Math 396. Matrix computations on the Exterior and Symmetric Powers

1. Let $V = \mathbf{R}^2$ be a vector space over $\mathbf{R}$, e_1, e_2 the standard basis of V and $T\colon V \to V$ the linear map represented by the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

(i) Compute $\mathrm{Sym}^2(T)\colon \mathrm{Sym}^2(V) \to \mathrm{Sym}^2(V)$.

(ii) Compute $\wedge^2(T)\colon \wedge^2(V) \to \wedge^2(V)$.

Solutions.

(i) We will compute the 3×3-matrix A of $\mathrm{Sym}^2(T)$ with respect to the basis

$$\{e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2\}$$

of $\mathrm{Sym}^2(V)$. Recall that $\mathrm{Sym}^2(T)$ is uniquely characterized by the property $\mathrm{Sym}^2(T)(v_1 \cdot v_2) = T(v_1) \cdot T(v_2)$. Hence, we compute

$$\begin{aligned} \mathrm{Sym}^2(T)(e_1 \cdot e_1) &= T(e_1) \cdot T(e_1) \\ &= (1e_1 + 3e_2) \cdot (1e_1 + 3e_2) \\ &= 1e_1 \cdot e_1 + 3e_1 \cdot e_2 + 3e_2 \cdot e_1 + 9e_2 \cdot e_2 \\ &= 1e_1 \cdot e_1 + 6e_1 \cdot e_2 + 9e_2 \cdot e_2. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{Sym}^2(T)(e_1 \cdot e_2) &= T(e_1) \cdot T(e_2) \\ &= (1e_1 + 3e_2) \cdot (2e_1 + 4e_2) \\ &= 2e_1 \cdot e_1 + 10e_1 \cdot e_2 + 12e_2 \cdot e_2. \end{aligned}$$

Lastly,

$$\begin{aligned} \mathrm{Sym}^2(T)(e_2 \cdot e_2) &= T(e_2) \cdot T(e_2) \\ &= (2e_1 + 4e_2) \cdot (2e_1 + 4e_2) \\ &= 4e_1 \cdot e_1 + 16e_1 \cdot e_2 + 16e_2 \cdot e_2. \end{aligned}$$

Thus, the matrix A is given by

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 10 & 16 \\ 9 & 12 & 16 \end{pmatrix}$$

(ii) The 1×1 matrix B of $\wedge^2(T)$ is given by $\det(T) = -2$.

2. Let $V = \mathbf{R}^3$ be a vector space over $\mathbf{R}$, e_1, e_2, e_3 the standard basis of V and $T\colon V \to V$ the linear map represented by the matrix

$$\begin{pmatrix} 1 & 7 & 8 \\ 4 & 3 & 6 \\ 1 & 0 & -8 \end{pmatrix}$$

(i) Compute $\mathrm{Sym}^2(T)\colon \mathrm{Sym}^2(V) \to \mathrm{Sym}^2(V)$.

(ii) Compute $\wedge^2(T)\colon \wedge^2(V) \to \wedge^2(V)$.

Solutions.

(i) We will compute the 6×6-matrix A of $\mathrm{Sym}^2(T)$ with respect to the basis

$$\{e_1 \cdot e_1, e_1 \cdot e_2, e_1 \cdot e_3, e_2 \cdot e_2, e_2 \cdot e_3, e_3 \cdot e_3\}$$

of $\mathrm{Sym}^2(V)$. The computation is as follows

$$\begin{aligned}\mathrm{Sym}^2(T)(e_1 \cdot e_1) =& T(e_1) \cdot T(e_1)\\ =&(1e_1 + 4e_2 + 1e_3) \cdot (1e_1 + 4e_2 + 1e_3)\\ =&1e_1 \cdot e_1 + 4e_1 \cdot e_2 + 1e_1 \cdot e_3 + 4e_2 \cdot e_1 +\\ &16e_2 \cdot e_2 + 4e_2 \cdot e_3 + 1e_3 \cdot e_1 + 4e_3 \cdot e_2 + 1e_3 \cdot e_3\\ =&1e_1 \cdot e_1 + 8e_1 \cdot e_2 + 2e_1 \cdot e_3 + 16e_2 \cdot e_2 + 8e_2 \cdot e_3 + 1e_3 \cdot e_3.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Sym}^2(T)(e_1 \cdot e_2) =& T(e_1) \cdot T(e_1)\\ =&(1e_1 + 4e_2 + 1e_3) \cdot (7e_1 + 3e_2)\\ =&7e_1 \cdot e_1 + 3e_1 \cdot e_2 + 28e_2 \cdot e_1 + 12e_2 \cdot e_2 + 7e_3 \cdot e_1 + 3e_3 \cdot e_2\\ =&7e_1 \cdot e_1 + 31e_1 \cdot e_2 + 7e_1 \cdot e_3 + 12e_2 \cdot e_2 + 3e_2 \cdot e_3.\end{aligned}$$

The remaining columns of A are computed in the same manner and the matrix A is given by

$$A = \begin{pmatrix} 1 & 7 & 8 & 49 & 56 & 64\\ 8 & 31 & 38 & 42 & 66 & 96\\ 2 & 7 & 0 & 0 & -56 & -128\\ 16 & 12 & 24 & 9 & 18 & 36\\ 8 & 3 & -26 & 0 & -24 & -96\\ 1 & 0 & -8 & 0 & 0 & 64 \end{pmatrix}$$

(ii) We will compute the 3×3 matrix B of $\wedge^2(T)$ with respect to the basis

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}.$$

Recall that the map $\wedge^2(T)$ is characterized by the property that $\wedge^2(T)(v_1 \wedge v_2) = T(v_1) \wedge T(v_2)$. We compute

$$\begin{aligned}\wedge(T)(e_1 \wedge e_2) &= T(e_1) \wedge T(e_2)\\ &= (1e_1 + 4e_2 + 1e_3) \wedge (7e_1 + 3e_2)\\ &= 7e_1 \wedge e_1 + 3e_1 \wedge e_2 + 28e_2 \wedge e_1 + 12e_2 \wedge e_2 + 7e_3 \wedge e_1 + 3e_3 \wedge e_2\\ &= 3e_1 \wedge e_2 - 28e_1 \wedge e_2 - 7e_1 \wedge e_3 - 3e_2 \wedge e_3\\ &= -25e_1 \wedge e_2 - 7e_1 \wedge e_3 - 3e_2 \wedge e_3.\end{aligned}$$

Similarly,

$$\begin{aligned}\wedge(T)(e_1 \wedge e_3) &= T(e_1) \wedge T(e_3)\\ &= (1e_1 + 4e_2 + 1e_3) \wedge (8e_1 + 6e_2 - 8e_3)\\ &= 6e_1 \wedge e_2 - 8e_1 \wedge e_3 + 32e_2 \wedge e_1 - 32e_2 \wedge e_3 + 8e_3 \wedge e_1 + 6e_3 \wedge 1\\ &= 6e_1 \wedge e_2 - 8e_1 \wedge e_3 - 32e_1 \wedge e_2 - 32e_2 \wedge e_3 - 8e_1 \wedge e_3 - 6e_2 \wedge e_3\\ &= -26e_1 \wedge e_2 - 16e_1 \wedge e_3 - 38e_2 \wedge e_3\end{aligned}$$

Finally,

$$\begin{aligned}
\wedge(T)(e_2 \wedge e_3) &= T(e_2) \wedge T(e_3) \\
&= (7e_1 + 3e_2) \wedge (8e_1 + 6e_2 - 8e_3) \\
&= 42e_1 \wedge e_2 - 56e_1 \wedge e_3 + 24e_2 \wedge e_1 - 24e_2 \wedge e_3 \\
&= 18e_1 \wedge e_2 - 56e_1 \wedge e_3 - 24e_2 \wedge e_3
\end{aligned}$$

Thus, the resulting matrix B is given by

$$B = \begin{pmatrix} -25 & -26 & 18 \\ -7 & -16 & -56 \\ -3 & -38 & -24 \end{pmatrix}$$