Math 396. Hodge-star operator

In the theory of pseudo-Riemannian manifolds there will be an important operator (on differential forms) called the Hodge star; this operator will be an essential ingredient in the formulation of Stokes' theorem as a theorem concerning integration and vector fields on oriented Riemannian manifolds (the viewpoint of Math 216) as opposed to just integration of differential forms on oriented manifolds (which is the perspective through which the theorem is really proved). The Hodge star operator also arises in the coordinate-free formulation of Maxwell's equations in flat spacetime (viewed as a pseudo-Riemannian manifold with signature $(3,1)$).

As with orientations, the Hodge star arises from certain notions in linear algebra, applied to tangent and cotangent spaces of manifolds. The aim of this handout is to develop the relevant foundations in linear algebra, and the globalization on manifolds will be given later in the course. The final calculation in this handout shows that the theory of the vector cross product on $\mathbf{R}^3$ is best understood through the perspective of the Hodge star operator.

All vector spaces are assumed to be finite-dimensional in what follows.

1. Definitions

Let $(V, \langle\cdot,\cdot\rangle, \mu)$ be an oriented non-degenerate quadratic space over $\mathbf{R}$ with dimension $d > 0$. In the positive half-line of $\wedge^d(V)$ there is a unique unit vector, also called the volume form determined by μ. We denote it $\mathrm{Vol}_\mu \in \wedge^d(V)$, so $\mathrm{Vol}_{\mu^\vee}$ in $\wedge^d(V^\vee)$ is what was called the volume form in $\wedge^d(V^\vee)$ below Theorem 2.6 in the handout on orientations. (The notation Vol_μ is slightly abusive since Vol_μ depends on not just the orientation μ but also on the quadratic form.) For any $1 \le r \le d$ we write $\langle\cdot,\cdot\rangle_r$ to denote the non-degenerate symmetric bilinear form induced on $\wedge^r(V)$, so for elementary wedge products we have

$$\langle v_1 \wedge \cdots \wedge v_r, v'_1 \wedge \cdots \wedge v'_r\rangle_r = \det(\langle v_i, v'_j\rangle).$$

This symmetric bilinear form induces a canonical isomorphism $\wedge^r(V) \simeq (\wedge^r V)^\vee$ via $\eta \mapsto \langle \eta, \cdot\rangle_r = \langle \cdot, \eta\rangle_r$. However, there is a completely different and largely algebraic way to compute $(\wedge^r V)^\vee$ in terms of V:

Lemma 1.1. *Let V be a nonzero finite-dimensional vector space over a field F, with $d = \dim V$. For $1 \le r < d$, the unique bilinear pairing*

$$\wedge^r(V) \times \wedge^{d-r}(V) \xrightarrow{\wedge} \wedge^d(V)$$

satisfying

$$(v_1 \wedge \cdots \wedge v_r, v_{r+1} \wedge \cdots \wedge v_d) \mapsto v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots \wedge v_d$$

is a perfect pairing with values in the 1-dimensional space $\wedge^d(V)$.

Proof. The existence and uniqueness of such a bilinear pairing (ignoring the perfectness aspects) was carried out in the handout on tensor algebra. For the perfectness, we work with bases. Letting $\{e_i\}$ be an ordered basis of V, we consider the basis of $\wedge^r(V)$ given by elementary wedge products $e_I = e_{i_1} \wedge \cdots \wedge e_{i_r}$ for $I = \{i_1, \dots, i_r\}$ a strictly increasing sequence of r integers between 1 and d. Likewise, $\wedge^{d-r}(V)$ has a basis given by the elementary wedge products $e_{I'}$ for strictly increasing sequences I' consisting of $n - r$ integers between 1 and d. Let $\theta = e_1 \wedge \cdots \wedge e_d$, so this is a basis of $\wedge^d(V)$. For any two I and I' as above, $e_I \wedge e_{I'} \in \wedge^d(V)$ vanishes if I and I' are not complementary (as otherwise this d-fold elementary wedge product has e_i appearing twice for $i \in I \cap I'$) whereas $e_I \wedge e_{I'} = \pm\theta$ in the complementary case. Hence, upon identifying $\wedge^d(V)$ with F via the basis θ the given pairing makes the basis $\{e_I\}$ of $\wedge^r(V)$ dual to the basis $\{e_{I'}\}$ of $\wedge^{d-r}(V)$ up to some signs. In particular, we get perfectness. ∎

In our situation, the orientation μ and the non-degenerate symmetric bilinear form on V provide a *canonical* basis vector for $\wedge^d(V)$, namely the positive unit vector Vol_μ, and so for $1 \le r < d$ the perfect pairing

$$\wedge^r(V) \times \wedge^{d-r}(V) \xrightarrow{\wedge} \wedge^d(V) = \mathbf{R}\mathrm{Vol}_\mu \simeq \mathbf{R}$$

sets up a natural isomorphism $\wedge^{d-r}(V) \simeq (\wedge^r(V))^\vee$; concretely, for $\eta \in \wedge^{d-r}(V)$ the associated functional ℓ_η on $\wedge^r(V)$ is given by

$$\eta' \wedge \eta = \ell_\eta(\eta')\mathrm{Vol}_\mu \tag{1}$$

in $\wedge^d(V)$ for $\eta' \in \wedge^r(V)$. In the special cases $r = d$ and $r = 0$ we make the same construction via the pairings

$$\wedge^0(V) \times \wedge^d(V) \to \wedge^d(V), \;\; \wedge^d(V) \times \wedge^0(V) \to \wedge^d(V)$$

defined by scalar multiplication against $\wedge^0(V) = \mathbf{R}$ (on which we take $\langle \cdot, \cdot \rangle_0$ to be the standard inner product $\langle x, y \rangle_0 = xy$). Thus, ℓ_η satisfies (1) even in these special cases.

Definition 1.2. For an oriented non-degenerate quadratic space V over $\mathbf{R}$ with dimension $d > 0$ and for any $0 \le r \le d$, the *rth Hodge star* operator

$$\star_r : \wedge^r(V) \simeq \wedge^{d-r}(V)$$

is defined as the composite of the isomorphisms $\wedge^r(V) \simeq (\wedge^r(V))^\vee \simeq \wedge^{d-r}(V)$ constructed above. Equivalently, for any $\omega, \eta \in \wedge^r(V)$,

$$\omega \wedge (\star_r \eta) = \langle \omega, \eta \rangle_r \mathrm{Vol}_\mu$$

in $\wedge^d(V)$.

2. Calculations

Lest the definition look like a mouthful of complications, we shall now show that in an orthonormal oriented frame the calculation of $\star_r$ is largely a matter of being careful with signs. First we treat two silly cases, $r = 0$ and $r = d$.

Example 2.1. We claim that $\star_0(1) = \mathrm{Vol}_\mu$ but $\star_d(\mathrm{Vol}_\mu) = (-1)^{d-a}$ in $\wedge^0(V) = \mathbf{R}$ if $\langle \cdot, \cdot \rangle$ on V has signature $(a, d-a)$. For the calculation of $\star_0(1) \in \wedge^d(V)$, since $1 \wedge \omega = \omega$ in $\wedge^n(V)$ for any $n \ge 0$ and any $\omega \in \wedge^n(V)$ (viewing $1 \in \mathbf{R} = \wedge^0(V)$) we have the identity

$$\star_0(1) = 1 \wedge \star_0(1) = \langle 1, 1 \rangle_0 \mathrm{Vol}_\mu = \mathrm{Vol}_\mu.$$

Taking $\omega = \eta = \mathrm{Vol}_\mu$ in the formula characterizing the Hodge star, we see that $\star_d(\mathrm{Vol}_\mu) \in \mathbf{R}$ is equal to $\langle \mathrm{Vol}_\mu, \mathrm{Vol}_\mu \rangle_d$. If $\{e_i\}$ is an orthonormal positive basis of V with $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$ then $\mathrm{Vol}_\mu = e_1 \wedge \cdots \wedge e_d$ has self-pairing $\prod \varepsilon_i = (-1)^{d-a}$, so indeed $\star_d(\mathrm{Vol}_\mu) = (-1)^{d-a}$. In particular, in the positive-definite case we have $d - a = 0$ and hence $\star_d(\mathrm{Vol}_\mu) = 1$ in such cases.

We wish to push this calculation further by computing $\star_r$ on elementary wedge products of r of the e_i's for $1 \le r \le d-1$ as well:

Example 2.2. Let $\{e_1, \dots, e_d\}$ be a positively oriented orthonormal basis of V, with $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$ and $\langle e_i, e_j \rangle = 0$ for $i \ne j$. In particular, $\mathrm{Vol}_\mu = e_1 \wedge \cdots \wedge e_d$. Fix $0 \le r \le d$ and an ordered set I pairwise distinct (not necessarily strictly increasing) indices $1 \le i_1, \dots, i_r \le d$. Let $I' = \{j_1, \dots, j_{d-r}\}$ be an enumeration of the complementary set in $\{1, \dots, d\}$ (not necessarily in strictly increasing order), and let $\varepsilon_{I,I'}$ denote the sign such that

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-r}} = \varepsilon_{I,I'} e_1 \wedge \cdots \wedge e_d = \varepsilon_{I,I'} \mathrm{Vol}_\mu.$$

For example, if $1 \le r \le d-2$ then we can always permute the set I' to arrange that $\varepsilon_{I,I'} = 1$; however, if $r = d-1$ then there is nothing to permute non-trivially and we may not be able to force $\varepsilon_{I,I'}$ to equal 1 without modifying the original enumeration I (which in practice is convenient to leave unchanged). Moreover, even if $\varepsilon_{I,I'} = 1$ it may happen that $\varepsilon_{I',I} \neq 1$: it is a simple exercise with wedge products, given shortly, to deduce that $\varepsilon_{I,I'}\varepsilon_{I',I} = (-1)^{r(d-r)}$. (Concretely, we have $\varepsilon_{I,I'} = 1$ precisely when the j_t's are ordered so that $\{i_1, \dots, i_r, j_1, \dots, j_{d-r}\}$ is an *even* permutation of $\{1, \dots, d\}$. Of course, it could then clearly happen that such evenness is destroyed upon moving the i_k's to the end of the sequence, and so $\varepsilon_{I',I}$ may fail to equal 1 even when $\varepsilon_{I,I'} = 1$.) For such i_k's and j_h's we claim that there is an identity

$$\star_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \varepsilon_{I,I'} \cdot (\varepsilon_{i_1} \cdots \varepsilon_{i_r}) \cdot e_{j_1} \wedge \cdots \wedge e_{j_{d-r}} \tag{2}$$

where empty products (in case $r = 0$ or $r = d$) are understood to be 1 and $\varepsilon_k = \langle e_k, e_k\rangle = \pm 1$. As a special case of (2), when $\langle \cdot, \cdot \rangle$ is positive-definite (so $\varepsilon_i = 1$ for all i) then

$$\star_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \varepsilon_{I,I'} e_{j_1} \wedge \cdots \wedge e_{j_{d-r}}.$$

We also claim that in general $\star_{d-r} \circ \star_r$ is multiplication by $(-1)^{r(d-r)+(d-a)}$, where $\langle \cdot, \cdot \rangle$ has signature $(a, d-a)$. (In practice one usually neglects to mention the subscript r and merely writes $\star$ with r understood from context, so one says $\star\star = (-1)^{r(d-r)+(d-a)}$, with the two $\star$'s not literally the same thing unless $2r = d$.) In particular, when $\langle \cdot, \cdot \rangle$ is positive-definite (so $d - a = 0$) then $\star_{d-r} \circ \star_r = (-1)^{r(d-r)}$.

The special cases $r = 0$ and $r = d$ were treated in the preceding example, so we can assume $r, d-r > 0$. By Corollary 2.3 in the handout on orientations, the r-fold wedge products $e_{i_1} \wedge \cdots \wedge e_{i_r}$ are an orthonormal basis of $\wedge^r(V)$. By definition of the Hodge star,

$$\omega \wedge \star_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \langle \omega, e_{i_1} \wedge \cdots \wedge e_{i_r}\rangle_r e_1 \wedge \cdots \wedge e_d \tag{3}$$

for all $\omega \in \wedge^r(V)$. We have

$$\star_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{1 \le k_1 < \cdots < k_{d-r} \le d} a_{k_1, \dots, k_{d-r}} e_{k_1} \wedge \cdots \wedge e_{k_{d-r}}$$

for coefficients to be determined. For each such strictly increasing sequence $\{k_1, \dots, k_{d-r}\}$, taking ω to be a wedge product of e_i's over all $i \notin K$ makes the left side of (3) equal $\pm a_{k_1, \dots, k_{d-r}}$ yet the right side vanishes when K is not the complement of $\{i_1, \dots, i_r\}$ (since the coefficient multiplier is zero for such an ω). Hence, the only surviving term is the one corresponding to the sequence $\{k_1, \dots, k_{d-r}\}$ complementary to $\{i_1, \dots, i_r\}$, and so

$$\star_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = c e_{j_1} \wedge \cdots \wedge e_{j_{d-r}}$$

for some c to be determined. By taking $\omega = e_{i_1} \wedge \cdots \wedge e_{i_r}$ in (3) we get

$$c e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-r}} = \det(\langle e_{i_a}, e_{i_b}\rangle)\varepsilon_{I,I'} e_1 \wedge \cdots \wedge e_d = \varepsilon_{I,I'} \cdot (\varepsilon_{i_1} \cdots \varepsilon_{i_r}) \cdot e_1 \wedge \cdots \wedge e_d$$

due to orthonormality of $\{e_1, \dots, e_d\}$ and how we defined $\varepsilon_{I,I'}$ in terms of the chosen enumeration $\{j_1, \dots, j_{d-r}\}$ of the set complementary to $\{i_1, \dots, i_r\}$. Hence, $c = \varepsilon_{I,I'}\varepsilon_{i_1} \cdots \varepsilon_{i_r}$, as desired.

Now we compute $\star_{d-r} \circ \star_r$. For any $\eta \in \wedge^i(V)$ and $\theta \in \wedge^j(V)$ we have $\eta \wedge \theta = (-1)^{ij}\theta \wedge \eta$, as one checks by using bilinearity to reduce to the case of elementary wedge products. Thus, we have

$$\begin{aligned} e_{j_1} \wedge \cdots \wedge e_{j_{d-r}} \wedge e_{i_1} \wedge \cdots \wedge e_{i_r} &= (-1)^{r(d-r)} e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-r}} \\ &= (-1)^{r(d-r)} \varepsilon_{I,I'} e_1 \wedge \cdots \wedge e_d. \end{aligned}$$

This implies $\varepsilon_{I',I} = (-1)^{r(d-r)}\varepsilon_{I,I'}$. When we compute $\star_{d-r}$ on $e_{j_1} \wedge \cdots \wedge e_{j_{d-r}}$ we apply the preceding paragraph with r and $\{i_1, \dots, i_r\}$ replaced by $d-r$ and $\{j_1, \dots, j_{d-r}\}$, so

$$\begin{aligned}(\star_{d-r} \circ \star_r)(e_{i_1} \wedge \cdots \wedge e_{i_r}) &= \varepsilon_{I,I'} \cdot (\varepsilon_{i_1} \cdots \varepsilon_{i_r}) \cdot \star_{d-r}(e_{j_1} \wedge \cdots \wedge e_{j_{d-r}}) \\ &= \varepsilon_{I,I'} \cdot (\varepsilon_{i_1} \dots \varepsilon_{i_r}) \cdot \varepsilon_{I',I}(\varepsilon_{j_1} \cdots \varepsilon_{j_{d-r}})(e_{i_1} \wedge \cdots \wedge e_{i_r}) \\ &= (-1)^{r(d-r)} \cdot \prod_{t=1}^{d} \varepsilon_t \cdot e_{i_1} \wedge \cdots \wedge e_{i_r}.\end{aligned}$$

This gives the desired result because $\prod_{t=1}^{d} \varepsilon_t = (-1)^{d-a}$ due to the orthonormality of the e_i's with respect to $\langle \cdot, \cdot \rangle$ and the definition of the signature of a quadratic form over $\mathbf{R}$.

Example 2.3. Let $V = \mathbf{R}^4$ with the standard ordered basis $\{e_1, \dots, e_4\}$, giving the standard orientation represented by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We pick $c > 0$ and give V the so-called Lorentz quadratic form $x_1^2 + x_2^2 + x_3^2 - c^2 x_4^2$, so the e_i's are pairwise orthogonal with e_1, e_2, e_3 unit vectors having self-pairing 1 whereas $\langle e_4, e_4 \rangle = -c^2$. In this case, $d = 4$ and the signature $(a, d-a)$ is $(3, 1)$. Since $c > 0$, for $e_4' = e_4/c$ the basis $\{e_1, e_2, e_3, e_4'\}$ is a positive orthonormal basis. Therefore the volume form is $e_1 \wedge e_2 \wedge e_3 \wedge e_4'$.

We wish to compute $\star_r$ for $1 \leq r \leq 3$. Since $\star_{d-r} \circ \star_r = (-1)^{r(d-r)+(d-a)} = (-1)^{r+1}$, we can restrict attention to $r = 1, 2$. Using the general calculations above, we get

$$\star_1(e_1) = e_2 \wedge e_3 \wedge e_4', \;\; \star_1(e_2) = -e_1 \wedge e_3 \wedge e_4', \;\; \star_1(e_3) = e_1 \wedge e_2 \wedge e_4', \;\; \star_1(e_4) = ce_1 \wedge e_2 \wedge e_3$$

in $\wedge^3(V)$. Likewise,

$$\star_2(e_1 \wedge e_2) = e_3 \wedge e_4', \;\; \star_2(e_1 \wedge e_3) = -e_2 \wedge e_4', \;\; \star_2(e_1 \wedge e_4) = -ce_2 \wedge e_3,$$

$$\star_2(e_2 \wedge e_3) = e_1 \wedge e_4', \;\; \star_2(e_2 \wedge e_4) = ce_1 \wedge e_3, \;\; \star_2(e_3 \wedge e_4) = -ce_1 \wedge e_2$$

in $\wedge^2(V)$. By direct calculation we see $\star_2^2 = -1 = (-1)^{2+1}$, in accordance with the general theory.

Example 2.4. Let $(V, \mu, \langle \cdot, \cdot \rangle)$ be an oriented non-degenerate quadratic space with dimension $d = 3$. The special feature of this case is that $d - 1 = 1 + 1$, so for $v, v' \in V = \wedge^1(V)$ we have $v \wedge v' \in \wedge^2(V) = \wedge^{d-1}(V)$, so $\star_{d-1}(v \wedge v') \in \wedge^1(V) = V$. We define the *cross product*

$$v \times v' = \star_2(v \wedge v') \in V.$$

This is an $\mathbf{R}$-bilinear alternating pairing $V \times V \to V$ since $\star_2$ is $\mathbf{R}$-linear and $(v, v') \mapsto v \wedge v'$ is $\mathbf{R}$-bilinear and alternating. In the special case that $\langle \cdot, \cdot \rangle$ is *positive-definite*, for a positive orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ Example 2.2 gives

$$\mathbf{i} \times \mathbf{j} = \star_2(\mathbf{i} \wedge \mathbf{j}) = \mathbf{k}, \;\; \mathbf{j} \times \mathbf{k} = \star_2(\mathbf{j} \wedge \mathbf{k}) = \mathbf{i}, \;\; \mathbf{k} \times \mathbf{i} = \star_2(\mathbf{k} \wedge \mathbf{i}) = \mathbf{j}$$

because $\mathbf{k} \wedge \mathbf{i} \wedge \mathbf{j} = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k} = \mathbf{j} \wedge \mathbf{k} \wedge \mathbf{i} = \mathrm{Vol}_\mu$ in $\wedge^3(V)$.

Returning to the general case (i.e., without positive-definiteness restrictions), let us verify a property of the $\mathbf{R}$-bilinear pairing $[v, v'] = v \times v'$: though it deserves to be called a "product" because it is distributive over addition in each variable (more specifically, it is $\mathbf{R}$-bilinear), it is *not* an associative composition law on V. Instead, it satisfies a higher-order relation, the (cyclic) *Jacobi identity* from the theory of Lie algebras:

$$[v, [v', v'']] + [v', [v'', v]] + [v'', [v, v']] = 0$$

in V. To prove this we can apply the isomorphism $\star_1$ without harm, so since $\star_1 \circ \star_2 = 1$ it is equivalent to say

$$v \wedge \star_2(v' \wedge v'') + v' \wedge \star_2(v'' \wedge v) + v'' \wedge \star_2(v \wedge v') = 0$$

in $\wedge^2(V)$. To verify this vanishing, note that trilinearity reduces us to the case when v, v', v'' lie in a fixed basis. The case when two of them coincide is trivial (as one term vanishes and the other two are negatives of each other), so it suffices to work with $\{v, v', v''\}$ a single fixed basis. Pick an orthogonal basis, so since $v \wedge v' \wedge v''$ is a nonzero multiple of the volume form it follows from the orthogonality and the definition of $\star_2$ that $\star_2(v' \wedge v'')$ is a multiple of v, whence $v \wedge \star_2(v' \wedge v'') = 0$. The same argument applied to the ordered triple $\{v', v'', v\}$ and $\{v'', v, v'\}$, so all three terms vanish for such special bases.