Math 396. The Möbius strip in $\mathbf{R}^3$

1. Introduction

In the handout on quotients by group actions, the Möbius strip M_a (with height $2a$) was defined as an abstract smooth manifold made as a quotient of $(-a,a)\times S^1$ by a free and properly discontinuous action by the group of order 2. Our purpose here is to work out some tangent space calculations to verify that the explicit "definition" of the Möbius strip via trigonometric parameterization as given in the course text (on page 10) is really a smooth embedding of our abstract Möbius strip of height $2a$ into $\mathbf{R}^3$.

Using the C^∞ isomorphism between $\mathbf{R}/2\pi\mathbf{Z}$ and the circle $S^1 \subseteq \mathbf{R}^2$ via $\theta \mapsto (\cos\theta, \sin\theta)$ (which carries $\theta \mapsto \pi + \theta$ over to $w \mapsto -w$ on S^1), we consider the standard parameter $\theta \in \mathbf{R}$ as a local coordinate on S^1. For finite $a > 0$, consider the C^∞ map $f : (-a,a)\times S^1 \to \mathbf{R}^3$ defined by

$$(t,\theta) \mapsto (2a\cos 2\theta + t\cos\theta\,\cos 2\theta, 2a\sin 2\theta + t\cos\theta\,\sin 2\theta, t\sin\theta).$$

Since $f(-t, \pi+\theta) = f(t,\theta)$ by inspection, it follows from the universal property of the quotient map $(-a,a)\times S^1 \to M_a$ that f unique factors through this via a C^∞ map $\overline{f} : M_a \to \mathbf{R}^3$. Our goal is to prove that $\overline{f}$ is an embedding and to use this viewpoint to understand some basic properties of the Möbius strip.

2. Embedding

Theorem 2.1. *The map $\overline{f}$ is an immersion.*

Note that we do not yet claim $\overline{f}$ is an embedding (i.e., also injective and even a homeomorphism onto its image).

Proof. We first reduce the problem to working with f, as f is given by a simple explicit formula across its entire domain (M_a does not have global coordinates). Of course, working locally for $\overline{f}$ is "the same" as working locally for f, so the reduction step to working with f isn't really necessary if one says things a little differently. However, it seems a bit cleaner to just make the reduction right away and so to thereby work with the map f that feels a bit more concrete than the map $\overline{f}$ at the global level). Let $p : (-a,a)\times S^1 \to M_a$ be the natural quotient map. Each point in M_a has the form $p(\xi_0)$ for some ξ_0 and the Chain Rule gives that the injection $\mathrm{d}f(\xi_0)$ factors as $\mathrm{d}\overline{f}(p(\xi_0)) \circ \mathrm{d}p(\xi_0)$ with $\mathrm{d}p(\xi_0)$ an isomorphism (as p is a local C^∞ isomorphism, via the theory of quotients by free and properly discontinuous group actions). Hence, the tangent map for $\overline{f}$ is injective at $p(\xi_0)$ if and only if the tangent map for f is injective at ξ_0. It is therefore enough (even equivalent!) to prove that f is an immersion.

For $\xi_0 = (t_0, \theta_0)$, $\mathrm{d}f(\xi_0)$ sends the basis vectors $\partial_t|_{\xi_0}$ and $\partial_\theta|_{\xi_0}$ of $\mathrm{T}_{\xi_0}((-a,a)\times S^1)$ to the following respective vectors in $\mathrm{T}_{f(\xi_0)}(\mathbf{R}^3) \simeq \mathbf{R}^3$ (using the ordered basis $\{\partial_x|_{f(\xi_0)}, \partial_y|_{f(\xi_0)}, \partial_z|_{f(\xi_0)}\}$ of $\mathrm{T}_{f(\xi_0)}(\mathbf{R}^3)$):

$$(\cos\theta_0\,\cos 2\theta_0, \cos\theta_0\,\sin 2\theta_0, \sin\theta_0),$$

and

$$(-4a\sin 2\theta_0 + t_0(-\sin\theta_0\,\cos 2\theta_0 - 2\cos\theta_0\,\sin 2\theta_0), 4a\cos 2\theta_0 + t_0(-\sin\theta_0\,\sin 2\theta_0 + 2\cos\theta_0\,\cos 2\theta_0), t_0\cos\theta_0).$$

A direct calculation shows that these two vectors in $\mathbf{R}^3$ are perpendicular with respective squared lengths 1 and $4a+2t_0\cos\theta_0$. Thus, $\mathrm{d}f(\xi_0)$ sends a basis of $\mathrm{T}_{\xi_0}((-a,a)\times S^1)$ to a pair of independent vectors in $\mathrm{T}_{f(\xi_0)}(\mathbf{R}^3)$, so f is an immersion. ∎

Theorem 2.2. *The map $\overline{f}$ is an embedding.*

Proof. To prove that $\overline{f}$ is injective, it suffices to prove that $f(t,\theta) = f(t',\theta')$ if and only if $(t',\theta') = (t,\theta)$ or $(t',\theta') = (-t,\pi+\theta)$. By direct calculation, if f_1, f_2, f_3 are the component functions of f then $f_1^2 + f_2^2 = (2a + t\cos\theta)^2$, and so since $2a + t\cos\theta > 0$ (as $|t| < a$) we have $2a + t\cos\theta = \sqrt{f_1^2 + f_2^2}$. Hence, $f(t,\theta) = f(t',\theta')$ implies $t\cos\theta = t'\cos\theta'$. Using the third component function, $t\sin\theta = t'\sin\theta'$. Squaring and adding, $t^2 = t'^2$, so $t' = \pm t$. From the definition of f, if $t = t' = 0$ then $\cos 2\theta = \cos 2\theta'$ and $\sin 2\theta = \sin 2\theta'$, so $2\theta - 2\theta' \in 2\pi\mathbf{Z}$. Hence, in such cases $\theta - \theta' \in \pi\mathbf{Z}$, which is to say $\theta' = \theta$ or $\theta' = \theta + \pi$ (as the angular coordinate only matters up to adding $2\pi\mathbf{Z}$). If instead t and t' are both nonzero then $t' = \varepsilon t$ for a unique sign $\varepsilon = \pm 1$, and cancelling the nonzero t gives $\cos\theta' = \varepsilon\cos\theta$ and $\sin\theta' = \varepsilon\sin\theta$. In the case $\varepsilon = 1$ (i.e., $t' = t$) this gives $\theta' = \theta$, and in the case $\varepsilon = -1$ (i.e., $t' = -t$) this gives $\theta' = \theta + \pi$. This verifies the injectivity of $\overline{f}$.

To prove that the injective immersion $\overline{f}$ is an embedding, we have to prove that it is a homeomorphism onto its image. We may use the sequential criterion for continuity of the inverse (as our spaces are second countable and Hausdorff), so we have to prove that if $f(t_n,\theta_n) \to f(t,\theta)$ in $\mathbf{R}^3$ then the images of the (t_n,θ_n)'s in M_a converge to the image of (t,θ) in M_a. By changing the choices of representatives in $(-a,a)\times S^1$, we may assume $t_n \geq 0$ for all n and $t \geq 0$. We shall first prove $(t_n,\theta_n) \to (t,\theta)$ in $(-a,a)\times S^1$ if $t > 0$. Since

$$t_n^2 = (t_n\cos\theta_n)^2 + (t_n\sin\theta_n)^2 = (\sqrt{f_1(t_n,\theta_n)^2 + f_2(t_n,\theta_n)^2} - 2a)^2 + f_3(t_n,\theta_n)^2,$$

we have $t_n^2 \to t^2$, so $t_n = |t_n| \to |t| = t$. Also, we likewise get $t_n\cos\theta_n \to t\cos\theta$ and $t_n\sin\theta_n \to t\sin\theta$. So far this works if $t \geq 0$. If $t \neq 0$ then $t_n/t \to 1$, so $\cos\theta_n \to \cos\theta$ and $\sin\theta_n \to \sin\theta$. By trigonometry, this forces $\theta_n \to \theta$ in S^1. If $t = 0$ then by the same method as above we can still infer $t_n \to 0$, and so the product $s_n t_n$ tends to 0 for any *bounded* sequence $\{s_n\}$. Thus, from the definition of f we may infer from the convergence

$$f(t_n,\theta_n) \to f(t,\theta) = f(0,\theta) = (2a\cos 2\theta, 2a\sin 2\theta, 0)$$

and the condition $t_n \to 0$ that $2a\cos 2\theta_n \to 2a\cos 2\theta$ and $2a\sin 2\theta_n \to 2a\sin 2\theta$. Since $2a \neq 0$, it follows via trigonometry that $2\theta_n \to 2\theta$ in S^1, whence for large n each $\theta_n - \theta \in \mathbf{R}/2\pi\mathbf{Z}$ is very close to either 0 mod $2\pi\mathbf{Z}$ or π mod $2\pi\mathbf{Z}$. We may change each θ_n by an arbitrary integral multiple of π (at the expense of perhaps negating t_n, which does not affect the condition $t_n \to 0$), so we get $\theta_n - \theta \to 0$ in $\mathbf{R}/2\pi\mathbf{Z}$. Thus, $\theta_n \to \theta$ in S^1 after making this modification in our initial choices of representatives in $(-a,a)\times S^1$ for the chosen sequence in M_a. ∎

3. A bit of topology

As an application of our knowledge that the "explicit" Möbius strip in $\mathbf{R}^3$ (via the parameteric formulas for the common images of $\overline{f}$ and f, coupled with the picture from page 10 in the text that shows this really is the Möbius strip from real life), let's see how to explain the elementary observation that cutting a paper model of a Möbius strip along its central line does not cause the piece of paper to fall into two pieces (as one might initially expect). We would like to understand mathematically what is going on.

The C^∞ inclusion $S^1 \to (-a,a)\times S^1$ via $\theta \mapsto (0,\theta)$ is compatible with the antipodal map on S^1 and with the map $(t,\theta) \mapsto (-t,\pi+\theta)$ on $(-a,a)\times S^1$, so we get an induced C^∞ map on quotients that is a closed C^∞ submanifold (by the general good behavior of "nice" group-action quotients and closed submanifolds, as explained in the handout on quotients by group actions). Near the end of the handout on quotients by group actions, it was shown that the squaring map $w \mapsto w^2$ from S^1 to S^1 gives a C^∞ isomorphism of S^1 with the quotient of S^1 by the antipodal map $w \mapsto -w$. Thus, we get a quotient circle C as a C^∞ closed submanifold in M_a (the image of $\{0\}\times S^1 \subseteq (-a,a)\times S^1$).

Inside of the "real world" model $\overline{f}(M_a)$, the central circle is $\overline{f}(C)$ (why?), and so the assertion of interest is that $\overline{f}(M_a) - \overline{f}(C)$ is connected. Since $\overline{f}$ is a homeomorphism onto its image, it is equivalent to say that the abstract complement $M_a - C$ is *connected*. Note that it is crucial we worked with $\overline{f}$ and not f, since $C = f(\{0\} \times S^1)$ yet

$$(-a, a) \times S^1 - \{0\} \times S^1 = ((-a, a) - \{0\}) \times S^1$$

is disconnected. (There is no inconsistency here, since f is not even injective, let alone an embedding, so it could well carry a disconnected subset of its source onto a connected subset of its image.)

To see the geometry of $M_a - C$, we look at the map

$$((-a, a) - \{0\}) \times S^1 \to M_a - C.$$

This map is the quotient by $(t, \theta) \mapsto (-t, \theta + \pi)$, so the formation of this quotient simply involved identifying $(-a, 0) \times S^1$ with $(0, a) \times S^1$ via $(-t, \theta) \leftrightarrow (t, \pi + \theta)$ for $0 < t < a$. More specifically, the connected component $(0, a) \times S^1$ maps onto $M_a - C$ via a bijective C^∞ local isomorphism, so this map is necessarily a C^∞ isomorphism. Thus, $M_a - C$ is connected since $(0, a) \times S^1$ is connected. Note that the subset $\overline{f}(M_a) - \overline{f}(C)$ is exactly $f((0, a) \times S^1)$, with the map $f : (0, a) \times S^1 \to \overline{f}(M_a) - \overline{f}(C)$ a homeomorphism (and even a C^∞ isomorphism).