Math 396. Bundle pullback and transition matrices

1. Motivation

Let $f : X' \to X$ be a C^p mapping between C^p premanifolds with corners $(X, \mathscr{O})$ and $(X', \mathscr{O}')$, $0 \leq p \leq \infty$. Let $\pi : E \to X$ and $\pi' : E' \to X'$ be C^p vector bundles. Consider a C^p bundle morphism

$$\begin{tikzcd}
E' \arrow[r, "T"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

so for each $x' \in X'$ we get $\mathbf{R}$-linear maps $T|_{x'} : E'(x) \to E(f(x'))$. A basic example is the case when X, X', and f are of class C^{p+1} and $T = \mathrm{d}f : TX' \to TX$ is the induced total derivative mapping that is the old tangent map $\mathrm{d}f(x) : \mathrm{T}_{x'}(X') \to \mathrm{T}_{f(x')}(X)$ on fibers.

For each $x' \in X'$ we may use E and f to obtain a vector space $E(f(x'))$ determined by f and E, and it is natural to ask if these can be "glued" together to be the fibers of some C^p vector bundle $f^*(E) \to X'$ equipped with a bundle morphism $\widetilde{f} : f^*E \to E$ over $f : X' \to X$ that is the identity map $(f^*E)(x') = E(f(x')) \to E(f(x'))$ on fibers. (Strictly speaking, the notation $\widetilde{f}$ is abusive since it depends on E and not just f; hopefully this will not cause confusion.) Such a pair $(f^*E \to X', \widetilde{f})$ will be called a *pullback bundle* when it satisfies the universal property that for any C^p bundle morphism $T : E' \to E$ over $f : X' \to X$ there is a unique C^p vector bundle morphism $T' : E' \to f^*E$ over X' giving a factorization

$$\begin{tikzcd}
E' \arrow[r, "T'"'] \arrow[rr, bend left, "T"] \arrow[d, "\pi'"'] & f^*E \arrow[r, "\widetilde{f}"'] \arrow[d] & E \arrow[d, "\pi"] \\
X' \arrow[r, equal] & X' \arrow[r, "f"'] & X
\end{tikzcd}$$

so $T'|_{x'} : E'(x') \to (f^*E)(x') = E(f(x'))$ is exactly the map $T|_{x'}$. (The content is that the set-theoretic mapping T' is a C^p mapping.) In other words, the pullback bundle should promote bundle morphisms T between bundles over different base spaces to bundle morphisms T' between bundles over a common base space.

Note that it does not make sense to go in the reverse direction: if we are given a vector bundle on X' then most points in X are either not hit by any point of X' or are hit by more than one point (or perhaps infinitely many points) of X', so there is no reasonable way to use f to associate vector spaces to points of X by means of a vector bundle on X'. Our first aim in this handout is to develop the "pullback" construction, and to give some examples. The second aim of this handout is to provide a very classical description of C^p vector bundles of constant rank in terms of what are called "transition matrices". This is used quite a lot in the more advanced study of vector bundles, and is conveniefnt for explicitly describing many "linear algebra" bundle constructions via operations on matrices.

2. Pullback of bundles

Let $f : X' \to X$ be a C^p mapping between C^p premanifolds with corners, $0 \leq p \leq \infty$. Let $\pi : E \to X$ be a C^p vector bundle. We want to construct a C^p vector bundle $f^*(\pi) : f^*E \to X'$

equipped with a C^p vector bundle morphism

$$\begin{tikzcd}
f^*E \arrow[r, "\widetilde{f}"] \arrow[d, "f^*(\pi)"'] & E \arrow[d, "\pi"] \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

that is universal in the following sense: for any C^p vector bundle $\pi' : E' \to X'$ and any bundle morphism $T : E' \to E$ over $f : X' \to X$ there is a unique way to fill in a commutative diagram

$$\begin{tikzcd}
E' \arrow[r, "T'"'] \arrow[rr, bend left, "T"] \arrow[d, "\pi'"'] & f^*E \arrow[r, "\widetilde{f}"'] \arrow[d, "f^*(\pi)"] & E \arrow[d, "\pi"] \\
X' \arrow[r, equal] & X' \arrow[r, "f"'] & X
\end{tikzcd}$$

with T' a C^p bundle morphism over X'.

We define f^*E to be the disjoint union of sets

$$f^*E = \coprod_{x' \in X'} E(f(x'))).$$

A typical point in f^*E is denoted (x', v) with $v \in E(f(x'))$. There may be many points $x' \in X'$ with the same image $x \in X$, and the associated vector space $E(f(x'))$ sitting in f^*E is abstractly "the same" for all such $x' \in f^{-1}(x)$. For this reason, we must keep track of the indexing parameter $x' \in X'$ and not just the "bare" vector space $E(f(x'))$ when describing operations on points in f^*E. We define the map $f^*(\pi) : f^*E \to X'$ by $(x', v) \mapsto x' \in X'$.

We need to topologize f^*E and give it a C^p-structure such that (i) $f^*(\pi) : f^*E \to X'$ is a C^p map, (ii) the linear structure on the fibers $(f^*(\pi))^{-1}(x') = E(f(x'))$ satisfies the local triviality condition to make f^*E a C^p bundle over X', and (iii) the proposed universal property holds. Our procedure will be very similar to the method used in an earlier handout to define the vector bundle $V_{\mathscr{M}}$ associated to a locally free finite-rank $\mathscr{O}$-module $\mathscr{M}$ (by putting a suitable topology and C^p-structure on the disjoint union $\coprod_{x \in X} \mathscr{M}(x)$). Before we get into the details, we wish to emphasize that the local picture will be quite simple: if $\{U_i\}$ is an open covering of X for which there are C^p bundle isomorphisms $E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i}$ corresponding to a trivializing frame $\{s_k^{(i)}\}$ in $E(U_i)$, then for the opens $U_i' = f^{-1}(U_i)$ that cover X' the restriction $(f^*E)|_{U_i'}$ has a C^p trivialization given by the sections $u_i' \mapsto s_k^{(i)}(f(u_i')) \in E(f(u_i')) = (f^*E)(u_i')$ to $f^*(\pi) : f^*E \to X'$ over $U_i' \subseteq X'$. Roughly speaking, the "only" problem is to cleanly build the right bundle f^*E over X' giving rise to such local trivializations on fibers over $f^{-1}(U_i)$'s. This construction problem can be solved in several ways (all of which give answers that are uniquely isomorphic in accordance with the universal property), and in what follows we have chosen the construction that seems most elegant (in terms of minimizing non-canonical choices and the intervention of matrices) in view of our present knowledge.

To define the topology on f^*E we will use the method of gluing topologies. Consider pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi : E|_U \simeq U \times \mathbf{R}^n$ is a C^p isomorphism of bundles. (Of course, n may depend on U.) For the open set $U' = f^{-1}(U)$ in X' we get a bijection $\phi' : (f^*(\pi))^{-1}(U') \simeq U' \times \mathbf{R}^n$ over U' by using the linear isomorphism $(f^*(\pi))^{-1}(u') = (f^*E)(u') = E(f(u')) \simeq \mathbf{R}^n$ defined by $\phi|_{f(u')}$ over each $u' \in U'$. We wish to use ϕ' to transfer the topology of the product $U' \times \mathbf{R}^n$ to a topology on $(f^*(\pi))^{-1}(U')$, and to glue these topologies to topologize f^*E. Note that as we vary (ϕ, U) the opens U do cover X and hence the open preimages U' do cover X'.

To glue, as usual we have to verify two properties: for any (ϕ_1, U_1) and (ϕ_2, U_2) such that $U_1 \cap U_2$ is non-empty, we must prove (i) the overlap $(f^*(\pi))^{-1}(U_1') \cap (f^*(\pi))^{-1}(U_2') = (f^*(\pi))^{-1}(U_1' \cap U_2')$ is open in each of $(f^*(\pi))^{-1}(U_1')$ and $(f^*(\pi))^{-1}(U_2')$, and (ii) this overlap inherits the same subspace topology from both. (Note that if $U_1 \cap U_2$ is empty then so is its preimage $U_1' \cap U_2'$ under f.) Since U_1 meets U_2, the constant ranks for E on U_1 and U_2 are equal, say n. The open subset property (i) just says that ϕ_i' carries the overlap to an open subset of $U_i' \times \mathbf{R}^n$, and indeed it carries the overlap to the subset $(U_1' \cap U_2') \times \mathbf{R}^n$ in $U_i' \times \mathbf{R}^n$ that is obviously open. The agreement of subspace topologies in (ii) is equivalent to the assertion that the transition mapping

$$\phi_2' \circ (\phi_1')^{-1} : (U_1' \cap U_2') \times \mathbf{R}^n \to (U_1' \cap U_2') \times \mathbf{R}^n$$

is a homeomorphism. Explicitly, the map is given by $(u', v) \mapsto (u', (\phi_2 \circ \phi_1^{-1})|_{f(u')}(v))$ where

$$\phi_2 \circ \phi_1^{-1} : (U_1 \cap U_2) \times \mathbf{R}^n \simeq E|_{U_1 \cap U_2} \simeq (U_1 \cap U_2) \times \mathbf{R}^n$$

is the transition isomorphism over $U_1 \cap U_2$ for E. For $u \in U$ this latter map is given on u-fibers by a matrix $L(u) \in \mathrm{GL}_n(\mathbf{R})$ such that $L : U \to \mathrm{GL}_n(\mathbf{R})$ is a C^p mapping (this encodes that $\phi_2 \circ \phi_1^{-1}$ is a C^p mapping). Thus,

$$\phi_2' \circ (\phi_1')^{-1} : (u', v) \mapsto (u', ((L \circ f)(u'))(v)). \tag{1}$$

Since L is a continuous mapping and f is continuous, so is $L \circ f$. Thus, the formula for evaluation of a matrix on a vector implies (via (1)) that the self-map $\phi_2' \circ (\phi_1')^{-1}$ of $(U_1' \cap U_2') \times \mathbf{R}^n$ has continuous component functions and so is continuous; of course, the same argument applies to the inverse map (swap the roles of (ϕ_1, U_1) and (ϕ_2, U_2)), so we get the homeomorphism result.

We have put a topology on f^*E such that for each local trivialization $\phi : E|_U \simeq U \times \mathbf{R}^n$ as above, with n the constant rank for E over U, the subset $(f^*(\pi))^{-1}(U') \subseteq f^*E$ is open and the bijection $\phi' : (f^*(\pi))^{-1}(U') \to U' \times \mathbf{R}^n$ over U' induced by the linear fibral isomorphism

$$(f^*E)(u') = E(f(u')) \overset{\phi|_{f(u')}}{\to} \mathbf{R}^n$$

for all $u' \in U'$ is a homeomorphism. It follows that the projection $f^*(\pi) : f^*E \to X'$ is continuous since for each of the (U, ϕ)'s the part of f^*E over the open $U' \subseteq X'$ is homeomorphic to $U' \times \mathbf{R}^n$ (via ϕ') in a manner that carries the restriction of $f^*(\pi)$ over to the continuous standard projection $U' \times \mathbf{R}^n \to U'$. Thus, we have given f^*E a structure of topological vector bundle over X' because the ϕ''s are homeomorphisms that are linear on fibers (i.e., they provide local topological trivializations).

Note that when $U_1 \cap U_2$ is non-empty (so the constant ranks for E on U_1 and U_2 are equal, say n), the transition isomorphism $\phi_2' \circ (\phi_1')^{-1}$ is C^p (since by (1) it is given in terms of the composite $L \circ f : U_1' \cap U_2' \to \mathrm{GL}_n(\mathbf{R})$, with $f : U' \to U$ a C^p map and the matrix-valued function L describing the transition mapping $\phi_2 \circ \phi_1^{-1}$ also C^p on $U_1 \cap U_2$). Hence, the hypotheses for gluing C^p-structures (as in Lemma 1.2 in the handout on the equivalence between bundles and $\mathscr{O}$-modules) are satisfied for the topological space f^*E with its open cover by the $(f^*(\pi))^{-1}(U')$'s and the homeomorphisms given by the ϕ''s. We thereby get a unique C^p structure on f^*E with respect to which the maps $\phi' : (f^*E)|_{U'} \to U' \times \mathbf{R}^{n_U}$ are C^p isomorphisms for each C^p trivialization $\phi : E|_U \simeq U \times \mathbf{R}^{n_U}$. It follows (by working over the opens U' in X') that $f^*(\pi) : f^*E \to X'$ is a C^p mapping and the linear structures on the fibers $(f^*E)(x')$ make the ϕ''s be C^p trivializations that ensure f^*E is a C^p vector bundle over X'. This completes the construction of the C^p vector bundle structure on $f^*(\pi) : f^*(E) \to X'$.

Remark 2.1. By construction, if $U \subseteq X$ is *any* open subset (not necessarily with $E|_U$ trivial) and $U' \subseteq f^{-1}(U)$ is *any* open subset (not necessarily all of $f^{-1}(U)$), we have $(f^*E)|_{U'} = f^*_{U',U}(E|_U)$ with $f_{U',U} : U' \to U$ the C^p restriction of f. In this sense, the formation of f^*E is local on X and X'.

Lemma 2.2. *The set-theoretic mapping* $\widetilde{f} : f^*E \to E$ *over* $f : X' \to X$ *given on fibers by the linear identity map* $(f^*E)(x') = E(f(x')) \to E(f(x'))$ *is a* C^p *mapping.*

Proof. This problem is local over X and over X', so by Remark 2.1 we may assume that E is trivial. Let $\phi : E \simeq X \times \mathbf{R}^n$ be a choice of trivialization. By the construction of the C^p-structure on f^*E, the bijection $\phi' : f^*E \to X' \times \mathbf{R}^n$ given on fibers over $x' \in X'$ by

$$(f^*E)(x') = E(f(x')) \overset{\phi|_{f(x')}}{\simeq} \mathbf{R}^n$$

is a C^p trivialization over X'. Calculating on fibers shows that the C^p composite

$$f^*E \simeq X' \times \mathbf{R}^n \overset{f\times 1}{\to} X \times \mathbf{R}^n \simeq E$$

over $f : X' \to X$ is exactly the map $\widetilde{f}$, so indeed $\widetilde{f}$ is C^p. ■

Example 2.3. Before we verify that the pair $(f^*E \to X, \widetilde{f})$ satisfies the desired universal mapping property, we give two trivial examples. If $E \to X$ is a C^p vector bundle and $x \in X$ is a point, for the inclusion mapping $i : \{x\} \to X$ the pullback i^*E is the vector bundle over the 1-point space $\{x\}$ given by the vector space $E(x)$, with $\widetilde{i} : i^*E \to E$ the inclusion onto the fiber at x. If $U \subseteq X$ is an open subset and $i : U \to X$ is the inclusion map, then the vector bundle $i^*E \to U$ is the restriction $E|_U \to U$ (and $\widetilde{i} : i^*E \to E$ is the inclusion onto $E|_U$).

To give interesting examples with minimal mess, we verify the universal mapping property:

Theorem 2.4. *Let* $f : X' \to X$ *be a* C^p *mapping. Let* $E \to X$ *be a* C^p *vector bundle, and let* $E' \to X'$ *be a* C^p *vector bundle. If* $T : E' \to E$ *is a* C^p *bundle morphism over* $f : X' \to X$ *then there is a unique* C^p *bundle morphism* $T' : E' \to f^*E$ *over* X' *such that* $\widetilde{f} \circ T' = T$.

Set-theoretically, since $\widetilde{f}$ on x'-fibers induces the identity map on $E(f(x'))$ there is only one possibility for the fibral map $T' : E'(x') \to (f^*E)(x') = E(f(x'))$, namely $T|_{x'}$. Hence, the content of the theorem is that this T' is a C^p mapping.

Proof. The C^p problem is local on X', so by Remark 2.1 we may work locally on X' and X. Hence, the problem is reduced to the case when E and E' are trivial. Choose trivializations $\phi : E \simeq X \times \mathbf{R}^n$ and $\phi' : E' \simeq X' \times \mathbf{R}^{n'}$ over X and X' respectively. As we saw in the proof of Lemma 2.2, we get a C^p trivialization $f^*(\phi) : f^*(E) \simeq X' \times \mathbf{R}^n$ over X' that is the map $\phi|_{f(x')} : (f^*E)(x') = E(f(x')) \simeq \mathbf{R}^n$ on x'-fibers for each $x' \in X'$.

The C^p composite mapping

$$X' \times \mathbf{R}^{n'} \overset{\phi'^{-1}}{\simeq} E' \overset{T}{\to} E \overset{\phi}{\simeq} X \times \mathbf{R}^n$$

is given by $(x', v') \mapsto (f(x'), (T(x'))(v'))$ for a linear map $T(x') : \mathbf{R}^{n'} \to \mathbf{R}^n$, say given by a matrix $(a_{ij}(x'))$. The C^p property of this mapping is exactly the property that the matrix-valued mapping

$$[T] : X' \to \mathrm{Mat}_{n\times n'}(\mathbf{R})$$

given by $x' \mapsto (a_{ij}(x'))$ is C^p (i.e., the matrix-entries $a_{ij} : X' \to \mathbf{R}$ are C^p functions). Indeed, sufficiency follows from the C^p property of f and the formula for evaluating a matrix on a vector in Euclidean space, and necessity follows from chasing (x', e'_j) for standard basis vectors $e'_j \in \mathbf{R}^{n'}$. Define the C^p mapping $T' : E' \to f^*E$ to be the composite

$$E' \overset{\phi'}{\simeq} X' \times \mathbf{R}^{n'} \to X' \times \mathbf{R}^{n} \overset{f^*(\phi)^{-1}}{\simeq} f^*E$$

with middle map given by $(x', v') \mapsto (x', (a_{ij}(x'))(v'))$. This is a C^p map (over X') because each $a_{ij} : X' \to \mathbf{R}$ is C^p, and on fibers it gives the map $E'(x') \to (f^*E)(x') = E(f(x'))$ that is exactly $T|_{x'}$. Hence, we have built the desired C^p mapping. ■

An important feature of bundle pullback is that it is well-behaved with respect to bundle morphisms over the initial base space X:

Corollary 2.5. *Let $f : X' \to X$ be a C^p mapping and let $T : E_1 \to E_2$ be a C^p bundle morphism over X. There is a unique map $f^*(T) : f^*(E_1) \to f^*(E_2)$ between C^p bundles over X' such that on fibers over each $x' \in X$ it is the $\mathbf{R}$-linear map*

$$(f^*(E_1))(x') = E_1(f(x')) \overset{T|_{f(x')}}{\to} E_2(f(x')) = (f^*(E_2))(x').$$

(*In particular, if $E_2 = E_1$ and T is the identity then $f^*(T)$ is the identity.*) *The formation of $f^*(T)$ behaves well with respect to composition of C^p bundle morphisms in the sense that if $\widetilde{T} : E_2 \to E_3$ is another C^p bundle map, then $f^*(\widetilde{T} \circ T) = f^*(\widetilde{T}) \circ f^*(T)$ as maps from $f^*(E_1)$ to $f^*(E_3)$ over X'.*

Proof. The uniqueness of $f^*(T)$ is clear on the set-theoretic level, and calculating on fibers also verifies the compatibility with composition in T. The only real problem is to prove that the map $f^*(T)$ defined set-theoretically to be $T|_{f(x')}$ on x'-fibers is in fact a mapping of C^p bundles over X'. But it is easy to directly construct $f^*(T)$ as a C^p bundle mapping by using the universal property of pullback bundles: consider the composite diagram of C^p bundle maps

$$\begin{tikzcd}
f^*E_1 \arrow[r, "\widetilde{f}"] \arrow[d] & E_1 \arrow[r, "T"] \arrow[d] & E_2 \arrow[d] \\
X' \arrow[r, "f"'] & X \arrow[r, equal] & X
\end{tikzcd}$$

By the universal property of the pullback bundle f^*E_2, this composite map uniquely factors through a C^p vector bundle morphism $f^*E_1 \to f^*E_2$ over X', and checking on fibers shows that this morphism is $f^*(T)$. ■

3. Examples

Let $f : X' \to X$ be a C^p mapping between C^p premanifolds with corners, $0 \leq p \leq \infty$, and let $\pi : E \to X$ be a C^p vector bundle.

Example 3.1. We identify C^p-sections $s \in E(X)$ with C^p vector bundle morphisms $[s] : X \times \mathbf{R} \to E$ over X (given by $1 \mapsto s(x)$ on x-fibers). By the universal property, the composite bundle morphism

$$X' \times \mathbf{R} \overset{f \times 1}{\to} X \times \mathbf{R} \overset{[s]}{\to} E$$

over f uniquely factors through a bundle morphism $X' \times \mathbf{R} \to f^*E$ over X'. This latter map corresponds to a C^p-section in $(f^*E)(X')$ that we denote $f^*(s)$ and call the *pullback section.*

Concretely, on fibers we have $(f^*(s))(x') \in (f^*E)(x') = E(f(x'))$ is $s(f(x'))$. This fibral calculation shows that the resulting map $f^* : E(X) \to (f^*E)(X')$ is $\mathbf{R}$-linear, and even linear over the $\mathbf{R}$-algebra map $\mathscr{O}(X) \to \mathscr{O}'(X')$ given by composition with f. (That is, for $h \in \mathscr{O}(X)$,

$f^*(h \cdot s) = (h \circ f) \cdot f^*(s)$.) In the special case $E = X \times \mathbf{R}$ we have $f^*E = X' \times \mathbf{R}$ and via the equalities $\underline{E} = \mathscr{O}$ and $\underline{f^*E} = \mathscr{O}'$ the map $f^* : \mathscr{O}(X) = E(X) \to (f^*E)(X') = \mathscr{O}'(X')$ is $h \mapsto h \circ f$. In general, $f^* : E(X) \to (f^*E)(X')$ is *neither injective nor surjective* (even for $E = X \times \mathbf{R}$, let alone for more interesting examples such as $E = TX$).

For any open $U \subseteq X$ we have $(f^*E)|_{f^{-1}U} = f_U^*(E|_U)$ with $f_U : f^{-1}(U) \to U$ the C^p restriction of f (Remark 2.1), so we may apply the same construction to f_U instead of f to define $f^*(s) \in (f^*E)(f^{-1}U)$ for any $s \in E(U)$. Calculating on fibers shows that the formation of $f^*(s)$ is local: for open $U_0 \subseteq U$ and $s \in E(U)$, $(f^*s)|_{f^{-1}(U_0)} = f^*(s|_{U_0})$ in $(f^*E)(f^{-1}(U_0))$.

Example 3.2. Fiber calculations show that in the setup of Example 3.1, if $\{s_j\}$ is a trivializing frame for $E|_U$ then $\{f^*(s_j)\}$ is a trivializing frame for $(f^*E)(f^{-1}(U))$. That is, the vectors $(f^*(s_j))(x') = s_j(f(x')) \in (f^*E)(x') = E(f(x'))$ give a basis for each $x' \in f^{-1}(U)$ when the vectors $s_j(x) \in E(x)$ give a basis for each $x \in U$.

The next two (rather lengthy) examples rest on the theory of the tangent bundle, to be developed in a few lectures. Postpone reading these examples until that time; skip ahead to Example 3.5.

Example 3.3. Assume $1 \leq p \leq \infty$, and consider the total derivative mapping $\mathrm{d}f : TX' \to TX$ of C^{p-1} bundles (over $f : X' \to X$). By the universal property of pullback, we arrive at a unique C^{p-1} bundle morphism $TX' \to f^*(TX)$ over X' whose restriction to fibers over $x' \in X'$ is the old tangent map

$$\mathrm{d}f(x') : \mathrm{T}_{x'}(X') \simeq (TX')(x') \to (f^*(TX))(x') = (TX)(f(x')) \simeq \mathrm{T}_{f(x')}(X).$$

Hence, we may consider the C^{p-1} bundle morphism $TX' \to f^*(TX)$ over X' as merely another global repackaging of the collection of tangent mappings arising from $f : X' \to X$. In certain settings (but not all!) it is more convenient to work with this map instead of the C^{p-1} bundle morphism $\mathrm{d}f : TX' \to TX$ over $f : X' \to X$. As an important example, when f is an immersion then the mapping $TX' \to f^*(TX)$ of bundles over X' has fiber map $\mathrm{T}_{x'}(X') \hookrightarrow \mathrm{T}_{f(x')}(X)$ over $x' \in X'$, so this bundle map over X' encodes how the tangent spaces to X' "move" inside of the tangent spaces of X. This example will be a partial motivation for the notion of *subbundle* that we shall find to be very useful later on.

We must warn the reader of a common source of confusion. Passing to X'-sections, $\mathrm{d}f$ gives a mapping $(TX')(X') \to (f^*(TX))(X')$ that associates to any C^{p-1} vector field $\vec{v}'$ on X' the X'-section $f^*(\vec{v}')$ of $f^*(TX)$ whose value in each fiber $(f^*(TX))(x') = TX(f(x')) = \mathrm{T}_{f(x')}(X)$ is $\mathrm{d}f(x')(\vec{v}'(x'))$. This pullback section $f^*(\vec{v}')$ is just a repackaging of the data of the $\mathrm{d}f(x')(\vec{v}'(x'))$'s in the tangent spaces $\mathrm{T}_{f(x')}(X)$ for varying $x' \in X'$ and it has *nothing* to do with any global vector field on X. More specifically, although we have two maps

$$(TX')(X') \to (f^*(TX))(X'), \ \ (TX)(X) \to (f^*(TX))(X')$$

to the same target, these have *nothing* to do with each other: the first encodes the tangent mapping arising from f (and so it is a special construction adapted to the fact that our vector bundles are tangent bundles) whereas the second is a "general nonsense" mapping that makes sense with TX replaced by any C^{p-1} vector bundle on X (it does not encode any information related to tangent maps, for example). We remind the reader again that C^{p-1} vector fields on X (resp. X') do *not* "give rise" to C^{p-1} vector fields on X' (resp. X); the pullback bundle $f^*(TX)$ on X' is an abstract thing whose general X'-sections have *no* interpretation via vector fields on either X or X' (*cf.* the comments on f^* generally being neither injective nor surjective in Example 3.1).

Example 3.4. Just as the tangent bundle globalizes pointwise tangent spaces, we can use the concept of pullback to globalize the pointwise linear isomorphism

$$\mathrm{T}_{(x_1,x_2)}(X_1 \times X_2) \simeq \mathrm{T}_{x_1}(X_1) \oplus \mathrm{T}_{x_2}(X_2) \tag{2}$$

for tangent spaces on products of C^p premanifolds with corners X_1 and X_2 ($1 \leq p \leq \infty$). Let $\pi_j : X_1 \times X_2 \to X_j$ be the standard C^p projection map. We have C^{p-1} bundle diagrams

$$\begin{tikzcd} T(X_1 \times X_2) \arrow[r, "\mathrm{d}\pi_j"] \arrow[d] & T(X_j) \arrow[d] \\ X_1 \times X_2 \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

and so by the universal property of pullback we get C^{p-1} bundle morphisms $T(\pi_j) : T(X_1 \times X_2) \to \pi_j^*(T(X_j))$ over $X_1 \times X_2$. As is worked out in the handout on direct sums for vector bundles, these behave with respect to bundle morphisms exactly as direct sums of vector spaces behave with respect to linear maps. Thus, we get a C^{p-1} bundle morphism

$$T(\pi_1) \oplus T(\pi_2) : T(X_1 \times X_2) \to \pi_1^*(T(X_1)) \oplus \pi_2^*(T(X_2))$$

over $X_1 \times X_2$. On fibers over $(x_1, x_2) \in X_1 \times X_2$ this recovers the pointwise map

$$\mathrm{d}\pi_1(x_1, x_2) \oplus \mathrm{d}\pi_2(x_1, x_2) : \mathrm{T}_{(x_1,x_2)}(X_1 \times X_2) \to \mathrm{T}_{x_1}(X_1) \oplus \mathrm{T}_{x_2}(X_2)$$

that is exactly the standard isomorphism (2). Hence, $T(\pi_1) \oplus T(\pi_2)$ is a C^{p-1} bundle morphism that is an isomorphism on fibers, and so it is a C^{p-1} bundle isomorphism.

Concretely, if $U_j \subseteq X_j$ is an open set over which we have C^p coordinates $\{x_1^{(j)}, \dots, x_{n_j}^{(j)}\}$, then $T(X_j)|_{U_j}$ has the trivializing frame $\{\partial_{x_i^{(j)}}\}_{1 \leq i \leq n_j}$, and so its pullback $\pi_j^*(T(X_j))|_{U_1 \times U_2}$ has the trivializing frame given by the sections $\pi_j^*(\partial_{x_i^{(j)}})$'s for $1 \leq i \leq n_j$ (Example 3.2). By looking in fibers over $U_1 \times U_2$ and using Example 3.3 for π_j, the inverse of the isomorphism $T(\pi_1) \oplus T(\pi_2)$ on $U_1 \times U_2$-sections carries $(\pi_j^*(\partial_{x_i^{(j)}}), 0)$ to $\partial_{x_i^{(j)} \circ \pi_j}$, where the $n_1 + n_2$ functions

$$x_1^{(1)} \circ \pi_1, \dots, x_{n_1}^{(1)} \circ \pi_1, x_1^{(2)} \circ \pi_2, \dots, x_{n_2}^{(2)} \circ \pi_2 : U_1 \times U_2 \to \mathbf{R} \tag{3}$$

are the induced C^p coordinates on $U_1 \times U_2$. These $n_1 + n_2$ pullback sections give a trivializing frame for $\pi_1^*(T(X_1)) \oplus \pi_2^*(T(X_2))$, and so via (2) we thereby recover the trivializing frame for $T(X_1 \times X_2)|_{U_1 \times U_2}$ given by the partials with respect to the C^p coordinate system (3) on $U_1 \times U_2$.

Example 3.5. Suppose that Γ is a group equipped with right C^p-actions on X and on E that are free and properly discontinuous, and assume that π is Γ-equivariant in the sense that $\pi(v.\gamma) = \pi(v).\gamma$ for all $v \in E$ and $\gamma \in \Gamma$. Assume also that the action of $\gamma \in \Gamma$ induces the fibral bijection $E(x) \to E(x.\gamma)$ that is a linear isomorphism for all $x \in X$. By the homework we know that the map of C^p quotients $\overline{\pi} : \overline{E} = E/\Gamma \to X/\Gamma = \overline{X}$ is a C^p vector bundle. There is a natural commutative diagram

$$\begin{tikzcd} E \arrow[r, "\eta_E"] \arrow[d, "\pi"'] & \overline{E} \arrow[d, "\overline{\pi}"] \\ X \arrow[r, "\eta_X"'] & \overline{X} \end{tikzcd}$$

where the horizontal maps are the natural local C^p isomorphisms onto the Γ-quotients. On fibers over $x \in X$ and its image $\overline{x} = \eta_X(x)$ the induced map $E(x) \to \overline{E}(\overline{x})$ is clearly linear, and even an isomorphism. In particular, by the universal property of pullback bundles we see that η_E uniquely factors through a bundle morphism $E \to \eta_X^*(\overline{E})$ over X, and this latter morphism must be an isomorphism since it is an isomorphism on fibers over X (due to the fibral isomorphism property for η_E). To summarize: the pullback of $E/\Gamma \to X/\Gamma$ by the quotient map $X \to X/\Gamma$ recovers the original bundle $E \to X$.

An especially interesting case is that of the Möbius strip with infinite height. Let $f : S^1 \to C$ be the quotient by the antipodal map and let $M_\infty \to C$ be the Möbius strip with infinite height, so M_∞ is the quotient of the trivial bundle $\mathbf{R} \times S^1 \to S^1$ via the involution $(t, \theta) \mapsto (-t, \theta + \pi)$ that lies over the antipodal map $\theta \mapsto \theta + \pi$ on S^1. We have seen that M_∞ is a C^∞ line bundle over C that is not topologically trivial. However, its pullback $f^*(M_\infty) \to S^1$ is the trivial line bundle $\mathbf{R} \times S^1 \to S^1$. Thus, a C^∞ bundle with positive rank and no non-vanishing continuous sections may become a trivial C^∞ bundle after a very mild pullback (such as the "double covering" $S^1 \to C$).

Another property of bundle pullback that is very useful in practice is that it is well-behaved with respect to composition in the map along which we are forming the pullback:

Corollary 3.6. *Let $g : X'' \to X'$ and $f : X' \to X$ be C^p mappings between C^p premanifolds with corners, $0 \le p \le \infty$, and let E be a C^p vector bundle on X. There is a unique isomorphism of C^p vector bundles $c_{g,f,E} : (f \circ g)^*(E) \simeq g^*(f^*(E))$ over X'' given on fibers over each $x'' \in X''$ by the composite linear isomorphism*

$$((f \circ g)^*(E))(x'') \simeq E((f \circ g)(x'')) = E(f(g(x''))) \simeq (f^*(E))(g(x'')) \simeq (g^*(f^*E))(x''). \tag{4}$$

(As an example, for opens $U \subseteq X$ and $s \in E(U)$ this carries $(f \circ g)^*(s)$ to $g^*(f^*(s))$ on the level of sections over $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$.)

Moreover, these isomorphisms are transitive in the sense that if $h : X''' \to X''$ is a third C^p mapping then the two composite isomorphisms

$$(f \circ g \circ h)^*E \simeq h^*((f \circ g)^*E) \simeq h^*g^*f^*E, \quad (f \circ g \circ h)^*E \simeq (g \circ h)^*f^*E \simeq h^*g^*f^*E$$

coincide. That is, $h^*(c_{g,f,E}) \circ c_{h,f \circ g,E} = c_{h,g,f^*(E)} \circ c_{g \circ h,f,E}$ as isomorphisms from $(f \circ g \circ h)^*(E)$ to $h^*(g^*(f^*(E)))$.

Proof. The transitivity condition may be checked on fibers, where it is just the associativity of composition for maps of sets. Also, the uniqueness of $c_{g,f,E}$ is immediate because its effect on fibers is specified. The problem is therefore to prove that the set-theoretic map $c_{g,f,E}$ that is linear on fibers over X'' is a C^p mapping. Once again, we use the universal property of bundle pullback to recreate $c_{g,f,E}$ as a C^p mapping: the composite bundle morphism $g^*(f^*E) \to f^*E \to E$ over $f \circ g : X'' \to X$ uniquely factors through a bundle morphism $g^*(f^*E) \to (f \circ g)^*(E)$ over X''. Checking on x''-fibers gives the linear isomorphism inverse to (4), so we have built a C^p vector bundle isomorphism whose C^p inverse is $c_{g,f,E}$. ■

4. Transition matrices

Fix a positive integer n. Perhaps the most concrete (and classical) way to describe a rank-n C^p vector bundle on X is through what are called transition matrices. This is a vector-bundle analogue of local C^p charts on C^p premanifolds with corners; the purpose of the intervention of matrices is to encode the linear structure on the fibers of vector bundles over the base space X.

Let $\pi : E \to X$ be a rank-n vector bundle of class C^p, $0 \le p \le \infty$, and let $\{U_i\}$ be a trivializing cover for E. Pick C^p isomorphisms of C^p vector bundles $\phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^n$ over U_i. Over $U_{ij} = U_i \cap U_j$ we have two trivializing isomorphisms via the restrictions of ϕ_i and ϕ_j. That is, we have two C^p bundle isomorphisms

$$\phi_i : E|_{U_{ij}} \simeq U_{ij} \times \mathbf{R}^n, \quad \phi_j : E|_{U_{ij}} \simeq U_{ij} \times \mathbf{R}^n.$$

We thereby get a "transition isomorphism" of C^p bundles

$$\phi_{ji} = \phi_j \circ \phi_i^{-1} : U_{ij} \times \mathbf{R}^n \simeq E|_{U_{ij}} \simeq U_{ij} \times \mathbf{R}^n.$$

This induces linear fibral isomorphisms $T_{ji}(x) : \mathbf{R}^n \simeq \mathbf{R}^n$ over each $x \in U_{ij}$, so we have

$$\phi_j \circ \phi_i^{-1} : (x, v) \mapsto (x, T_{ji}(x)(v))$$

with $T_{ji}(x) \in \mathrm{GL}_n(\mathbf{R})$.

Lemma 4.1. *Fix i and j. The map of sets $T_{ji} : U_{ij} \to \mathrm{GL}_n(\mathbf{R})$ given by $x \mapsto T_{ji}(x)$ is a C^p mapping.*

Proof. Writing $T_{ji}(x) = (a_{rs}(x))_{1 \leq r,s \leq n}$ for functions $a_{rs} : U_{ij} \to \mathbf{R}$, we have to prove that these functions are C^p on U_{ij}. Consider the inclusion $\iota_s : \mathbf{R} \to \mathbf{R}^n$ onto the sth coordinate axis and the standard projection $\pi_r : \mathbf{R}^n \to \mathbf{R}$ onto the rth coordinate axis. The composite C^p mapping

$$U_{ij} \times \mathbf{R} \overset{1 \times \iota_s}{\to} U_{ij} \times \mathbf{R}^n \overset{\phi_{ji}}{\simeq} U_{ij} \times \mathbf{R}^n \overset{1 \times \pi_r}{\to} U_{ij} \times \mathbf{R}$$

is exactly $(x, c) \mapsto (x, a_{rs}(x)c)$, so using the C^p mapping $x \mapsto (x, 1)$ from U_{ij} to $U_{ij} \times \mathbf{R}$ we get that the map $x \mapsto (x, a_{rs}(x))$ from U_{ij} to $U_{ij} \times \mathbf{R}$ is a C^p mapping. Composing with the C^p projection $U_{ij} \times \mathbf{R} \to \mathbf{R}$ gives that the function $a_{rs} : U_{ij} \to \mathbf{R}$ is C^p. ■

To summarize, using the trivializing open cover $\{U_i\}$ and the choices of C^p isomorphisms of bundles $\phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^n$ we have built a collection of C^p mappings

$$T_{ji} : U_i \cap U_j \to \mathrm{GL}_n(\mathbf{R})$$

that we call the "transition matrices" for the trivialization of the $E|_{U_i}$'s via the ϕ_i's. *Note that this has nothing to do with local C^p-charts on the base space X.* These collections of matrix-valued C^p mappings T_{ji} are not unrelated to each other: they satisfy the *triple overlap* condition

$$T_{ij} \cdot T_{jk} = T_{ik}$$

as matrix-valued mappings $U_i \cap U_j \cap U_k \to \mathrm{GL}_n(\mathbf{R})$. Indeed, this comes down to the elementary associativity calculation

$$(\phi_i \circ \phi_j^{-1}) \circ (\phi_j \circ \phi_k^{-1}) = \phi_i \circ \phi_k^{-1}$$

on $(U_i \cap U_j \cap U_k) \times \mathbf{R}^n$ and the fact that matrix multiplication encodes composition of linear maps.

The next result shows that this procedure can be reversed:

Theorem 4.2. *Let $\{U_i\}$ be an open covering of X, and let $T_{ji} : U_i \cap U_j \to \mathrm{GL}_n(\mathbf{R})$ be C^p mappings that satisfy the triple overlap condition $T_{ij}(x) \cdot T_{jk}(x) = T_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$ for all i, j, k.*

There exists a rank-n C^p vector bundle $\pi : E \to X$ with trivializations $\phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^n$ satisfying $\phi_j \circ \phi_i^{-1} = T_{ji}$ on $(U_i \cap U_j) \times \mathbf{R}^n$ for all i and j. Moreover, the data consisting of E and the ϕ_i's is unique in the following sense: if $\pi' : E' \to X$ with trivializations $\phi_i' : E'|_{U_i} \simeq U_i \times \mathbf{R}^n$ is another such structure likewise giving rise to the T_{ji}'s, then there is a unique C^p bundle isomorphism $f : E' \simeq E$ over X such that $\phi_i \circ f|_{U_i} = \phi_i'$ as C^p bundle isomorphisms from $E'|_{U_i}$ to $U_i \times \mathbf{R}^n$ for all i.

Before we prove the theorem, we make some remarks. A real nuisance in this theorem is that the trivializing covering $\{U_i\}$ and the specific trivializing isomorphisms ϕ_i play such a prominent role even though it is the vector bundle that is the primary focus of interest. A full treatment of the approach to vector bundles through the language of transition matrices requires the characterization of exactly which changes in the U_i's and the ϕ_i's do not "change" (at least up to isomorphism) the bundle we construct from the data of the transition matrices. Moreover, to actually work with this viewpoint one has to certainly do more, such as express the notion of bundle morphism (and many operations with vector bundles) in terms of transition matrices.

This is a long story, and so we will not delve into it any further here. Our point here is to simply record that the viewpoint of transition matrices satisfying a triple overlap condition is adequate to construct all vector bundles with constant rank and it is very widely used in practice and in important computations with vector bundles.

Proof. Let us first prove uniqueness. Suppose that $\pi : E \to X$ and $\pi' : E' \to X$ with respective C^p trivializations $\phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i}$ and $\phi'_i : E'|_{U_i} \simeq U_i \simeq \mathbf{R}^{n_i}$ over each U_i are both solutions to our existence problem. We want to find the asserted unique C^p bundle isomorphism $f : E' \simeq E$ over X satsifying $\phi_i \circ f|_{U_i} = \phi'_i$ over U_i for each i. The restriction $f_i : E'|_{U_i} \simeq E|_{U_i}$ of f over U_i has no choice but to be $\phi_i^{-1} \circ \phi'_i$, so each f_i is uniquely determined and hence f is uniquely determined. To actually build f, we *define* the bundle isomorphism $f_i : E'|_{U_i} \simeq E|_{U_i}$ over U_i to be $\phi_i^{-1} \circ \phi'_i$ and we seek to glue the f_i's over X. Over $U_i \cap U_j$, we claim that the restrictions of f_i and f_j to C^p maps

$$E'|_{U_i \cap U_j} \to E|_{U_i \cap U_j} \subseteq E$$

do coincide. This says that $\phi_i^{-1} \circ \phi'_i = \phi_j^{-1} \circ \phi'_j$ as maps from $E'|_{U_i \cap U_j}$ to $E|_{U_i \cap U_j}$, or equivalently that $\phi'_i \circ (\phi'_j)^{-1} = \phi_i \circ \phi_j^{-1}$ as self-maps of $(U_i \cap U_j) \times \mathbf{R}^n$. By hypothesis on the data $(E, \{\phi_i\})$ and $(E', \{\phi'_i\})$, both of these latter two self-maps are equal to $(x, v) \mapsto (x, (T_{ji}(x))(v))$. Hence, we can indeed (uniquely) glue the C^p bundle maps f_i over U_i to a C^p mapping $f : E' \to E$ over X that is a bundle morphism (linearity may be checked on fibers over each $x \in X$, since each x lies in some U_i). The same procedure applies to the inverses f_i^{-1} and so gives a bundle map $f' : E \to E'$ of class C^p that is an inverse to f (as may be checked on fibers over X).

Having settled the uniqueness aspect of the problem, we now turn to existence. There is a general procedure called "gluing of topological spaces" that we have to use. In essence, what we want to do is to glue $U_i \times \mathbf{R}^n$ to $U_j \times \mathbf{R}^n$ by pasting the open set $(U_i \cap U_j) \times \mathbf{R}^n \subseteq U_i \cap \mathbf{R}^n$ onto the open set $(U_i \cap U_j) \times \mathbf{R}^n \subseteq U_j \times \mathbf{R}^n$ via the fibrally linear homeomorphism $(x, v) \mapsto (x, (T_{ji}(x))(v))$ over $U_i \cap U_j$ for all i and j. To make this precise, we need to introduce a big topological space with an equivalence relation.

Let $S = \coprod_i (U_i \times \mathbf{R}^n)$ be the disjoint union of the topological spaces $U_i \times \mathbf{R}^n$. We declare a subset of S to be *open* when it meets each $U_i \times \mathbf{R}^n$ in an open set. This is obviously a topology on S. We define an equivalence relation on S as follows: for two points $s = (u_i, v_i) \in U_i \times \mathbf{R}^n$ and $s' = (u_j, v_j) \in U_j \times \mathbf{R}^n$ in S, we say $s \sim s'$ if and only if u_i and u_j are equal to a common point $x \in U_i \cap U_j$ and $v_j = (T_{ji}(x))(v_i)$ in $\mathbf{R}^n$. Let us check that this is an equivalence relation on S. Certainly $s \sim s$ since $T_{ii}(x)$ is the identity matrix for all $x \in U_i$ (this follows from the triple overlap condition $T_{ii}(x) \cdot T_{ii}(x) = T_{ii}(x)$ and the invertibility of $T_{ii}(x)$), and likewise if $s \sim s'$ then $s' \sim s$ because for all $x \in U_i \cap U_j$ we have that $T_{ji}(x)$ and $T_{ij}(x)$ are inverse to each other (their product is $T_{ii}(x)$, the identity matrix). Finally, suppose $s = (u_i, v_i)$, $s' = (u_j, v_j)$, and $s'' = (u_k, v_k)$ satisfy $s \sim s'$ and $s' \sim s''$. The points u_i and u_j coincide in $U_i \cap U_j$ and the points u_j and u_k coincide in $U_j \cap U_k$, so all three points are equal to a common point $x \in U_i \cap U_j \cap U_k$. By hypothesis $v_j = (T_{ji}(x))(v_i)$ and $v_k = (T_{kj}(x))(v_j)$ in $\mathbf{R}^n$, so the triple overlap condition gives

$$v_k = (T_{kj}(x) \cdot T_{ji}(x))(v_i) = (T_{ki}(x))(v_i)$$

in $\mathbf{R}^n$. This gives $s \sim s''$ as desired. We therefore have an equivalence relation on S.

Let E denote the set of $\sim$-equivalence classes in S. The projections $\pi_i : U_i \times \mathbf{R}^n \to U_i \subseteq X$ have the property that if $s = (u_i, v_i)$ and $s' = (u_j, v_j)$ are points in S with $s \sim s'$ then $\pi_i(s) = \pi_j(s')$ in X. Hence, we get a well-defined map of sets $\pi : E \to X$ that sends a $\sim$-equivalence class to the common point $\pi_i(s) \in X$ for any representative point $s \in U_i \times \mathbf{R}^n \subseteq S$ in the equivalence class for any i. For $x \in X$, consider the fiber $\pi^{-1}(x)$ in E. We claim that this has a natural structure of $\mathbf{R}$-vector space. The representatives for the equivalence classes in $\pi^{-1}(x) \subseteq E$ are points of the form $(x, v) \in U_i \times \mathbf{R}^n$ with U_i containing x. Since two points $(x, v), (x, v') \in U_i \times \mathbf{R}^n$ are $\sim$-equivalent *if and only if* $v = v'$ in $\mathbf{R}^n$ (as $T_{ii}(x)$ is the identity matrix), we conclude that for each U_i containing x, every point $e \in \pi^{-1}(x)$ has a *unique* representative of the form $(x, v_i(e)) \in U_i \times \mathbf{R}^n$ with $v_i(e) \in \mathbf{R}^n$.

For any two points $e, e' \in \pi^{-1}(x)$ and $c, c' \in \mathbf{R}$, we wish to define $ce + c'e' \in \pi^{-1}(x)$ to be the $\sim$-equivalence in E class represented by $(x, cv_i(e) + c'v_i(e')) \in U_i \times \mathbf{R}^n \subseteq S$ for any U_i containing x. The crucial issue is to show that this definition does not depend on the choice of such U_i. If U_j also contains x then we have the relations $v_j(e) = (T_{ji}(x))(v_i(e))$ and $v_j(e') = (T_{ji}(x))(v_i(e'))$ by the *definition* of the equivalence relation $\sim$, so by $\mathbf{R}$-*linearity* of $T_{ji}(x) : \mathbf{R}^n \simeq \mathbf{R}^n$ we get

$$(T_{ji}(x))(cv_i(e) + c'v_i(e')) = c \cdot (T_{ji}(x))(v_i(e)) + c' \cdot (T_{ji}(x))(v_i(e')) = cv_j(e) + c'v_j(e'),$$

whence

$$(x, cv_i(e) + c'v_i(e')) \sim (x, cv_j(e) + c'v_j(e'))$$

in S. This confirms that our proposed definition of $ce + c'e' \in \pi^{-1}(x) \subseteq E$ is well-posed, and calculation with representatives (say using a fixed U_i containing x) shows that it defines a structure of n-dimensional $\mathbf{R}$-vector space on $\pi^{-1}(x)$.

So far we have constructed a map of sets $\pi : E \to X$ and we have put a structure of n-dimensional $\mathbf{R}$-vector space on the fibers of π. Consider the composite mapping of sets

$$U_i \times \mathbf{R}^n \hookrightarrow S \twoheadrightarrow E.$$

By the definition of π (and of the equivalence relation $\sim$), this composite map is a bijection from $U_i \times \mathbf{R}^n$ onto $\pi^{-1}(U_i)$. Moreover, the resulting bijection

$$\psi_i : U_i \times \mathbf{R}^n \simeq \pi^{-1}(U_i)$$

carries the standard projection $U_i \times \mathbf{R}^n \to U_i$ over to the restriction $\pi_i : \pi^{-1}(U_i) \to U_i$ of $\pi : E \to X$, and the induced bijection $\mathbf{R}^n \simeq \pi^{-1}(x)$ of fibers over any $x \in U_i$ is an $\mathbf{R}$-linear isomorphism. This linearity is due to how the $\mathbf{R}$-vector space structure on $\pi^{-1}(x)$ was defined.

We topologize E as follows: a subset $\Sigma \subseteq E$ is *open* if and only if its preimage in $S = \coprod_i (U_i \times \mathbf{R}^n)$ is open. This is clearly a topology on E.

Lemma 4.3. *The subsets $\pi^{-1}(U_i) \subseteq E$ are open and the maps ψ_i are homemorphisms.*

Proof. The preimage of $\pi^{-1}(U_i)$ in S meets each $U_j \times \mathbf{R}^n$ in the subset $(U_i \cap U_j) \times \mathbf{R}^n$ that is certainly open. Hence, all $\pi^{-1}(U_i)$'s are open in E. To prove that ψ_i is a homemorphism, we have to prove that a subset $\Sigma \subseteq U_i \times \mathbf{R}^n$ is open if and only if $\psi_i(\Sigma) \subseteq \pi^{-1}(U_i)$ is open, which is to say that $\psi_i(\Sigma) \subseteq E$ has preimage in S that meets each $U_j \times \mathbf{R}^n$ in an open subset. In view of how $\sim$ and ψ_i are defined, the preimage of $\psi_i(\Sigma)$ in $U_j \cap \mathbf{R}^n$ is the image of $\Sigma \cap ((U_i \cap U_j) \times \mathbf{R}^n)$ under the mapping

$$(U_i \cap U_j) \times \mathbf{R}^n \simeq (U_i \cap U_j) \times \mathbf{R}^n$$

defined by $(x, v) \mapsto (x, (T_{ji}(x))(v))$. This mapping is a homeomorphism, and even a C^p isomorphism, because it is obviously C^p and using $T_{ij}(x)$ instead of $T_{ji}(x)$ gives a C^p inverse. Hence, since a subset of $(U_i \cap U_j) \times \mathbf{R}^n$ is open in this product if and only if it is open in $U_j \cap \mathbf{R}^n$, our problem is equivalent to the obvious assertion that a subset $\Sigma \subseteq U_i \times \mathbf{R}^n$ is open if and only if Σ meets $(U_i \cap U_j) \times \mathbf{R}^n \subseteq U_i \cap \mathbf{R}^n$ in an open subset for every j (including $j = i$!). ■

We have given E a topology such that the bijections $\psi_i : U_i \times \mathbf{R}^n \to \pi^{-1}(U_i)$ over U_i are homeomorphisms that are $\mathbf{R}$-linear isomorphisms on fibers over all points in U_i for all i. In fact, E also has a structure of topological vector bundle (with π as its structure map to the base X). To see that $\pi : E \to X$ is continuous, we note that the U_i's are an open cover of X with $\pi^{-1}(U_i) \subseteq E$ an open set, so the $\pi^{-1}(U_i)$'s cover E and so by the local nature of continuity it suffices to prove continuity for the restrictions $\pi_i : \pi^{-1}(U_i) \to U_i$ of π. Since the bijection $\psi_i : U_i \times \mathbf{R}^n \to \pi^{-1}(U_i)$ is a homeomorphism (with the source given its product topology), it is equivalent to show that $\pi_i \circ \psi_i : U_i \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous. By the definition of ψ_i and π_i, this map is the standard projection that is certainly continuous. With π now shown to be continuous, we likewise see that the ψ_i's provide local topological trivializations, so $\pi : E \to X$ is indeed a topological vector bundle.

The transition mapping $\psi_j^{-1} \circ \psi_i$ as a self-map of $(U_i \cap U_j) \times \mathbf{R}^n$ is exactly the map $(x, v) \mapsto (x, (T_{ji}(x))(v))$. Hence, if we can promote E to a C^p vector bundle over X such that the ψ_i's are C^p trivializations then we will have solved the existence problem because the transition matrices linking these trivializations ψ_i and ψ_j over $U_i \cap U_j$ are given by the map $T_{ji} : U_i \cap U_j \to \mathrm{GL}_n(\mathbf{R})$ for all i and j. (We take $\phi_i = \psi_i^{-1}$ in the statement of the existence problem.) To give the topological vector bundle E over X a structure of C^p bundle making the ψ_i's local C^p trivializations, we first note that (as has already been observed) the transition mappings $\psi_j^{-1} \circ \psi_i$ are C^p automorphisms of $(U_i \cap U_j) \times \mathbf{R}^n$ that are linear on fibers, and more explicitly are given by the mapping $(x, v) \mapsto (x, (T_{ji}(x))(v))$ with $T_{ji} : U_i \cap U_j \to \mathrm{GL}_n(\mathbf{R})$ a C^p mapping (i.e., its matrix-entry functions are C^p functions). The same applies to the inverse transition mapping, so we conclude that the transition mappings are C^p isomorphisms. Hence, by the gluing lemma for C^p-structures (from Lemma 1.2 in the handout on equivalence between bundles and $\mathscr{O}$-modules) there is a unique C^p-structure on E with respect to which the fibrally linear ψ_i's are C^p isomorphisms. Using this C^p-structure, it remains to check that $\pi : E \to X$ is a C^p map. This goes exactly as in the proof that π is continuous (working over the U_i's and now using that the ψ_i's are C^p isomorphisms and that each standard projection $U_i \times \mathbf{R}^n \to U_i$ is C^p). ■