MATH 396. TENSOR EXAMPLES

1. Let $V = \mathbf{R}^2$ be a vector space over $\mathbf{R}$. Suppose $S\colon V \to V$ and $T\colon V \to V$ are linear maps represented by the matrices

$$S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad T = \begin{pmatrix} 16 & 8 \\ 4 & -7 \end{pmatrix}.$$

Compute the 4 by 4 matrix for $S \otimes T$ with respect to the ordered basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$ of $\mathbf{R}^2 \otimes \mathbf{R}^2$ (with $e_1 = (1,0)$, $e_2 = (0,1)$).

Solution.

The computation is not mysterious in any way. It is done using the standard method to find the matrix of a linear map written with respect to an ordered basis. Recall that the tensor product of the linear maps S and T is the unique *linear* map characterized by the property $(S \otimes T)(v \otimes w) = S(v) \otimes T(w)$ for elementary tensors $v \otimes w \in V \otimes V$. Hence, we compute

$$\begin{aligned} (S \otimes T)(e_1 \otimes e_1) &= S(e_1) \otimes T(e_1) \\ &= (1e_1 + 3e_2) \otimes (16e_1 + 4e_2) \\ &= 16(e_1 \otimes e_1) + 4(e_1 \otimes e_2) + 48(e_2 \otimes e_1) + 12(e_2 \otimes e_2) \end{aligned}$$

and

$$\begin{aligned} (S \otimes T)(e_1 \otimes e_2) &= S(e_1) \otimes T(e_2) \\ &= (1e_1 + 3e_2) \otimes (8e_1 - 7e_2) \\ &= 8(e_1 \otimes e_1) - 7(e_1 \otimes e_2) + 24(e_2 \otimes e_1) - 21(e_2 \otimes e_2). \end{aligned}$$

Similarly,

$$(S \otimes T)(e_2 \otimes e_1) = 32(e_1 \otimes e_1) + 8(e_1 \otimes e_2) + 64(e_2 \otimes e_1) + 16(e_2 \otimes e_2)$$

and

$$(S \otimes T)(e_2 \otimes e_2) = 16(e_1 \otimes e_1) - 14(e_1 \otimes e_2) + 32(e_2 \otimes e_1) - 28(e_2 \otimes e_2).$$

Thus, relative to the *ordered* basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ of $\mathbf{R}^2 \otimes \mathbf{R}^2$, the matrix for $S \otimes T$ is given by

$$\begin{pmatrix} 16 & 8 & 32 & 16 \\ 4 & -7 & 8 & -14 \\ 48 & 24 & 64 & 32 \\ 12 & -21 & 16 & -28 \end{pmatrix}$$

Note that if we view this matrix in the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

then A_{ij} is given by $a_{ij}T$, where a_{ij} is the element in the ith row and jth column of the matrix S.

2. Let $\{v_i\}$ be a basis of a finite-dimensional vector space V over a field F. Prove that $x = \sum c_{ij} v_i \otimes v_j \in V \otimes V$ is an elementary tensor if and only if $c_{ij}c_{i'j'} = c_{ij'}c_{i'j}$ for all i, i', j, j'.

Solution.

If $x = v \otimes v'$ with $v = \sum c_i v_i$ and $v' = \sum c'_j v_j$ then $x = \sum c_i c'_j v_i \otimes v'_j$, so $c_{ij} = c_i c'_j$ and hence the proposed necessary identities do hold.

Now, we prove the converse. Suppose that the coefficients of x satisfy $c_{ij}c_{i'j'} = c_{ij'}c_{i'j}$ for all i, i', j, j'. To show $x = v \otimes v'$ for some $v, v' \in V$ we may certainly assume $x \neq 0$. Thus, we can scale x by $F^\times$ and assume $c_{i_0 j_0} = 1$ for some i_0, j_0. Consider $v = \sum c_i v_i$ and $v' = \sum c'_j v_j$ with

$c_{i_0} = c'_{j_0} = 1$ and all other coefficients unknown. The condition $x = v \otimes v'$ says $c_{ij} = c_i c'_j$ (to see this, just "multiply out") for all i and j. In particular, we must have

$$c'_j = c_{i_0} c'_j = c_{i_0 j} \text{ and } c_i = c_i c'_{j_0} = c_{i j_0}$$

for all i and j. This does give $c'_{j_0} = 1$ and $c_{i_0} = 1$ because

$$c'_{j_0} = c_{i_0 j_0} = 1 \text{ and } c_{i_0} = c_{i_0 j_0} = 1.$$

It must be proved that these values actually satisfy $c_{ij} = c_i c'_j$ for all i and j, which is to say $c_{ij} = c_{ij_0} c_{i_0 j}$. But this is now clear because

$$c_i c'_j = c_{ij_0} c_{i_0 j} = c_{ij} c_{i_0 j_0} = c_{ij}.$$

3. Let $V_1, \ldots, V_n$ be finite-dimensional vector spaces over a field, with $n \geq 2$.

(i) By considering multilinear pairings $V_1 \times \cdots \times V_n \to W$ to varying vector spaces W, adapt the method for $n = 2$ to prove the existence and uniqueness (up to unique isomorphism) of a universal such pairing

$$V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$$

(denoted $(v_1, \cdots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n$) If $\{e_{1,j}, \cdots, e_{d_j,j}\}$ is a basis of V_j with $d_j = \dim V_j$, prove that the $\prod d_i$ elements $e_{i_1,1} \otimes \cdots \otimes e_{i_n,n}$ are a basis of $V_1 \otimes \cdots \otimes V_n$. (Treat the case when some $V_j = 0$ separately.) In the special case $V_1 = \cdots = V_n = V$, this is denoted $V^{\otimes n}$.

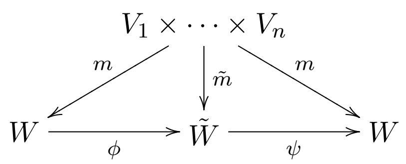
(ii) For linear maps $T_j : V_j \to W_j$, define and uniquely characterize (via elementary tensors) a linear map

$$T_1 \otimes \cdots \otimes T_n : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_n,$$

and discuss its behavior with respect to composites with linear maps $W_j \to U_j$. Also describe its matrix in terms of bases as in (i) and the corresponding matrices of the T_j's. In the special case $V_i = V$ and $W_i = W$ and $T_i = T$ for all i, the map is denoted $T^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$.

Solutions.

(i) The proof of uniqueness up to unique isomorphism for such a universal object is exactly the same as in the case $n = 2$, replacing the word "bilinear" with "multilinear" everywhere. Recall the usual diagram:



where the pairs (W, m) and $(\tilde{W}, \tilde{m})$ both have the universal property of the tensor product of $V_1, \ldots, V_n$. By the universal property, there exists a linear map $\phi \colon W \to \tilde{W}$ uniquely satisfying $\phi \circ m = \tilde{m}$ because $\tilde{m}$ is multilinear and m is "universal". In a similar manner, we get the linear map $\psi \colon \tilde{W} \to W$ uniquely satisfying $\psi \circ \tilde{m} = m$. Note that both $(\psi \circ \phi) \circ m = m$ and $\mathrm{id}_W \circ m = m$. So by uniqueness, $\mathrm{id}_W = \psi \circ \phi$. Using the same idea, $\phi \circ \psi = \mathrm{id}_{\tilde{W}}$. Hence we see that the structure we seek is uniquely determined up to unique isomorphism.

As for the existence aspect, we choose bases $\{e_{ij}\}_{1 \leq i \leq d_j}$ for $1 \leq j \leq n$, and we note that (as with bilinear pairings) a multilinear map

$$\mu : V_1 \times \cdots \times V_n \to W$$

is both determined by the values $\mu(e_{i_1,1},\dots,e_{i_n,n})\in W$ and may be defined by such values arbitrarily assigned (since for any $w_{i_1,\dots,i_n}\in W$ with $1\le i_j\le d_j$ the formula

$$\mu(\sum_{i_1} a_{i_1}e_{i_1,1},\dots,\sum_{i_n} a_{i_n}e_{i_n,n}) = \sum a_{i_j,j}w_{i_1,\dots,i_n}\in W$$

is a multilinear pairing of the V_j's into W with $\mu(e_{i_1,1},\dots,e_{i_n,n})=w_{i_1,\dots,i_n}$). We define T to be the Euclidean space F^S with S given as the finite set

$$S=\{1,\dots,d_1\}\times\cdots\times\{1,\dots,d_n\}.$$

For $I=(i_1,\dots,i_n)\in S$, let $e_I\in F^S$ be the assumed "standard basis" vector. The multilinear pairing $V_1\times\cdots\times V_n\to T$ given by

$$(\sum_{i_1} a_{i_1}e_{i_1,1},\dots,\sum_{i_n} a_{i_n}e_{i_n,n})\mapsto \sum_I\prod_{j=1}^n a_{i_j,j}\cdot e_I$$

is universal; the proof of universality (including the uniqueness aspect) is identical to the case $n=2$ in view of the mechanism we have outlined above for both uniquely characterizing as well as defining all possible multilinear μ's.

If we write $(v_1,\dots,v_n)\mapsto v_1\otimes\cdots\otimes v_n$ to denote the universal multilinear pairing into T, then with the model as constructed above we have $e_I=e_{i_1,1}\otimes\cdots\otimes e_{i_n,n}$. Hence, the basis assertion follows. (Of course, as in the case $n=2$ we can prove this basis assertion without reverting to the construction process, instead arguing by "pure thought" in terms of the universal property alone.)

(*ii*) The map $T\colon V_1\times\cdots\times V_n\to W_1\otimes\cdots\otimes W_n$ given by

$$(v_1,\dots,v_n)\mapsto T_1(v_1)\otimes\cdots\otimes T_n(v_n)$$

is clearly multilinear in the v_j's (as the T_j's are linear), so by the universal property of the tensor product of the V_j's we get the desired *linear* map $\tilde{T}=T_1\otimes\cdots\otimes T_n$ that is uniquely characterized by the condition $v_1\otimes\cdots\otimes v_n\mapsto T_1(v_1)\otimes\cdots\otimes T_n(v_n)$:

$$\begin{array}{ccc} V_1\times\cdots\times V_n & & \\ \Big\downarrow m & \searrow^{T} & \\ V_1\otimes\cdots\otimes V_n & \xrightarrow[\tilde{T}]{} & W_1\otimes\cdots\otimes W_n \end{array}$$

If $T'_j: W_j\to U_j$ are linear maps, then

$$(T'_1\otimes\cdots\otimes T'_n)\circ(T_1\otimes\cdots\otimes T_n)=(T'_1\circ T_1)\otimes\cdots\otimes(T'_n\circ T_n)$$

as linear maps from $V_1\otimes\cdots\otimes V_n$ to $U_1\otimes\cdots\otimes U_n$, as can be checked by working with the elementary n-fold tensors (by the universal property, or because they span the space) as for $n=2$.