is, $0^*(\mathrm{d}(p_X \circ p_i))$ as a global section of $0^*(T^*E) = T^*X \oplus (X \times \mathrm{T}_0(\mathbf{R}^n)^\vee)$ vanishes along the first factor and is constant section $\mathrm{d}x_i(0)$ along the second factor. This says exactly that the dual mapping $TX \oplus (X \times \mathrm{T}_0(\mathbf{R}^n)) \simeq 0^*(TE) \stackrel{\theta_E}{\to} E = X \times \mathbf{R}^n$ carries TX into the zero-section and carries the constant sections $\partial_{x_i}|_0$ along the second factor to the constant sections $\underline{e}_i \in (X \times \mathbf{R}^n)(X)$ on the standard basis of $\mathbf{R}^n$. That is, θ_E has kernel bundle TX and is a bundle surjection onto E. ■

Proof. (of Theorem 3.1): The manifold E has constant dimension $n + r$. Recall Whitney's embedding theorem: any smooth manifold of dimension d admits a smooth embedding into $\mathbf{R}^{2d+1}$. By Whitney's embedding theorem *applied to the manifold* E, there is a smooth embedding $E \hookrightarrow \mathbf{R}^N$ for $N = 2\dim E + 1 = 2(n + r) + 1$. *Choose* such an embedding. By Theorem 3.11, $E \simeq \mathrm{N}_{X/E}$ over X. But X is a submanifold of E which in turn is a submanifold of $\mathbf{R}^N$, so by Lemma 3.10 the normal bundle $\mathrm{N}_{X/E}$ over X is a subbundle of $\mathrm{N}_{X/\mathbf{R}^N}$. If $i : X \to \mathbf{R}^N$ is the smooth composite embedding (the zero-section of E followed by the chosen embedding of E into $\mathbf{R}^N$ as smooth manifolds), then by definition $\mathrm{N}_{X/\mathbf{R}^N}$ is a quotient bundle of $i^*(T(\mathbf{R}^N))$. But the rank-N bundle $\mathrm{T}(\mathbf{R}^N) \to \mathbf{R}^N$ is *trivial*, so its rank-N pullback $i^*(T(\mathbf{R}^N)) \to X$ over X is trivial. That is, we have exhibited $E \simeq \mathrm{N}_{X/E}$ as a subbundle of a quotient $\mathrm{N}_{X/\mathbf{R}^N}$ of a trivial bundle $X \times \mathbf{R}^N$.

Rather generally, I claim that any C^∞ quotient bundle of $X \times \mathbf{R}^N$ is also C^∞-isomorphic to a C^∞ subbundle of $X \times \mathbf{R}^N$. Indeed, if E' is such a quotient and E'' is the kernel subbundle of the quotient mapping $X \times \mathbf{R}^N \twoheadrightarrow E'$ then (using Lemma 3.5 with $V = \mathbf{R}^N$ having the standard inner product) consider the orthogonal subbundle $(E'')^\perp$ in $X \times \mathbf{R}^N$ using the standard inner product on $\mathbf{R}^N$; fiberwise $(E'')^\perp(x) \subseteq \mathbf{R}^N$ is the orthogonal complement to $E''(x) \subseteq \mathbf{R}^N$. The composite C^∞ bundle mapping $E'' \to X \times \mathbf{R}^N \to E'$ over X is an isomorphism on fibers over X, and so it is an isomorphism of C^∞ bundles over X. This settles the claim.

We conclude that $\mathrm{N}_{X/\mathbf{R}^N}$ has a structure of C^∞-subbundle of $X \times \mathbf{R}^N$, so E is C^∞-isomorphic to a C^∞ subbundle of a C^∞ subbundle of $X \times \mathbf{R}^N$, whence the C^∞ vector bundle $E \to X$ has a structure (in fact, many such!) of a rank-r C^∞ subbundle of $X \times \mathbf{R}^N$. By the universal property of $G = \mathbf{G}_{N-r}(\mathbf{R}^N)$ equipped with its universal subbundle W of rank r inside of $G \times \mathbf{R}^N$, it follows that there is a unique smooth map $f : X \to G$ such that $f^*(W) = E$ inside of $X \times \mathbf{R}^N$. (If we change how we embed E into $\mathbf{R}^N$ via Whitney's theorem, then the structure on E of rank-r subbundle of $X \times \mathbf{R}^N$ would change and so the map f would change. That is, f is *not* intrinsic to $E \to X$.) In particular, E is C^∞ isomorphic to a pullback f^*W of the universal subbundle W over the Grassmannian G, as desired. ■

Corollary 3.12. *In the setup of Theorem* 3.1*, there exist global sections* $s_1, \dots, s_{2n+2r+1} \in E(X)$ *such that* $\{s_i(x)\}$ *spans* $E(x)$ *for all* $x \in X$*. That is, there exists a* C^∞ *vector bundle surjection* $X \times \mathbf{R}^{2n+2r+1} \twoheadrightarrow E$ *over* X*.*

Proof. Let $N = 2n + 2r + 1$. Applying the proof of Theorem 3.1 to the dual bundle $E^\vee$, we see that $E^\vee$ admits a structure of subbundle of $X \times \mathbf{R}^N$. Hence, by dualizing and using the double duality isomorphism $E \simeq E^{\vee\vee}$, the bundle E admits a structure of quotient bundle of $(X \times \mathbf{R}^N)^\vee \simeq X \times (\mathbf{R}^N)^\vee \simeq X \times \mathbf{R}^N$. ■