Math 396. Orientations on bundles and manifolds

1. Orientation atlases and orientation forms

Recall that an *oriented trivializing atlas* for $E \to M$ is a trivializing atlas

$$\{(U_i, \phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i})\}$$

such that whenever $U_i \cap U_j$ is non-empty (so $n_i = n_j$) the ordered bases of $E(m)$ induced by ϕ_i and ϕ_j for each $m \in U_i \cap U_j$ lie in the same orientation class (i.e., the change of basis matrix for the m-fiber has positive determinant). In other words, a trivializing atlas is oriented if it gives a well-defined orientation on each fiber $E(m)$ (a non-trivial condition only for those m in a double overlap $U_i \cap U_j$ for $i \neq j$). Thus, when given an oriented trivializing atlas we get an orientation on every $E(m)$. (Note that this would not make sense if we allowed $E(m) = 0$.) We say that two oriented trivializing atlases are *equivalent* if, for each $m \in M$, they put the same orientation on the fiber $E(m)$. As we saw in class, this really is an equivalence relation, and in each equivalence class there is a unique maximal element (in the sense of containing all trivializing frames from all oriented atlases in the equivalence class). These maximal elements are called *orientation atlases* for $E \to M$. Thus, two orientation atlases are equal if and only if they define the same orientation on each fiber $E(m)$ (as equivalence is the same as equality for the maximal elements).

We define an *orientation form* on $E \to M$ to be a nowhere-vanishing global section ω of $\det E$, which is to say a trivializing section of the line bundle $\det E$. The reason for the name is that for each $m \in M$ the nonzero fiber value $\omega(m) \in \det(E(m))$ specifies a connected component of $\det(E(m)) - \{0\}$ and so puts an orientation on the vector space $E(m)$. Two orientation forms ω and ω' on $E \to M$ are non-vanishing C^p sections of the same line bundle $\det(E)$, and so $\omega = f\omega'$ for a unique non-vanishing C^p function f on M. We say ω and ω' are *equivalent* (denoted $\omega \sim_+ \omega'$) if f is everywhere positive. This says exactly that the orientations on $E(m)$ specified by $\omega(m)$ and $\omega'(m)$ coincide for every $m \in M$. It is clear that $\sim_+$ is an equivalence relation, and the orientation forms in a common $\sim_+$-equivalence class put the same orientation on each fiber $E(m)$.

We wish to set up a natural bijection between $\sim_+$-equivalence classes of orientation forms on $E \to M$ and orientation atlases on $E \to M$. The comparison will be via the orientations induced on the fibers $E(m)$ from each piece of data.

Theorem 1.1. *Pick an orientation atlas on $E \to M$, and let μ_m be the resulting orientation on $E(m)$ for each $m \in M$. There is a unique $\sim_+$-equivalence class of orientation forms that induce the orientation μ_m on $E(m)$ for all $m \in M$.*

Pick a $\sim_+$-equivalence class of orientation forms on $E \to M$, and let μ'_m be the resulting orientation on $E(m)$ for each $m \in M$. There is a unique orientation atlas on $E \to M$ that induces the orientation μ'_m on $E(m)$ for all $m \in M$.

These procedures define inverse bijections between the set of orientation atlases and the set of $\sim_+$-equivalence classes of orientation forms on $E \to M$

Proof. Since equality of orientation atlases and $\sim_+$-equivalence of orientation forms may be checked by considering orientations on fibers of $E \to M$, the final part of the theorem (concerning inverse bijections) follows from the rest. Similarly, the uniqueness aspects in the first two claims in the theorem are clear; the only real issue is that of existence.

Pick an orientation form $\omega \in (\det E)(M)$, and let μ_m be the resulting orientation on $E(m)$ for each $m \in M$. (This orientation of $E(m)$ only depends on ω up to $\sim_+$-equivalence.) We seek to find an oriented trivializing atlas for $E \to M$ that also induces the orientation μ_m on $E(m)$ for each $m \in M$. Consider connected open sets $U \subseteq M$ over which E is trivial; such opens do cover M. Let $\{U_i\}$ be a collection of such opens that cover M. For each i, let $\{s_{1i}, \dots, s_{n_i,i}\}$ in $E(U_i)$ be a trivialization of $E|_{U_i}$, so $s_{1i} \wedge \dots \wedge s_{n_i,i}$ is a non-vanishing section of $(\det E)(U_i)$. Thus, $s_{1i} \wedge \dots \wedge s_{n_i,i} = f_i \omega|_{U_i}$ for a non-vanishing C^p function f_i on U_i. But U_i is *connected*, so f_i has constant sign. Replacing s_{1i} with $-s_{1i}$ if necessary, we can arrange that $f_i > 0$ on U_i, so we have built a trivializing frame over U_i that induces the orientation μ_m on $E(m)$ for each $m \in U_i$. Let $\phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i}$ be the trivialization of $E|_{U_i}$ that we have just built.

I claim that the data $\{(U_i, \phi_i : E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i})\}$ defined by the above trivializations is an oriented trivializing atlas that induces the orientation μ_m on $E(m)$ for each $m \in M$ (and so the associated orientation atlas gives what we need). Since two ordered bases of a vector space have change-of-basis matrix with positive determinant if and only if they give rise to the same orientation, the trivializing atlas is oriented if for each U_i and $m \in U_i$, the orientation put on $E(m)$ by the trivialization ϕ_i over U_i depends only on m and not on the particular i such that U_i contains m (so if $m \in U_i \cap U_j$ then the ith and jth trivializations put the same orientation on $E(m)$). We can do better, as is required: the ith trivialization puts the orientation μ_m on $E(m)$ for all $m \in U_i$. This follows from how we built the trivialization of $E|_{U_i}$ above. This completes the passage from equivalence classes of orientation forms to orientation atlases.

To go in the reverse direction, we pick an orientation atlas on $E \to M$, say putting orientation μ_m on $E(m)$ for each $m \in M$, and we seek to build a trivializing section $\omega \in (\det E)(M)$ such that $\omega(m) \in \det(E(m)) - \{0\}$ lies in the component distinguished by μ_m for each $m \in M$. We first consider the local version of the problem, and then we will globalize using a C^p partition of unity (hence the need to assume M is second countable and Hausdorff, not merely a premanifold with corners). Let $\{U_i\}$ be the opens in the chosen orientation atlas, so they are an open covering of M and we have trivializations $E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i}$ giving rise to the orientation μ_m on $E(m)$ for each $m \in U_i$. Passing to the top exterior power, we get a trivialization $(\det E)|_{U_i} \simeq U_i \times \wedge^{n_i}(\mathbf{R}^{n_i})$ such that for each $m \in U_i$ the induced isomorphism on m fibers is $\det(E(m)) \simeq \wedge^{n_i}(\mathbf{R}^{n_i})$ with the μ_m-component on the left going over to the "standard" component on the right (for $e_1 \wedge \dots \wedge e_{n_i}$, with $\{e_j\}$ the standard basis of $\mathbf{R}^{n_i}$). Thus, the constant section $\underline{e}_1 \wedge \dots \wedge \underline{e}_{n_i}$ goes over to a trivializing section ω_i of $(\det E)|_{U_i}$ such that on each fiber over $m \in U_i$ the vector $\omega_i(m) \in \det(E(m)) - \{0\}$ lies in the component for μ_m. In particular, $\omega_i|_{U_i \cap U_j} = f_{ij} \omega_j|_{U_i \cap U_j}$ with $f_{ij} > 0$ for all i, j.

It now suffices to find $\omega \in (\det E)(M)$ such that for each i and each $m \in U_i$ the point $\omega(m) \in \det(E(m))$ is a *positive* multiple of $\omega_i(m)$ (as then $\omega(m)$ is nonzero, and is in the component for μ_m for every $m \in M$). We certainly cannot expect the ω_i's to extend to global sections of the line bundle $\det E \to M$, but we can use a C^p partition of unity to get enough for our purposes. More generally, consider the following problem. Let $L \to M$ be a C^p line bundle and let $\{U_i\}$ be an open covering for which there exist trivializing sections $s_i \in L(U_i)$ such that for each $m \in U_i \cap U_j$ the nonzero points $s_i(m)$ and $s_j(m)$ in the line $L(m)$ lie in the same half-line; that is, the unique C_p function $f_{ij} : U_i \cap U_j \to \mathbf{R}^\times$ satisfying $s_i|_{U_i \cap U_j} = f_{ij} \cdot s_j|_{U_i \cap U_j}$ is *positive* on the (possibly disconnected!) overlap $U_i \cap U_j$ for all i, j. We claim that there exists $s \in L(M)$ such that for all i and all $m \in U_i$, the nonzero elements $s(m), s_i(m) \in L(m)$ are positive multiples of each other. Applying such a result to $L = \det E$ and the ω_i's would complete the proof.

To build such an s, we shall use a C^p partition of unity. More specifically, let $\{\phi_\alpha\}$ be such a partition of unity subordinate to the cover $\{U_i\}$, so the supports K_α of the ϕ_α's are compact and form a locally finite collection in M with $K_\alpha \subseteq U_{i(\alpha)}$ for some $i(\alpha)$ for each α. In particular, the C^p section $\phi_\alpha s_{i(\alpha)}$ on $U_{i(\alpha)}$ has support in the compact (hence closed!) subset K_α in M, whence we may glue it with the 0-section over the open $M - K_\alpha$ to build an element in $L(M)$ that we also suggestively denote $\phi_\alpha s_{i(\alpha)}$. (In other words, we "extend by zero".) The sum $s = \sum_\alpha \phi_\alpha s_{i(\alpha)}$ is locally finite over M (since the collection of K_α's is locally finite in M), so it makes sense and gives an element in $L(M)$. I claim that this does the job.

Pick any i and $m \in U_i$. Since $\sum \phi_\alpha(m) = 1$, some $\phi_{\alpha_0}(m)$ is positive and hence $m \in K_{\alpha_0} \subseteq U_{i(\alpha_0)}$. Thus, $m \in U_i \cap U_{i(\alpha)}$, so in $L(m)$ the elements $s_i(m)$ and $s_{i(\alpha_0)}(m)$ are in the same half-line. Hence, to show $s(m)$ is a positive multiple of $s_i(m)$ in $L(m)$ it is equivalent to show that $s(m)$ is a positive multiple of $s_{i(\alpha_0)}(m)$. We may therefore instead pick any α_0 and $m \in U_{i(\alpha_0)}$ such that $\phi_{\alpha_0}(m) > 0$, and it suffices to prove that $s(m)$ is a positive multiple of $s_{i(\alpha_0)}(m)$. (In particular, $s(m) \neq 0$!) By definition, $s(m)$ is a sum of the finitely many nonzero elements $\phi_\alpha(m)s_{i(\alpha)}(m)$ with $\phi_\alpha(m) \neq 0$ (so $m \in K_\alpha \subseteq U_{i(\alpha)}$ and hence $s_{i(\alpha)}(m)$ *makes sense*). Each such $\phi_\alpha(m)$ is positive, and since $m \in U_{i(\alpha)} \cap U_{i(\alpha_0)}$ it follows that $s_{i(\alpha)}(m), s_{i(\alpha_0)}(m) \in L(m) - \{0\}$ lie in the same half-line. In particular, $s_{i(\alpha)}(m)$ is a positive multiple of $s_{i(\alpha_0)}(m)$. Thus, each of the finitely many nonzero terms in the sum defining $s(m)$ are positive multiples of $s_{i(\alpha_0)}(m)$, whence their sum $s(m)$ in $L(m)$ is also a positive multiple of $s_{i(\alpha_0)}(m)$ as desired. ■

2. Orientation on manifolds with corners

Let M be a C^p manifold with corners, $1 \leq p \leq \infty$. In class we defined an *orientation* on M to be the specification of a $\sim_+$-equivalence of orientation forms on the C^{p-1} tangent bundle (a certain equivalence class of nonvanishing global sections of $\det(TM)$). Over connected components M_i of dimension 0 (i.e., isolated points), we have $\det(TM_i) = M_i \times \mathbf{R}$ by definition and hence an orientation form is just a non-vanishing function up to everywhere-positive multiplier. But M_i is just an isolated point, so we see that on 0-dimensional components the data of an orientation is just the data of a sign: positive or negative. This is important in order that the general Stokes' Theorem in the context of curves with boundary will recover the Fundamental Theorem of Calculus.

By the preceding work, we see that on positive-dimensional components of M, the data of an orientation is "the same" as the data of an orientation atlas on TM, or equivalently an oriented trivializing atlas for TM taken up to equivalence (respecting the orientations on the fibers $\mathrm{T}_m(M)$). This is all "general nonsense" that makes sense for arbitrary vector bundles (not just TM) over manifolds with corners. But the tangent bundle is special: on positive-dimensional components we can use local coordinates on the manifold to define special kinds of local trivializations of the bundle: when given a C^p chart $\{x_i\}$ on an open subset $U \subseteq M$, we get a trivialization of the C^{p-1} vector bundle $TM|_U$ via the ordered frame $\{\partial_{x_i}\}$. Hence, it is natural (and very useful!) in the *special case* of the tangent bundle to find a translation of the notion of orientation in terms of certain kinds of C^p atlases on the base manifold M (at least ignoring isolated points). We shall now carry out this translation.

We consider a C^p manifold with corners M with constant positive dimension n, and $p \geq 1$ (what follows may be applied separately on the positive-dimensional connected components of a general M). We consider C^p atlases $\mathscr{A}$ for M. We call such an atlas *oriented* if for any two members $(U, \{x_i\})$ and $(U', \{x'_j\})$ in $\mathscr{A}$ the transition matrix $(\partial_{x'_j} x_i) : U \cap U' \to \mathrm{GL}_n(\mathbf{R})$ has everywhere-positive determinant on the (possibly disconnected!) overlap $U \cap U'$. A C^p atlas $\mathscr{A} = \{(U_\alpha, \{x_i^{(\alpha)}\})\}$ of M gives rise to a trivializing atlas for the C^{p-1} vector bundle TM via the trivializing frame $\{\partial_{x_i^{(\alpha)}}\}$ on $TM|_{U_\alpha}$ for all α. We call this the trivializing atlas the one *associated* to $\mathscr{A}$, and its transition matrix over $U_\alpha \cap U_\beta$ is $(\partial_{x_j^{(\beta)}}(x_i^{(\alpha)}))$. Thus, $\mathscr{A}$ is an oriented atlas if and only if its associated trivializing atlas for $TM \to M$ is oriented.

If $\mathscr{A}$ and $\mathscr{B}$ are two oriented C^p-atlases on M, we call them *equivalent* if for any two members $(U, \{x_i\})$ of $\mathscr{A}$ and $(V, \{y_j\})$ of $\mathscr{B}$, over the (possibly disconnected) overlap $U \cap V$ the non-vanishing determinant $\det(\partial_{y_j} x_i)$ is everywhere positive. This *is* an equivalence relation, as it is exactly the property that the associated oriented trivializing atlases for $TM \to M$ are equivalent. In particular, we see that if $\mathscr{A}$ and $\mathscr{B}$ are oriented atlases then to check if they are equivalent it suffices to check positivity on overlaps from collections of charts in each of $\mathscr{A}$ and $\mathscr{B}$ that cover M. Moreover, when equivalence holds then the combined atlas $\mathscr{A} \cup \mathscr{B}$ is oriented as well. Likewise, if we take the union of all oriented C^p atlases in a fixed equivalence class, we arrive at another oriented atlas in the same class. Thus, each equivalence class of oriented C^p atlases has a unique maximal member (analogous to the notion of maximal atlas on a manifold, except that we are imposing an extra positivity condition on the overlaps of the charts). In this way, by passing from an oriented C^p chart to the associated oriented trivializing atlas for the C^{p-1} vector bundle $TM \to M$, we get an *injection* from the set of maximal oriented C^p atlases of M to the set of orientation atlases on the tangent bundle $TM \to M$. The C^{p-1} vector bundle TM admits many trivializing atlases aside from those that come from C^p-charts on M, does every equivalence class of oriented trivializing atlases for TM (i.e., local trivializations whose transition matrices on double overlaps have everywhere-positive determinant) contain one that comes from a C^p chart? Yes:

Theorem 2.1. *Each oriented trivializing atlas $\{(U_i, \phi_i : TM|_{U_i} \simeq U_i \times \mathbf{R}^n)\}$ for the C^{p-1} vector bundle $TM \to M$ is equivalent to one coming from an oriented C^p atlas of M.*

Proof. Let μ_m be the orientation on $\mathrm{T}_m(M)$ arising from the chosen oriented trivializing atlas for $TM \to M$. We seek to build an oriented C^p atlas $\{(U'_\alpha, \{x_j^{(\alpha)}\})\}$ on M such that for all α the ordered basis $\partial_{x_j^{(\alpha)}}|_m$ of $\mathrm{T}_m(M)$ is in the orientation class μ_m for each $m \in U_\alpha$. Pick a C^p-chart $(U, \{x_j\})$ with U a connected open subset of some U_i; such U's cover M. We have two trivializations for the C^{p-1} vector bundle TM over $U \subseteq U_i$, namely $\{\partial_{x_j}\}$ and $\phi_i|_U : TM|_U \simeq U \times \mathbf{R}^n$. The transition matrix relating the two trivializations has non-vanishing determinant on the open set U, and so by connectivity of U it follows that this determinant has constant sign across U. By replacing x_1 with $-x_1$ if necessary we can arrange that this sign is positive. Hence, for each $m \in U$ the ordered basis $\{\partial_{x_j}|_m\}$ of $\mathrm{T}_m(M)$ is in the same orientation class as the ordered basis determined by the isomorphism $\phi_i|_m : \mathrm{T}_m(M) \simeq \mathbf{R}^n$. By definition, this latter orientation class is μ_m, and hence the C^p chart $(U, \{x_j\})$ is "compatible" with the μ_m's for $m \in U$.

Such C^p charts cover M, and so give an atlas $\mathscr{A}$ such that the associated trivializing atlas for the C^{p-1} vector bundle $TM \to M$ has each trivialization defining the orientation μ_m on every point m in its domain, whence this trivializing atlas is oriented. Hence, $\mathscr{A}$ is an oriented C^p atlas of M, and its associated oriented trivializing atlas for $TM \to M$ is clearly equivalent to the one chosen at the outset (inducing μ_m on the fiber $\mathrm{T}_m(M)$ for each $m \in M$). ■

We conclude that in the case of a C^p manifold with corners M without isolated points, $1 \le p \le \infty$, the notion of orientation for M can be described in two ways: in the language of oriented C^p charts on M and the language of oriented C^{p-1} trivializating atlases for $TM \to M$.

Remark 2.2. It is a perhaps surprising fact that orientability of a C^p manifold in fact only depends on the underlying topological manifold and not on the differentiable structure. Of course, if the manifold is not at least C^1 then there is no tangent bundle, and so in such cases the approach to orientation of manifolds through tangent bundles does not make sense. Using certain techniques in algebraic topology, it is possible to give a definition of "orientation" for topological manifolds that is equivalent to the definition above when a C^p structure is given. It is beyond the level of this course to discuss this topic any further.

It turns out that although there are non-orientable manifolds (such as the Möbius strip), manifolds that arise as tangent bundles are always orientable in a *canonical* manner:

Theorem 2.3. *Let M be a C^p manifold with corners, $p \geq 2$. The C^{p-1} manifold with corners TM is orientable, and in fact has a canonical orientation.*

Before we give the proof, it is instructive to conbsider a simple example. Take $M = V$ to be a positive-dimensional vector space, so TM is naturally isomorphic to $V \oplus V$ as a smooth manifold. Such a manifold has a canonical orientation: if $\mathbf{v} = \{v_1, \dots, v_n\}$ is an ordered basis of V, then we consider the ordered basis $\{(v_1, 0), \dots, (v_n, 0), (0, v_1), \dots, (0, v_n)\}$ of $V \oplus V$ as providing global linear coordinates. If $\mathbf{w} = \{w_1, \dots, w_n\}$ is another ordered basis of V, then we get an analogous ordered basis of $V \oplus V$. What is the transition matrix relating these? It is the $2n \times 2n$ block matrix whose upper-left and lower-right $n \times n$ blocks are the transition matrix T from $\mathbf{v}$ to $\mathbf{w}$ coordinates, and whose off-diagonal $n \times n$ blocks vanish. Hence, its determinant is $(\det T)^2 > 0$. Thus, all such linear coordinates on $V \oplus V$ built in this way define the same orientation class on $V \oplus V$, whence $V \oplus V$ is canonically oriented as a smooth manifold.

Proof. We may assume M is connected, or more generally has constant dimension n. If $n = 0$ then TM is the zero-bundle over M and so $\det(TM) = M \times \mathbf{R}$. This is canonically oriented: we just the sign $+$ at each point. Now we shall assume $n > 0$. Let $(U, \{x_i\})$ be a C^p chart on M, so the open set $TM|_U$ in TM gets a C^{p-1} chart given by the associated trivialization $TM|_U \simeq U \times \mathbf{R}^n$ using the ∂_{x_i}'s. The standard ordered system of linear coordinates on $\mathbf{R}^n$ and the x-coordinates on U thereby give a $2n$-tuple of coordinates on $TM|_U$. By the very construction of TM, this is a C^{p-1} chart for TM; the coordinates are suggestively denoted $\{x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}\}$. Namely, for any point $\xi \in TM|_U$ over $u \in U$ corresponding to a vector $\vec{v} \in \mathrm{T}_u(M)$, the associated coordinates of ξ are the $x_i(u)$'s followed by the ordered list of coefficients in the linear expansion of $\vec{v}$ with respect to the ordered basis of $\partial_{x_j}|_u$'s in $\mathrm{T}_u(M)$.

As we vary $(U, \{x_i\})$ through all C^p charts on M, this procedure gives a C^{p-1} atlas for TM (not generally maximal). It is a canonical C^{p-1} atlas on the manifold with corners TM, and we claim it is oriented as such (and so gives a canonical orientation). If $(U, \{x_i\})$ and $(U', \{x'_j\})$ are two C^p coordinate charts on M, then we get C^{p-1} coordinate charts on $TM|_U$ and $TM|_{U'}$ via the above recipe, so on the overlap $TM|_{U \cap U'}$ there is a Jacobian transition matrix relating the restrictions of the two C^{p-1} coordinate systems. One calculates, much like in the example considered above this proof, that the transition matrix (in the appropriate direction) is a $2n \times 2n$ matrix whose upper-left and lower-right $n \times n$ blocks are the same matrix, $(\partial_{x'_j}(x_i))$, and whose other two off-diagonal $n \times n$ blocks vanish. Hence, the determinant is $\det(\partial_{x'_j} x_i)^2 > 0$. Thus, the canonical C^{p-1} atlas we have built on TM is an oriented atlas. ■

3. Orientation on a torus

We now take up a concrete example by studying orientations on the torus $S^1 \times S^1$ viewed as a smooth submanifold T in $\mathbf{R}^3$ via the trigonometric embedding

$$(\theta, \psi) \mapsto ((a + r\cos\theta)\cos\psi, (a + r\cos\theta)\sin\psi, r\sin\psi)$$

with $0 < |a| < r$ for fixed a and r. In $\mathbf{R}^3$, the surface T is centered at $(0, 0, 0)$ and parallel to the xy-plane with axis of symmetry given by the z-axis; the inner radius is $r - |a|$ and the outer radius is $r + |a|$. The parameter θ is the angle measure for the circles of radius r that wrap around T from the "inner" circle to the "outer" circle and back again, whereas the parameter ψ measures the angle for the projection of the point onto the xy-plane.

We give $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ its "counterclockwise" orientation corresponding to the trivializing section ∂_θ of $T(S^1) = \det(T(S^1))$, so on the product $S^1 \times S^1$ we get a product orientation associated to the trivializing frame of vector fields $\{\partial_\theta, \partial_\psi\}$. There is also another orientation, namely $\{\partial_\psi, \partial_\theta\}$, that is opposite the first one. Since T is connected, these give the two distinct orientations on T. The chosen embedding into $\mathbf{R}^3$ distinguishes the two factors, and with the embedding as we have chosen it we want to work out which of the two orientations "corresponds" to the outward-normal orientation (via the recipe from class for relating orientations on a hypersurface in an oriented Riemannian manifold and unit normal fields on the hypersurface).

We shall relate the orientation $\{\partial_\psi, \partial_\theta\}$ on $S^1 \times S^1 \simeq T$ to a "unit normal field" along T in $\mathbf{R}^3$, and it will turn out to be the "outward" one. Using the above formula for embedding $S^1 \times S^1$ into $\mathbf{R}^3$, we compute that at each point of T in the associated tangent space of $\mathbf{R}^3$ we have

$$\partial_\theta = -r\sin\theta\cdot\cos\psi\partial_x - r\sin\theta\cdot\sin\psi\partial_y + r\cos\theta\partial_z, \;\; \partial_\psi = -(a+r\cos\theta)\sin\psi\partial_x + (a+r\cos\theta)\cos\psi\partial_y.$$

Hence, at each point $\xi \in T$ we use the inner product $\langle\cdot,\cdot\rangle_\xi$ on $\mathrm{T}_\xi(\mathbf{R}^3) \simeq \mathbf{R}^3$ to compute

$$\langle\partial_\theta, \partial_\theta\rangle = r^2, \;\; \langle\partial_\theta, \partial_\psi\rangle = 0, \;\; \langle\partial_\psi, \partial_\psi\rangle = (a + r\cos\theta)^2.$$

(The middle equality reflects the fact that the integral curves for the vector fields ∂_θ and ∂_ψ meet orthogonally in the surface $T \subseteq \mathbf{R}^3$, or rather that their velocity vectors at a crossing point are orthogonal in the tangent plane to T viewed inside of the tangent space to $\mathbf{R}^3$ at that point.) The associated metric tensor on T is therefore given by

$$\mathrm{d}s^2 = r^2\mathrm{d}\theta\otimes\mathrm{d}\theta + (a + r\cos\theta)^2\mathrm{d}\psi\otimes\mathrm{d}\psi.$$

Note that pullback along the rotation $(\theta, \psi) \mapsto (\theta, \psi + \psi_0)$ preserves the metric tensor, but pullback along the rotation $(\theta, \psi) \mapsto (\theta + \theta_0, \psi)$ generally distorts it. This reflects the *geometry* of how T sits in $\mathbf{R}^3$ (or rather, the geometric meaning of the map from $S^1 \times S^1$ into $\mathbf{R}^3$), namely that rotation in the ψ-direction is really rotation about the axis of symmetry (z-axis) of T and rotation in the θ-direction moves circular curves (integral curves for ∂_θ) on the "inner part" of T to the "outer part" and vice-versa, certainly distorting lengths in the process.

For $\xi \in T$, the vector cross product $\partial_\psi|_\xi \times \partial_\theta|_\xi$ in $\mathrm{T}_\xi(\mathbf{R}^3) \simeq \mathbf{R}^3$ is readily computed to be

$$r(a + r\cos\theta)\cos\phi\cos\psi\partial_x + r(a + r\cos\theta)\cos\theta\sin\psi\partial_y + r\sin\theta\cdot(a + r\cos\theta)\partial_z.$$

This has length $r(a + r\cos\theta)$, so the associated "unit normal vector" $\vec{N}(\xi)$ is

$$\vec{N}(\xi) = \cos\theta(\xi)\cos\psi(\xi)\partial_x|_\xi + \cos\theta(\xi)\sin\psi(\xi)\partial_y|_\xi + \sin\theta(\xi)\partial_z|_\xi.$$

By the "right-hand rule" for vector cross-products, the ordered basis $\{\partial_\psi|_\xi, \partial_\theta|_\xi, \vec{N}(\xi)\}$ is a positive basis of $\mathrm{T}_\xi(\mathbf{R}^3) \simeq \mathbf{R}^3$ with respect to the standard orientation $\{\partial_x, \partial_y, \partial_z\}$ on $\mathbf{R}^3$ (or, if we prefer to make mathematics be independent of how many hands we have, we may simply compute the determinant of the 3×3 transition matrix between these bases of $\mathrm{T}_\xi(\mathbf{R}^3)$ to be $r(a + r\cos\theta) > 0$). We conclude that the ordered frame $\{\vec{N}(\xi), \partial_\psi|_\xi, \partial_\theta|_\xi\}$ of $\mathrm{T}_\xi(\mathbf{R}^3)$ is also positive for each $\xi \in T$.

Hence, under the recipe for passing between orientation on a hypersurface and a unit normal field (when working in an oriented Riemannian manifold), we conclude that the unit normal field $\vec{N}$ along T in $\mathbf{R}^3$ is the one that "corresponds" to the orientation $\{\partial_\psi, \partial_\theta\}$ on $S^1 \times S^1$ with coordinates (θ, ψ) via the identification $S^1 \times S^1 \simeq T$ chosen at the outset. Is this the outward unit normal field or the inward one? At the right-most point $\xi = (a + r, 0, 0)$ on the x-axis (corresponding to $(\theta, \psi) = (0, 0)$, which is to say the point $(1, 1) \in S^1 \times S^1$ when S^1 is viewed in $\mathbf{C}^\times$ in the usual manner), we have $\vec{N}(\xi) = \partial_x|_\xi$. This is visibly the outward direction at ξ on T, and so we have built the orientation corresponding to the outward unit normal on T (via the chosen parameterization of T by $S^1 \times S^1$).

4. The manifold underlying a bundle

We conclude our general tour through orientability by addressing the following natural question: if $E \to M$ is a vector bundle, then how does orientability for E as a vector bundle over M relate to orientability of the manifold with corners underlying E? That is, if we view E separately as a vector bundle or as an abstract manifold with corners then do we get different answers to the (different!) questions "is it orientable"? In the case when the base manifold M is orientable (which is to say, has orientable tangent bundle TM, or more concretely admits an oriented coordinate atlas) then there is no ambiguity:

Theorem 4.1. *Let M be an orientable smooth manifold with corners, and $E \to M$ a C^∞ vector bundle. Then E is orientable in the vector-bundle sense if and only if its underlying smooth manifold with corners is orientable in the sense of abstract smooth manifolds with corners.*

Proof. We may assume that E has everywhere positive rank, as there is nothing to do over the connected components on which E has rank 0.

First assume that E is orientable as a vector bundle, and fix an orientation as such. We choose an oriented coordinate atlas on M, taking the coordinate domains $U_i \subseteq M$ to be sufficiently small (as we may) so that each $E|_{U_i}$ admits a trivialization compatible with the chosen orientation on the bundle $E \to M$. On each $E|_{U_i}$ we now have a coordinate system: the concatenation of the oriented coordinates on U_i and followed by the dual linear coordinates in the "vertical" direction via the oriented trivialization of the bundle $E|_{U_i}$. These two coordinate systems (in the horizontal and vertical directions) have nothing to do with each other, so the transition matrices (on the open overlaps $E|_{U_i \cap U_j} = (E|_{U_i}) \cap (E|_{U_j})$ in E) are block matrices whose off-diagonal blocks vanish and whose two diagonal blocks are the transition matrices for the individual coordinate systems. That is, the relevant determinant is the product of the Jacobian determinant for change of coordinates on $U_i \cap U_j$ and the determinant of the matric of functions that expresses the transformation of one bundle trivialization into another from our oriented bundle atlas on $E \to M$ over the U_i's. Both such determinants are positive, so their product is positive. Hence, we have found an oriented coordinate atlas on E as an abstract manifold with corners, so it is orientable as such.

Now we go in the reverse direction, assume E is oriented as an abstract manifold with corners. We need to construct an oriented trivialization for the bundle $E \to M$. Fix an orientation μ on E as an abstract manifold with corners, so all open subsets of E are likewise oriented. We fix an oriented coordinate atlas for M whose open coordinate domains U_i are so small that each $E|_{U_i}$ is trivial. We want to find a trivialization of each such $E|_{U_i} \to U_i$ so that the transition matrices between bundle trivialization over the possibly disconnected overlap $U_i \cap U_j$ is everywhere positive. But note that the manifold with corners $E|_{U_i}$ is *connected* because it is isomorphic to a product of the connected U_i with a vector space. We have a specified orientation μ_i on this abstract manifold with corners, since it is open in the abstract manifold with corners E on which we have already chosen an orientation μ. Hence, there are exactly *two* orientations on $E|_{U_i}$ considered as a manifold with corners. If we choose a trivialization for the bundle $E|_{U_i} \to U_i$ then together with the chosen coordinate system on U_i we get a global coordinate system on the connected $E|_{U_i}$ (following the concatenation recipe as in the first part of the proof) and this puts an orientation on $E|_{U_i}$. It is either μ_i or $-\mu_i$. Since E has everywhere positive rank, we may negate the first member of the trivialization to negate the orientation if necessary to that it recovers μ_i.

I claim that the trivialization we have chosen for the bundles $E|_{U_i} \to U_i$ are an oriented trivializing atlas for $E \to M$, whence this is oriented as a vector bundle (as desired). Let us see what is happening on $E|_{U_i \cap U_j}$. We have picked two trivializations of each fiber $E(m)$ for $m \in U_i \cap U_j$ and we must prove that the transition matrix relating these has positive determinant. Consider the product of this determinant with the determinant of the Jacobian matrix at m relating the two coordinate systems on M chosen over U_i and U_j. This Jacobian determinant is positive since our choice of coordinate atlas on M is oriented. Hence, multiplying against this Jacobian determinant does not affect the sign we are seeking to determine. But this product calculates a determinant δ for a transition matrix between a coordinate system on $E|_{U_i}$ that induces the orientation μ_i and a coordinate system on $E|_{U_j}$ that induces the orientation μ_j. Since μ_i and μ_j are induced by the global orientation μ on the manifold with corners E, it follows that δ is a determinant for a Jacobian transition matrix between two coordinate charts on E that are compatible with μ, so $\delta > 0$. ■

Example 4.2. Consider the Möbius strip M_∞ with infinite height. It is a line bundle over the circle S^1, and as such it has no non-trivial global section (by Homework 5, Exercise 3(*iii*)). Thus, it is non-orientable as a line bundle. But the base manifold S^1 is orientable, so by the preceding theorem it follows that M_∞ is not orientable as a manifold. For any finite $a > 0$, the Möbius strip M_a with height $2a$ is naturally an open submanifold of M_∞, and it is even C^∞ isomorphic to M_a. Indeed, we have an isomorphism $(-a, a) \times S^1 \simeq \mathbf{R} \times S^1$ via $(t, z) \mapsto (\tan(2t/\pi a), z)$ that is compatible with negation on both $\mathbf{R}$ and $(-a, a)$. Hence, this isomorphism respect the involutions on both sides whose quotients are the respective Möbius strips, so we get an induced isomorphism on such quotients. It follows that M_a is a non-orientable manifold for any $a > 0$.

This non-orientability can be also seen in a more direct way that uses the geometry underlying the construction of Möbius strips and so bypasses the above weirdness of having to invoke the function $\tan(x)$ from trigonometry. The argument goes as follows. On M_a and M_∞ there is an evident direct sum decomposition of the tangent bundle into a direct sum of "horizontal" and "vertical" line subbundles, with the "horizontal" subbundle given by pullback of the trivial line bundle $T(S^1)$ over the circle (trivialized by ∂_θ). See Example 2.2 in the handout on integral manifolds for more rigor on this fact. But for a pair of line bundles L, L' over a manifold with corners, $\det(L \oplus L') \simeq L \otimes L'$. Hence, if L is trivial then $\det(L \otimes L') \simeq L'$. It follows that the determinant of the tangent bundle for M_a (resp. for M_∞) is isomorphic to the line bundle of "vertical" vector fields, and it was shown in Example 2.2 in the handout on integral manifolds that this line bundle is *not* trivial.

Remark 4.3. If we make no orientability hypothesis on the base space M but we assume that the vector bundle $E \to M$ is orientable as both a vector bundle *and* as a manifold with corners, then it can be deduced that M *must* be orientable. This is shown by a method similar to the arguments used in the proof of Theorem 4.1. We leave this as an exercise for the interested reader.

5. Time orientation

There is a variant on the notion of orientation that is very important in General Relativity, to determine a globally consistent local sense of future and past in the 4-dimensional Lorenztian spacetime manifold $\mathbf{U}$. The point is that time in General Relativity (unlike in Newtonian mechanics) is *not* determined globally, but rather it is determined by every object locally depending on how in moves through spacetime; nonetheless, we need a global consistency to the sense of *direction* of time. More precisely, each tangent space $\mathrm{T}_u(\mathbf{U})$ is endowed with a Lorentz metric having signature $(3, 1)$, and the "velocity" (or rather, energy-momentum) vector $\vec{v} \in \mathrm{T}_u(\mathbf{U})$ arising from the trajectory of a particle through spacetime is required to satisfy two conditions: $\langle \vec{v}, \vec{v} \rangle_u \leq 0$ (this is related to the classical-sounding statement that speed of motion cannot exceed the speed of light; equality here is only for massless particles) and $\vec{v}$ must be a nonzero vector "pointing to the future". What does this latter condition mean? Well, consider the so-called local time cone at u:

$$\{v \in \mathrm{T}_u(\mathbf{U}) \,|\, \langle v, v \rangle_u < 0\}.$$

As you may perhaps convince yourself by drawing the loci $x^2 < y^2$ in $\mathbf{R}^2$ and $x^2 + y^2 - z^2 < 0$ in $\mathbf{R}^3$, this should have two connected components (which are interchanged by negation). Perhaps you should be cautious in beleiving this in the 4-dimensional case, since in contrast that the positive locus $x^2 + y^2 - z^2 > 0$ in $\mathbf{R}^3$ is connected! Granting such disconnectedness for a moment, distinguishing one of the two negative components and requiring the nonzero energy-momentum vector to lie in its closure is the device by which the physical theory forces everyone's sense of time to always point in the same direction, up to the fact that this choice of component in the time cone in the tangent space must be done in a "continuous" manner as we vary across spacetime. The concept of time orientation is the mathematical notion that makes this idea precise. Before we give a mathematical definition, we first give a basic lemma:

Lemma 5.1. *Let (V, q) be a non-degenerate quadratic space of finite dimension over $\mathbf{R}$, and assume that the signature is (r, s) with $s > 0$. The open subset $T = \{v \in V \mid q(v) < 0\}$ is connected if $s > 1$, and it has two connected components if $s = 1$. Moreover, if $s = 1$ and $q(v_1), q(v_2) < 0$, then the orthogonal projection of V onto $\mathbf{R} \cdot v_2$ with respect to the associated non-degenerate bilinear form $B_q(v, v') = q(v+v') - q(v) - q(v')$ carries v_1 to cv_2 for some $c \neq 0$ and v_1 lies in the same connected component as v_2 if and only if $c > 0$. In particular, negation in V carries T to itself and swaps the connected components.*

Note that even though q is indefinite, since it is non-degenerate if $v \in V$ is any nonzero vector then $H_v = (\mathbf{R} \cdot v)^\perp = \ker B_q(v, \cdot)$ is indeed a hyperplane, and if $q(v) \neq 0$ then $\mathbf{R} \cdot v \oplus H_v = V$ by dimension reasons (since $v \notin H_v$). Hence we do have a good notion of orthogonal projection from V onto the line $\mathbf{R} \cdot v$ if $q(v) \neq 0$. (Explicitly, $B_q(v, v) = 2q(v) \neq 0$ and the projection operator is $B_q(v, \cdot)v / B_q(v, v)$.) In contrast, if $q(v) = 0$ then $v \in H_v$ and there is no good notion of orthogonal projection onto $\mathbf{R} \cdot v$. Also, by switching to r in the role of s we see that this lemma then works the same way if we replace "< 0" with "> 0" throughout (e.g., just replace q with $-q$). We formulate the lemma in terms of negative-definite vectors because this is the case that is of greater interest in General Relativity (if one follows the mathematician's convention that the Lorentz metric has signature $(3, 1)$; for some reason I do not understand it seems more typical among physicists to make the signature in relativity theory be $(1, 3)$).

Proof. By Gramm–Schmidt, we can find a basis $\{e_i\}$ of V with respect to which q has the coordinatized form

$$q(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_r^2) - (x_{r+1}^2 + \cdots + x_n^2).$$

Hence, the locus T of interest is certainly non-empty and is obvious open (by continuity of q on V). Consider a point $u \in T$. Clearly the path

$$t \mapsto ((1-t)x_1(u), \ldots, (1-t)x_r(u), x_{r+1}(u), \ldots, x_n(u))$$

in V lies in T for all $t \in [0, 1]$, so to track the connected components of T it suffices to consider points $u \in U$ whose first r coordinates are 0. We must have $x_j(u) \neq 0$ for some $r + 1 \leq j \leq n$, so if we now similarly deform all coordinates other than the jth linearly to 0, and then scale the remaining jth coordinate linearly to its sign, we never leave T and so we conclude that each connected component of U contains $\pm e_j$ for some $r + 1 \leq j \leq n$. If $s > 1$, then for any two such distinct indices j and j' and any sign $\varepsilon = \pm 1$, the path $t \mapsto te_j + (1-t)\varepsilon e_{j'}$ is contained entirely within T, whence e_j and $\varepsilon e_{j'}$ lie in the same connected component. Thus, e_j, $e_{j'}$, and $-e_{j'}$ all lie in the same connected component for any $r + 1 \leq j \neq j' \leq n$. Letting j and j' vary in this way, all $\pm e_j$ must therefore be in the same connected component if $s > 1$. This gives that T is connected if $s > 1$.

Now suppose $s = 1$, so there are just the two basis vectors $\pm e_n$ to play with. This at least shows that there are at most two connected components. The problem is to prove that T really is disconnected. We need a continuous function to "separate" the components. Consider the function $f = B_q(e_n, \cdot) : V \to \mathbf{R}$. In coordinates,

$$B_q(x,y) = q(x+y) - q(x) - q(y) = 2\sum_{i=1}^{r} x_i y_i - 2\sum_{i=1}^{s} x_{r+i} y_{r+i}.$$

Thus, $f(y) = -2y_n$. Clearly $f(e_n) < 0 < f(-e_n)$ and f is continuous. Also, f is non-vanishing on T because since $s = 1$ the only way $q(y) < 0$ can happen is if $y_n \neq 0$ (though this necessary condition is far from sufficient). Hence, by the intermediate value theorem, T cannot be connected. (Explicitly, $T \cap \{f < 0\}$ and $T \cap \{f > 0\}$ is a non-trivial separation, and so these must be the two connected components.)

Finally, suppose $v_1, v_2 \in T$ and the orthogonal projection of v_1 into $\mathbf{R} \cdot v_2$ is cv_2. To prove $c \neq 0$, consider the hyperplane $H_{v_2} = (\mathbf{R} \cdot v_2)^{\perp} = \ker B_q(v_2, \cdot)$. By negative-definiteness of v_2 and non-degeneracy of q it follows from the orthogonal decomposition $V = \mathbf{R}v_2 \oplus H_{v_2}$ that q must have non-degenerate restriction to H_{v_2}. Hence, by well-definedness of signature and the negativity of $q(v_2)$, the restriction of q to H_{v_2} has to be positive-definite (as $s = 1$). The negative-definite vector v_1 therefore cannot lie in H_{v_2}, which says exactly that $c \neq 0$. It remains to show that v_1 is in the same connected component of T as v_2 if and only if $c > 0$. To see this, it seems simplest to bring in coordinates. Choose an orthonormal basis for q as above, with $e_n = v_2$. We therefore have $v_1 = a_1 e_1 + \cdots + a_{n-1} e_{n-1} + c e_n$. The deformation argument as above (i.e., linearly scale the a_j's to 0) shows that v_1 is in the same connected component of T as is ce_n. But we can likewise linearly scale c to $\varepsilon = \mathrm{sign}(c)$ without leaving T, so our problem is to show that εe_n is in the same connected component as e_n if and only if $\varepsilon = 1$. This was seen already. ■

We now focus our attention on the case of signature $(n-1, 1)$ with $n \geq 1$. For such an n-dimensional quadratic space (V, q), we call the disconnected open locus T of $v \in V$ with $q(v) < 0$ the *time cone*, and its two connected components are called *half-cones*. A *time orientation* of (V, q) is a choice of one of these two half-cones. The chosen half-cone is called the *future* half-cone, and its negative is called the *past* half-cone. Clearly each such (V, q) admits exactly two time orientations, and these are swapped by negation.

In the definition of a *spacetime*, we need more than just a (connected) 4-dimensional smooth Lorentzian manifold. We need to equip it with a continuously varying orientation and time orientation on each tangent space. The first of these two notions just means that the manifold should be oriented. (That is, we fix an orientation on the tangent bundle.) To make the second idea precise, we study the general problem of vector bundles endowed with a Lorentzian metric tensor. The key is the following theorem:

Theorem 5.2. *Let M be a non-empty C^p manifold with corners, $0 \leq p \leq \infty$, and let $\pi : E \to M$ be a C^p vector bundle with rank $n > 0$ that is equipped with a C^p metric tensor in $(E^{\vee} \otimes E^{\vee})(M)$ having signature $(n-1, 1)$ on fibers. The subset*

$$T_E = \{v \in E \mid \langle v, v \rangle_{\pi(v)} < 0\}$$

is open in E and if M is connected then it has at most two connected components. If M is connected and T_E has two connected components, then these are swapped by negation and each of its connected components meets the time cone of each fiber in one of the connected components of the fiber.

We call T_E the *time cone* of the Lorentzian bundle E. As an open subset it has a natural structure of C^p manifold with corners; of course, it depends on the choice of metric tensor on E. There is an analogous result in case of signature (r, s) with $s > 1$, saying that the corresponding locus of interest is open in E and is connected when M is connected. However, this is rather uninteresting in practice and goes by pretty much the same method of proof, so we leave it to the imagination of the interested reader.

Proof. We may and do assume that M is connected. The case $\dim M = 0$, which is to say M is a point, is exactly the case treated in the preceding lemma. To bootstrap this, we first invoke a non-trivial result (Theorem 1.1) from the handout on why the universe cannot be S^4: by using an auxiliary C^p Riemannian metric on E, there exists a (non-canonical) C^p bundle decomposition $E = E^+ \oplus E^-$ such that the Lorentzian metric tensor has Riemannian restriction to E^+ and negative-definite restriction to E^-. Here, E^+ is a subbundle necessarily of rank $n-1$ and E^- must be a line subbundle.

Let us now first prove the result locally over M. More precisely, let $U \subseteq M$ be a connected open over which E^+ and E^- are each trivial. Upon picking trivializing frames and applying the bundle version of Gramm-Schmidt, we get orthonormal frames for each. Hence, we get a global frame $\{e_1, \dots, e_n\}$ such that $\langle e_i, e_j \rangle$ vanishes for $i \neq j$ and is equal to 1 (resp. -1) when $i \leq n-1$ (resp. $i = n$). It follows that under the trivialization isomorphism $E|_U \simeq U \times \mathbf{R}^n$ (considered as an isomorphism of C^p manifolds with corners), the subset $T_{E|_U}$ goes over to $U \times T$ where $T \subseteq \mathbf{R}^n$ is the open time cone for the standard Lorentzian quadratic form on $\mathbf{R}^n$. Since T is open in $\mathbf{R}^n$ and has two connected components and these are swapped by negation, the same goes for the product $U \times T$ in $U \times \mathbf{R}^n$ (via negation on fibers over U). Hence, the same holds for $T_{E|_U}$ in $E|_U$, which is the part of T_E lying over U. This gives the resulting over small connected opens in M. In particular, T_E is open in E. Since E is a topological manifold with corners, it is locally path-connected. The same therefore holds for the open subset T_E, so the connected components of T_E are open. Hence, the connected components of T_E are open and closed.

Let us fix a choice of connected component $C \subseteq T_E$. Let $U \subseteq M$ be a small connected open subset as considered above, so $T_{E|_U}$ has exactly two connected components. Since C is open and closed in T_E, the part C_U of C that lies over U is open and closed in $T_{E|_U}$. Thus, C_U is either empty, equal to a connected component of $T_{E|_U}$, or equal to all of $T_{E|_U}$. In particular, for each $m \in U$, the "structure" of the fiber $C(m) \subseteq T_E(m)$ is independent of m: it is either empty, equal to a half-cone, or equal to the entire time cone in $E(m)$. This shows that the following three subsets of M are open: the locus M_0 of $m \in M$ for which $C(m)$ is empty, the locus M_1 of $m \in M$ for which $C(m)$ is a connected component of $T_E(m)$, and the locus M_2 of $m \in M$ for which $C(m) = T_E(m)$. But these three open sets are pairwise disjoint, so they are also all closed in M. Since M is non-empty and connected, we cannot have $M = M_0$ since *some* fiber of C over M is non-empty (as C is non-empty). Thus, either $M = M_1$ or $M = M_2$. The case $M = M_2$ is the case $C = T_E$, which is to say that T_E is connected.

We may now assume $M = M_1$, which is to say that C meets each fiber of T_E over M in one of its two half-cones. Negation on the bundle E induces a C^p involution of E and hence of the open subset T_E. Thus, it carries C to a connected component of T_E that we shall denote $-C$, and fiberwise over M we see that C and $-C$ never meet. Hence, $-C \neq C$. But obvious $C \cup -C$ fills up all fibers of T_E over M, so we conclude that these two connected components of T_E are the only ones. ∎

This theorem finally permits us to make a definition:

Definition 5.3. Let M be a C^p manifold with corners, $0 \leq p \leq \infty$, and let $E \to M$ be a C^p vector bundle endowed with a C^p Lorentzian metric tensor. Let $T_E \subseteq E$ be the time cone, and assume that over each connected component of M the restriction of T_E has two connected components. A *time orientation* of E is a choice of one of these two components over each of the connected components of M.

If $p \geq 1$ and TM is endowed with a C^{p-1} Lorentz structure, then a *time orientation* on the C^{p-1} Lorentzian manifold with corners M is a time orientation of the Lorentzian bundle $TM \to M$ in the above sense.

Example 5.4. We conclude by giving an example of a smooth connected manifold with two different Lorentzian metrics, one giving rise to a connected time cone and the other a disconnected time cone. Hence, this shows that whether or not the time connected is connected (given that the base manifold is connected) is not at all determined by the underlying vector bundle alone.

Let $M = (\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{R}$ be an infinite cylinder with "coordinates" (θ, t). The metric tensor $\mathrm{d}\theta^{\otimes 2} - \mathrm{d}t^{\otimes 2}$ is visibly Lorentzian and the time cone is disconnected: TM is trivial with the frame $\{\partial_\theta, \partial_t\}$, under which the resulting isomorphism $TM \simeq M \times \mathbf{R}^2$ carries the time cone of the tangent bundle over to the product $M \times \{|x| < |y|\}$ that is disconnected. Another Lorentzian metric tensor is given by

$$\cos\theta\, \mathrm{d}\theta^{\otimes 2} + \sin\theta\, (\mathrm{d}t \otimes \mathrm{d}\theta + \mathrm{d}\theta \otimes \mathrm{d}t) - \cos\theta\, \mathrm{d}t^{\otimes 2}.$$

The associated 2×2 symmetric matrix (g_{ij}) has determinant -1, so this is indeed a Lorentzian metric tensor. How can we prove that the time cone in this case is connected? Consider the vector field $\vec{v}(x) = -\sin x\, \partial_\theta|_{(\overline{x},x)} + (1 + \cos x)\, \partial_t|_{(\overline{x},x)}$ where $\overline{x} = x \bmod 2\pi\mathbf{Z}$ for $x \in \mathbf{R}$. This is nonzero away from $x \in \pi + 2\pi\mathbf{Z}$, and has self-pairing $-2(1 + \cos x)$ that is negative away from $x \in \pi + 2\pi\mathbf{Z}$. Thus, taking $-\pi < x < \pi$ gives a path in the time cone of the tangent bundle. Let C be the connected component containing this path. By the method of proof of the preceding theorem, over any connected base a proper component of the time cone of the bundle meets exactly one half-cone in each fiber. In particular, if we work over a small connected open in the base over which the time cone *does* split into two connected pieces then if C meets both pieces we conclude that C must be the entire time cone of the metrized tangent bundle: that is, the time cone of the metrized tangent bundle is connected.

We wish to study how the C interacts with time cone over a small neighborhood around $p = (\pi, \pi)$ in the cylinder over which the time cone splits into two components. Actually, C is "abstract", so it is really the above 1-parameter vector field of timelike vectors (which is a path in C) that we will study. Consider the local vector field $v_2 = \partial_\theta$ near p, so $\langle v_2, v_2 \rangle = \cos\theta$ is approximately $\cos\pi = -1$ and so is in the time cone. Applying Gramm-Schmidt gives an orthogonal vector field:

$$v_1 = \partial_t - \frac{\langle \partial_t, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \partial_t - \tan\theta\, \partial_\theta.$$

We compute $\langle v_1, v_1 \rangle = -\sec\theta$. Thus, near p we see that $\{v_1/\sqrt{|\sec\theta|}, v_2/\sqrt{|\cos\theta|}\}$ is an orthonormal basis with the second one in the time cone. Hence, membership in connected components on fibers is determined by the sign of the coefficient for the projection onto the line spanned by the second basis vector in the fiber. That is, orthogonal projection on the fiber-line spanned by v_2 distinguishes the half-cones in fibers.

When orthogonally projecting $\vec{v}(x)$ onto the line spanned by $v_2(x)$ for x near π, the coefficient of $v_2(x)$ is

$$\frac{\langle \vec{v}(x), v_2(x) \rangle}{\langle v_2(x), v_2(x) \rangle} = \tan(x),$$

and for x slightly less than π this is negative whereas for x slightly more than π this is positive. Hence, we conclude that over a small open around $p = (\pi, \pi)$ our connected component C does indeed meet both connected components of the time cone of the local tangent bundle, so we are done.