Math 396. Operations with pseudo-Riemannian metrics

We begin with some preliminary motivation. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional vector space endowed with a non-degenerate symmetric bilinear form (perhaps not positive-definite). In linear algebra, we have seen how to carry out several operations in the presence of this structure. For example, if V' is a subspace of V then we can form its orthogonal complement ${V'}^{\perp}$ (the subspace of $v \in V$ such that $\langle v, v'\rangle = 0$ for all $v' \in V$), and this is also a complementary subspace: $V' \oplus {V'}^{\perp} \to V$ is an isomorphism. We have also seen in §2 of the old handout on orientations how to endow $V^{\otimes n}$, $\mathrm{Sym}^n(V)$, and $\wedge^n(V)$ with non-degenerate symmetric bilinear forms, as well as $V^\vee$ (the "dual" bilinear form), and that using the one on $V^\vee$ likewise gave rise to non-degenerate symmetric bilinear forms on $(V^\vee)^{\otimes n}$, $\mathrm{Sym}^n(V^\vee)$, and $\wedge^n(V^\vee)$ such that the natural isomorphisms

$$(V^\vee)^{\otimes n} \simeq (V^{\otimes n})^\vee,\ \ \mathrm{Sym}^n(V^\vee) \simeq (\mathrm{Sym}^n V)^\vee,\ \ \wedge^n(V^\vee) \simeq (\wedge^n V)^\vee$$

are compatible with the bilinear forms built on the left via the dual form on $V^\vee$ and the bilinear forms on the right that are dual to the ones built above on the tensor, symmetric, and exterior powers of V.

Rather concretely, on $V^{\otimes n}$, $\mathrm{Sym}^n(V)$, and $\wedge^n V$ the induced symmetric bilinear forms were characterized uniquely by the conditions

$$\langle v_1 \otimes \cdots \otimes v_n, v_1' \otimes \cdots \otimes v_n'\rangle_{V^{\otimes n}} = \prod_i \langle v_i, v_i'\rangle,\ \ \langle v_1 \cdots v_n, v_1' \cdots v_n'\rangle_{\mathrm{Sym}^n V} = \sum_{\sigma \in S_n} \prod_{i=1}^n \langle v_i, v'_{\sigma(i)}\rangle,$$

$$\langle v_1 \wedge \cdots \wedge v_n, v_1' \wedge \cdots \wedge v_n'\rangle_{\wedge^n V} = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n \langle v_i, v'_{\sigma(i)}\rangle = \det(\langle v_i, v_j'\rangle),$$

and on $V^\vee$ the dual bilinear form was defined by $\langle \ell, \ell'\rangle_{V^\vee} = \langle v, v'\rangle$ where $\ell = \langle v, \cdot\rangle$ and $\ell' = \langle v', \cdot\rangle$. For example, if $\{v_i\}$ is a basis of V then the element of $(\wedge^n V)^\vee \otimes \wedge^n(V)^\vee \simeq \wedge^n(V^\vee) \otimes \wedge^n(V^\vee)$ that corresponds to $\langle\cdot,\cdot\rangle_{\wedge^n(V)}$ is

$$\sum_{j_1 < \cdots < j_n, j_1' < \cdots < j_n'} \det(\langle v_{j_r}, v_{j_s'}\rangle_{(r,s)})(v_{j_1}^* \wedge \cdots \wedge v_{j_n}^*) \otimes (v_{j_1'}^* \wedge \cdots \wedge v_{j_n'}^*)$$

because of two facts: (i) $\det(\langle v_{j_r}, v_{j_s'}\rangle) = \langle v_{j_1} \wedge \dots v_{j_n}, v_{j_1'} \wedge \cdots \wedge v_{j_n'}\rangle_{\wedge^n(V)}$, and (ii) under the isomorphism $(\wedge^n V)^\vee \simeq \wedge^n(V^\vee)$ the basis of $\wedge^n(V^\vee)$ given by wedge products of v_j^*'s *is* the dual basis to the basis of $\wedge^n(V)$ given by wedge products of v_j's.

In Corollary 2.3 of the orientation handout, we saw explicitly how to use an *orthonormal* basis $\{e_i\}$ of V (i.e., $\langle e_i, e_j\rangle = 0$ for $i \neq j$ and $\langle e_i, e_i\rangle = \pm 1$ for all i) to build orthonormal bases of all of these auxiliary spaces, and so in particular when $\langle\cdot,\cdot\rangle$ is *positive-definite* it follows that the induced bilinear forms on all of these auxiliary spaces are also positive-definite. That is, if V is an inner product space then all of the spaces

$$V^{\otimes n}, \mathrm{Sym}^n(V), \wedge^n(V), V^\vee, (V^\vee)^{\otimes n}, \mathrm{Sym}^n(V^\vee), \wedge^n(V^\vee)$$

are naturally inner product spaces, and from an orthonormal basis of V we can build orthonormal bases on all of these other spaces. For example, the dual basis to an orthonormal basis is orthonormal, and tensor and wedge products of members of an orthonormal basis are orthonormal. Recall that for symmetric powers there were certain factorials that intervened in the formation of an orthonormal basis from one for V, ultimately due to the intervention of factorials in the description of the duality between $\mathrm{Sym}^n(V)$ and $\mathrm{Sym}^n(V^\vee)$ in terms of a basis of V and its dual basis in $V^\vee$.

Our aim here is to show how all of these matters can be carried out for vector bundles endowed with a pseudo-Riemannian metric. In a sense, the real work all took place in linear algebra: all we have to do here is chase some "universal" fibral formulas. The universal nature of the constructions in linear algebra is "why" everything will carry over to vector bundles.

The most important case of all (for geometric applications) is to use a Riemannian metric on the tangent bundle $E = TM$ to a Riemannian manifold with corners M to build Riemannian metrics on the bundles $\wedge^n(E^\vee) = \Omega^n_M$ of smooth differential forms, especially the line bundle $\det(T^*M)$ of top-degree differential forms. Indeed, with respect to the induced Riemannian metric on this line bundle and an "orientation" on the manifold M (to be discussed later), there will be a distinguished top-degree form that is just the "volume form" on fibers, as defined in the old handout on orientations. The reason for the name *volume form* in that old handout was exactly this application, where they glue together to give a special top-degree differential form on an oriented Riemannian manifold. In terms of the theory of integration to be developed for top-degree differential forms on arbitrary oriented manifolds, integration of this volume form will provide the definition for *volume* of open subsets of an oriented Riemannian manifold. Applied in the case of surfaces in $\mathbf{R}^3$ given the *induced metric tensor* (from the flat one on $\mathbf{R}^3$), this procedure will recover all of the classical formulas for surface area of open subsets of various classes of (oriented) surfaces with boundary (such as surfaces of revolution, or the parametric "$z = f(x, y)$" sort, etc.). In particular, we will arrive at the striking fact that the induced metric tensor on an embedded surface in $\mathbf{R}^3$ is all we need to determine a satisfactory (and physically reasonable) theory of area on the surface.

Even though all that follows will largely be used in this course only for the tangent bundle and bundles derived from it by the standard tensor operations and duality, in any deeper study of differential geometry one quickly finds it necessary to consider many other kinds of vector bundles and so both practical applications as well as conceptual clarity inspire the decision to develop the rest of this handout for arbitrary vector bundles with pseudo-Riemannian metric, and not just for the tangent bundle (where it may be too tempting to phrase everything in the language of vector fields and differential forms, which is rather irrelevant to the "vector bundle linear algebra" being carried out).

1. Basic construction and examples

Recall that a finite-dimensional quadratic space (V, q) over $\mathbf{R}$ is called *non-degenerate* when the associated symmetric bilinear form $B_q(v, v') = q(v + v') - q(v) - q(v')$ sets up a perfect pairing between V and itself: $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$ is a linear isomorphism from V to $V^\vee$. If $\dim V = n$, recall that we classified all such pairs (V, q) up to isomorphism by means of the *signature* (r, s), where $r, s \geq 0$ and in some system of linear coordinates $q = \sum_{i=1}^r x_i^2 - \sum_{j=1}^s x_{r+j}^2$ (so $r + s = n$); we showed that such a pair (r, s) is independent of the choice of linear coordinates that "diagonalizes" q. We also proved that the signature is the unique discrete invariant in the sense that in the topological space $\mathrm{Quad}_{\mathrm{nd}}(V)$ of all non-degenerate quadratic forms on V (viewed as an open subset of the finite-dimensional vector space $\mathrm{Quad}(V)$ of all quadratic forms on V) the connected components consist of precisely those non-degenerate q on V with a common signature. In the case of pseudo-Riemannian metric tensors on rank-n vector bundles, which is to say a "continuously varying family" of non-degenerate quadratic spaces of a fixed dimension n, the signature may not be the same on all fibers but it is locally constant (and hence the same on all fibers over a fixed connected component of the base space):

Lemma 1.1. *Let M be a C^p manifold with corners, $0 \leq p \leq \infty$, and let $E \to M$ be a C^p vector bundle with constant rank $n > 0$. Let $\langle \cdot, \cdot \rangle$ be a pseudo-Riemannian metric tensor on $E \to M$. For the ordered pairs of non-negative integers (n_+, n_-) such that $n_+ + n_- = n$, the pairwise disjoint* *subsets*

$$\{m \in M \mid \text{signature}(\langle \cdot, \cdot \rangle_m) = (n_+, n_-)\}$$

in M are all open, so each is also closed and hence is a union of connected components of M.

Proof. The problem is local on M, so by working locally we may assume that E admits a trivializing frame, say $\{e_i\}$. Thus, the metric tensor (a global section of $E^\vee \otimes E^\vee$) can be uniquely written in the form $\sum_{i,j} g_{ij} e_i^* \otimes e_j^*$ where $g_{ij} \in C^p(M)$, $g_{ji} = g_{ij}$, and $\det(g_{ij})$ is nowhere vanishing on M. The natural map $\varphi : M \to \text{Quad}(\mathbf{R}^n)$ to the space of quadratic forms on $\mathbf{R}^n$ defined by $m \mapsto \sum_{i,j} g_{ij}(m) X_i X_j$ is visibly continuous and lands in the open subset $\text{Quad}_{\text{nd}}(\mathbf{R}^n)$ of non-degenerate quadratic forms on $\mathbf{R}^n$. But we already know from an earlier handout on quadratic spaces (and we recalled above) that signature is locally constant on this latter space, whence by continuity of φ the signature of the metric tensor on any fiber $E(m_0)$ is the same as that on $E(m)$ for all $m \in M$ near m_0. This gives the desired openness result. ■

In the study of vector bundles one usually restricts attention to studying those with a constant rank n (an automatic condition when the base space is connected), and likewise in view of the preceding lemma it is traditional in the theory of pseudo-Riemannian metric tensors on vector bundles with constant rank n to restrict attention to those with constant signature (i.e., the same signature on all fibers). This restriction is not a restriction at all when the base space is connected, as we have already seen. (Even when the base space is disconnected, this restriction is so mild as to be essentially irrelevant.) Of course, the two most important examples of constant signature metric tensors are the positive-definite ones (Riemannian metrics) and the ones with signature $(n-1, 1)$ (*Lorentzian* metrics, with $n > 1$, whose importance is due to their ubiquity in General Relativity).

It is natural to wonder if a rank-n vector bundle E over a manifold with corners M, all of class C^p with $0 \leq p \leq \infty$, admits a C^p pseudo-Riemannian metric tensor with *any* desired signature (r, s) where $r, s \geq 0$ are non-negative integers satisfying $r + s = n$. For example, does every E admit a Riemannian metric, as well as a Lorentzian one? In fact, the answer is very subtle: there are genuine obstructions to the existence of metric tensors with mixed signature ($r, s > 0$). For example, for $M = S^{2d}$ and $E = TM$, there is no Lorentzian metric (even of class C^0, let alone C^∞) on E! The issue of this obstruction, and the worked case of tangent bundles to even-dimensional spheres, is addressed in the handout "Why the universe cannot be S^4". Fortunately, in the most important case of definite signature there is no obstruction:

Theorem 1.2. *Every C^p vector bundle $E \to M$ over a C^p manifold with corners, $0 \leq p \leq \infty$, admits a Riemannian metric. In particular, every smooth manifold with corners admits a structure of Riemannian manifold with corners.*

The main tool in the proof will be C^p partitions of unity, so it is amazing that Morrey and Grauert proved that every real-analytic vector bundle over a real-analytic manifold admits a real-analytic Riemannian metric tensor. The real-analytic case rests on serious input from the theory of several complex variables in order to circumvent the lack of real-analytic partitions of unity.

Proof. Let $\{U_i\}$ be a locally finite trivializing cover of E, say $E|_{U_i} \simeq U_i \times \mathbf{R}^{n_i}$. Using the flat metric tensor on this trivial bundle (i.e., declaring the choice of trivializing frame over U_i to be pointwise orthonormal), we get a Riemannian metric $\langle \cdot, \cdot \rangle_i$ on $E|_{U_i}$. Let $\{\phi_\alpha\}$ be a subordinate C^p partition of unity with compact supports, say $\text{supp}(\phi_\alpha) \subseteq U_{i(\alpha)}$, so $\phi_\alpha \langle \cdot, \cdot \rangle_{i(\alpha)}$ may be "extended by zero" to a symmetric bilinear form $\langle \cdot, \cdot \rangle_\alpha^\sim$ on E that is pointwise positive semi-definite, vanishing on fibers away from $\text{supp}(\phi_\alpha)$, and positive-definite on the locus $\{\phi_\alpha > 0\}$. This is a C^p global section of $E^\vee \otimes E^\vee$. The sum $\sum_\alpha \langle \cdot, \cdot \rangle_\alpha^\sim$ is a *locally finite* sum (as the supports of the ϕ_α's are a locally finite collection of subsets of M), whence it makes sense as a C^p global section of $E^\vee \otimes E^\vee$ and it is pointwise positive-definite (as it is pointwise a sum of finitely many positive semi-definite forms, at least one of which is positive-definite since at each $m \in M$ some ϕ_α is positive). Thus, this gives the desired Riemannian metric tensor on $E \to M$. ■

Let us now discuss the very important notion of *pullback* for metric tensors. There are two ways this arises: with abstract vector bundles and with tangent bundles. In the setting of abstract vector bundles, pullback means the following. We give ourselves a C^p vector bundle $E \to M$ over a C^p manifold with corners M, as well as a pseudo-Riemannian metric tensor B on E (of class C^p, as always) and a C^p map $f : M' \to M$ from another C^p manifold with corners. Viewing B as a section in $(E^\vee \otimes E^\vee)(M)$, we get a pullback section $B' = f^*(B)$ that is a global section over M' of the pullback bundle $f^*(E^\vee \otimes E^\vee) \simeq (f^*E)^\vee \otimes (f^*E)^\vee$. On the fiber over $m' \in M'$, the bilinear form $B'(m')$ on $(f^*E)(m') = E(f(m'))$ is exactly the non-degenerate symmetric bilinear form $B(f(m'))$ *precisely* because of the old observation from tensor algebra that if $T : V' \to V$ is a map between finite-dimensional vector spaces over a field and $B : V \times V \to F$ is a bilinear form then the induced bilinear form $B' = B \circ (T \times T)$ on V' corresponds to the element in ${V'}^\vee \otimes {V'}^\vee$ that is the image under $T^\vee \otimes T^\vee$ of the element in $V^\vee \otimes V^\vee$ corresponding to B. In particular, back in our vector bundle setting, we see that if B has some constant signature then so does $B' = f^*(B)$.

In the case of tangent bundles, there is another notion of pullback metric tensor as follows. We consider $f : M' \to M$ that is an immersion between C^p manifolds with corners (i.e., the tangent maps are injective), so $T(M')$ is naturally a subbundle of $f^*(TM)$ via the map $\mathrm{d}f : T(M') \to TM$ over the map $f : M' \to M$. We suppose TM is endowed with a choice of *Riemannian* metric tensor $\mathrm{d}s^2$. In this case $f^*(\mathrm{d}s^2)$ is a Riemannian metric tensor on $f^*(TM)$. In general a non-degenerate quadratic form on a finite-dimensional vector space need not have non-degenerate restriction to a subspace (e.g., $x^2 - y^2$ on $\mathbf{R}^2$ is non-degenerate but it restricts to 0 on the line $x = y$). However, in the positive-definite setting the restriction to a subspace is certainly positive-definite and hence necessarily non-degenerate. Thus, whereas a pseudo-Riemannian metric tensor on a vector bundle need not induce one on a subbundle (working on fibers by restriction of the bilinear form to the subspace arising from the subbundle), due to possible loss of non-degeneracy, for Riemannian metric tensors there is no such problem. Thus, restricting $f^*(\mathrm{d}s^2)$ on $f^*(TM)$ to the fibers of the subbundle $T(M')$ puts an inner product on each of the fibers $\mathrm{T}_{m'}(M')$.

It is important to work out this fibral restriction in local coordinates, both to see that it is not just a set-theoretic section to $T^*(M')^{\otimes 2} \to M'$ but even C^p, and for general computational purposes. (The resulting Riemannian metric tensor $\mathrm{d}{s'}^2$ on $T(M')$ is called the *induced metric tensor* on M' via the immersion $f : M' \to M$; by abuse of notation it may also be denoted $f^*(\mathrm{d}s^2)$!) Consider local C^p coordinates $\{x_i\}$ on an open $U \subseteq M$ with $\mathrm{d}s^2|_U = \sum g_{ij}\mathrm{d}x_i \otimes \mathrm{d}x_j$ (for $g_{ij} \in C^p(U)$ satisfying $g_{ji} = g_{ij}$ and $\det(g_{ij})$ nowhere vanishing on U), and let $U' \subseteq f^{-1}(U)$ be an open with local C^p coordinates $\{x'_i\}$. To compute $\mathrm{d}{s'}^2(u')$ for $u' \in U'$, the key point is that, as noted above, restriction of bilinear forms can be computed tensorially via the dual map. More since specifically, the dual map to the inclusion $\mathrm{d}f(u') : \mathrm{T}_{u'}(M') \to \mathrm{T}_{f(u')}(M)$ is the "pullback" map on cotangent spaces $f^* : \mathrm{T}^*_{f(u')}(M) \to \mathrm{T}_{u'}(M')$ (corresponding to pointwise evaluation of pullback of 1-forms), by working across all of U' at once we have

$$\mathrm{d}{s'}^2|_{U'} = \sum_{i,j} f^*(g_{ij}) f^*(\mathrm{d}x_i) \otimes f^*(\mathrm{d}x_j) = \sum_{i,j} (g_{ij} \circ f)\mathrm{d}f_i \otimes \mathrm{d}f_j$$

where $f_k = x_k \circ f$ are the component functions of $f : U' \to U$ with respect to the choice of local coordinates on the target U. Since $\mathrm{d}f_i = \sum_r (\partial f_i/\partial x'_r)\mathrm{d}x'_r$ and $\mathrm{d}f_j = \sum_s (\partial f_j/\partial x'_s)\mathrm{d}x'_s$, we get

$$\mathrm{d}s'^2|_{U'} = \sum_{i,j}(g_{ij}\circ f)\sum_{r,s}\frac{\partial f_i}{\partial x'_r}\cdot\frac{\partial f_j}{\partial x'_s}\mathrm{d}x'_r\otimes\mathrm{d}x'_s = \sum_{r,s}\left(\sum_{i,j}\frac{\partial f_i}{\partial x'_r}\cdot(g_{ij}\circ f)\cdot\frac{\partial f_j}{\partial x'_s}\right)\mathrm{d}x'_r\otimes\mathrm{d}x'_s.$$

In particular, the metric tensor coefficients $g'_{rs} := \langle \partial_{x'_r}, \partial_{x'_s}\rangle'$ for $\mathrm{d}s'^2|_{U'}$ with respect to the coordinate system $\{x'_k\}$ are given by the explicit formula

$$g'_{rs} = \sum_{i,j}\frac{\partial f_i}{\partial x'_r}\cdot(g_{ij}\circ f)\cdot\frac{\partial f_j}{\partial x'_s}.$$

(In the special case when f is the identity map, $U' = U$, and we are simply choosing two different coordinate systems on the open U, this formula is just the old transformation formula for the "matrix" of a symmetric bilinear form with respect to a change of basis, as $f_i = x_i$ in this case and $(\partial x_i/\partial x'_j)$ is the change of basis matrix between the trivializing frames $\{\partial_{x_i}\}$ and $\{\partial_{x'_j}\}$ for the tangent bundle over U.)

Example 1.3. On $M = \mathbf{R}^n$ the standard flat metric tensor $\mathrm{d}s^2$ is $\sum_i \mathrm{d}x_i^{\otimes 2}$. Consider the circle $C = \{x^2 + y^2 = r_0^2\}$ in $\mathbf{R}^2$ for some $r_0 > 0$. We have the usual global trivialization of TC via the vector field ∂_θ given at each $m \in C$ by $\partial_\theta|_m = -y(m)\partial_x|_m + x(m)\partial_y|_m$ in $\mathrm{T}_m(C) \subseteq \mathrm{T}_m(\mathbf{R}^2)$. Computing with the orthonormal frame $\{\partial_x|_m, \partial_y|_m\}$ in $\mathrm{T}_m(\mathbf{R}^2)$, clearly for each $m \in C$ we have $\langle \partial_\theta|_m, \partial_\theta|_m\rangle_m = (-y(m))^2 + x(m)^2 = r_0^2$, so the induced metric tensor is $r_0^2\mathrm{d}\theta^{\otimes 2}$. In particular, the metric tensor written in terms of the local angle coordinate has constant coefficients, so the circle is "flat" from the viewpoint of Riemannian geometry. (In contrast, S^2 equipped with its induced metric tensor from $\mathbf{R}^3$ is *not* flat: we will prove later that around any point of S^2 there is *no* local coordinate system with respect to which the metric tensor has constant coefficients.)

Example 1.4. The case of the metric tensor on smooth hypersurfaces $S = \{z = f(x, y)\}$ in $\mathbf{R}^3$ (using global coordinates $x' = x|_S$ and $y' = y|_S$) is worked out in detail in the handout on hypersurface metric tensors, where even the general case of hypersurface graphs in $\mathbf{R}^{n+1}$ and surfaces of revolution in $\mathbf{R}^3$ are treated. Here, let us treat the interesting case of the torus $T = S^1 \times S^1 \hookrightarrow \mathbf{R}^3$ with inner radius $a - r$ and outer radius $a + r$ for $0 < r < a$; this is the compact smooth submanifold given in Homework 4, Exercise 2. Using angle coordinates θ and ψ on the respective factors in $S^1 \times S^1$, we get a global trivializing frame $\{\partial_\theta, \partial_\psi\}$ for the tangent bundle of T. In the solution to Homework 4, Exercise 2(*ii*) we saw that these two vector fields are pointwise orthogonal (working with the inner product on the ambient tangent bundle of $\mathbf{R}^3$ pulled back to T), and the respective self inner products are $\langle \partial_\theta|_m, \partial_\theta|_m\rangle_m = r^2$ and $\langle \partial_\psi|_m, \partial_\psi|_m\rangle_m = (a + r\cos\theta(m))^2$. Hence, the induced metric tensor on T from the one on $\mathbf{R}^3$ is $r^2\mathrm{d}\theta^{\otimes 2} + (a + r\cos\theta)^2\mathrm{d}\psi^{\otimes 2}$.

2. Orthogonal bundles

Let E be a C^p vector bundle with pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ over a C^p manifold with corners M ($0 \le p \le \infty$). Let E' be a subbundle of E. For each $m \in M$, we get a $\langle\cdot,\cdot\rangle_m$-orthogonal complement $E'(m)^\perp$ of $E'(m)$ in $E(m)$. I claim that these fit together to be the fibers of a unique C^p subbundle $E'^\perp$ in E. Such a subbundle is unique since we are specifying its fibers (see Lemma 2.1 in the handout on subbundles and quotient bundles), so the only problem is existence. To proof existence, we "just" need an artful way to describe the construction of the orthogonal complement in linear algebra without mentioning bases. There are a couple of ways to do this. For our purposes, the simplest method is this: if $(V, \langle \cdot, \cdot \rangle)$ is a finite-dimensional vector space (over $\mathbf{R}$) endowed with a non-degenerate symmetric bilinear form, and $i : V' \hookrightarrow V$ is the inclusion map for a subspace, then we get a dual surjection $i^\vee : V^\vee \twoheadrightarrow V'^\vee$ and composing this with the natural isomorphism $V \simeq V^\vee$ defined by $v \mapsto \langle v, \cdot \rangle$ gives a linear surjection $V \twoheadrightarrow V'^\vee$ defined by $v \mapsto \langle v, \cdot \rangle|_{V'}$. Hence, the kernel is $V'^\perp$. This works for bundles:

Theorem 2.1. *With notation as above, there exists a C^p subbundle $E'^\perp$ in E whose m-fiber is $E'(m)^\perp \subseteq E(m)$ for all $m \in M$, and $E' \oplus E'^\perp \to E$ is an isomorphism.*

Proof. The first step is to uniquely promote the fibral duality mappings $E(m) \simeq E(m)^\vee$ given by $v \mapsto \langle v, \cdot \rangle_m$ to a bundle isomorphism $E \simeq E^\vee$. This can be done in two ways: computation with local frames or using the equivalence between bundles and $\mathscr{O}$-modules. For the sake of concreteness, we'll just compute with local frames (and we again recall that the "reason" all such things work is that at the fibral level they involve no choices and so are always described by "universal formulas" in terms of bases). If we chose a local trivializing frame $\{s_i\}$ for $E|_U$ and let $\{s_i^*\}$ be the dual trivializing frame for $E^\vee|_U$ then the metric tensor over U has the form $\sum g_{ij} s_i^* \otimes s_j^*$ with $g_{ij} \in C^\infty(U)$ satisfying $g_{ij} = g_{ji}$ and $\det(g_{ij}) \in C^\infty(U)$ non-vanishing. Thus, the bundle mapping $E|_U \to E^\vee|_U$ given in the trivializing frames by the matrix (g_{ij}) does the job on fibers over U (i.e., it recovers the natural isomorphism $v \mapsto \langle v, \cdot \rangle_m$ on fibers, and so in particular is an isomorphism of vector bundles).

Thus, the pseudo-Riemannian metric defines a bundle isomorphism $E \simeq E^\vee$ giving the usual duality $E(m) \simeq E(m)^\vee$ on fibers, and if $i : E' \to E$ is the subbundle inclusion then the dual map $i^\vee : E^\vee \to E'^\vee$ is a bundle surjection. Hence, we get a composite bundle surjection $E \simeq E^\vee \to E'^\vee$, whence (by Theorem 2.6 in the handout on subbundles and quotient bundles) there is a kernel C^p *subbundle* $E'' \subseteq E$ whose fibers $E''(m) \subseteq E(m)$ are the kernels of the fibral maps $E(m) \simeq E(m)^\vee \twoheadrightarrow E'(m)^\vee$. But inspecting the definition of this fibral map shows that the kernel is *exactly* the orthogonal complement $E'(m)^\perp$ with respect to $\langle \cdot, \cdot \rangle_m$. Hence, the subbundle E'' in E has the desired fibers (which determines it uniquely), and by linear algebra on the fibers we know that the natural bundle map $E' \oplus E'' \to E$ is an isomorphism on fibers and hence is an isomorphism. ■

An interesting consequence of this theorem is a result whose statement does not mention pseudo-Riemannian metrics:

Corollary 2.2. *Any C^p subbundle E' of a C^p vector bundle E over a C^p manifold with corners M admits a "complementary" subbundle* (*i.e., there exists a C^p subbundle E'' in E such that $E' \oplus E'' \to E$ is an isomorphism*).

Proof. As we saw in class, we can construct a Riemannian metric on E (the construction works in the C^p setting). Pick such a metric. Now just take E'' to be $E'^\perp$! (Of course, if we change the Riemannian metric then we get a different complementary subbundle in general.) ■

3. Tensors and pairings

Let $E \to M$ be a C^p vector bundle over a C^p manifold with corners, $0 \le p \le \infty$, and we suppose E is endowed with a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$. As we reviewed in the motivational discussion, for each $m \in M$ the vector spaces

$$E(m)^{\otimes n}, \mathrm{Sym}^n(E(m)), \wedge^n(E(m)), E(m)^\vee, (E(m)^\vee)^{\otimes n}, \mathrm{Sym}^n(E(m)^\vee), \wedge^n(E(m)^\vee)$$

are naturally endowed with induced non-degenerate symmetric bilinear forms via $\langle \cdot, \cdot \rangle_m$ on $E(m)$. These are all positive-definite if $\langle \cdot, \cdot \rangle_m$ is positive-definite. It is natural to ask if these constructions are the fibers of pseudo-Riemmannian metrics on the vector bundles $E^{\otimes n}$, $\mathrm{Sym}^n(E)$, $\wedge^n(E)$, $E^\vee$,

$(E^\vee)^{\otimes n}$, $\mathrm{Sym}(E^\vee)$, and $\wedge^n(E^\vee)$ respectively (and so all are Riemannian metrics if the one given on E is Riemannian). Of course the answer is yes:

Theorem 3.1. *The above fibral constructions on the dual/tensor operations for the $E(m)$'s arise as the fibral bilinear forms for unique pseudo-Riemannian metrics on the vector bundles $E^{\otimes n}$, $\mathrm{Sym}^n(E)$, $\wedge^n(E)$, $E^\vee$, $(E^\vee)^{\otimes n}$, $\mathrm{Sym}(E^\vee)$, and $\wedge^n(E^\vee)$ respectively.*

Proof. The uniqueness is clear since we are specifying the construction on fibers. Thus, the only problem is to actually compute the fibral "metric tensors" (for suitably trivializations) and make sure that they vary smoothly in m. In fact, they can all be given by "universal formulas" in the metric-tensor coefficients of E (for local trivializing frames), whence the smoothness drops out.

Let us make this precise. Since the induced forms on the fibers of the tensor operations of $E^\vee$ are given by the dual pairings to the ones on the fibers of the tensor operations of E, it suffices to just treat the cases of the dual bundle and the tensor operations on E (because this fibral statement is something we know from linear algebra, as we recalled in §1 above). The problem is local on M, so we can assume E admits a trivializing frame $\{s_i\}$. Let $\{s_i^*\}$ be the dual trivializing frame for $E^\vee$. Hence, the metric tensor is $\sum g_{ij} s_i^* \otimes s_j^*$ with smooth functions g_{ij} (and $\{s_i^* \otimes s_j^*\}$ a trivializing frame for $E^\vee \otimes E^\vee$). Using the unique bundle isomorphisms $(E^{\otimes n})^\vee \simeq (E^\vee)^{\otimes n}$, $\mathrm{Sym}^n(E)^\vee \simeq \mathrm{Sym}^n(E^\vee)$, and $(\wedge^n E)^\vee \simeq \wedge^n(E^\vee)$ inducing the natural isomorphisms on fibers, we get trivializing frames of these three bundles using tensor/symmetric/exterior products of members of the dual frame $\{s_i^*\}$. Thus, the problem is reduced to showing that the coefficients of the fibral metric tensors with respect to these trivializing frames are smooth in the base parameter. These coefficients are just the "universal formulas"

$$\prod_{r=1}^{n} \langle s_{j_r}(m), s_{j'_r}(m)\rangle_m, \quad \sum_{\sigma \in S_n} \prod_{r=1}^{n} \langle s_{j_r}(m), s_{j'_{\sigma(r)}}(m)\rangle_m, \quad \det(\langle s_i(m), s_j(m)\rangle_m)$$

on m-fibers, or in other words

$$\prod_{r=1}^{n} g_{j_r, j'_r}(m), \quad \sum_{\sigma \in S_n} \prod_{r=1}^{n} g_{j_r, j'_{\sigma(r)}}(m), \quad \det(g_{ij}(m)).$$

These are all smooth in m since the g_{ij}'s are smooth functions on the base. ■

We conclude our discussion with one other bundle generalization of a construction from linear algebra: the Gram-Schmidt algorithm. In linear algebra, recall that Gram-Schmidt took as input an inner product space $(V, \langle\cdot,\cdot\rangle)$ and an ordered basis $\{v_1, \dots, v_n\}$ of V, and it spit out an orthonormal basis $\{e_1, \dots, e_n\}$ characterized uniquely by the property that $\{e_1, \dots, e_i\}$ has the same span as $\{v_1, \dots, v_i\}$ for $1 \le i \le n$ and $\langle e_i, v_i\rangle > 0$ for all i. The procedure does not really require positive-definiteness: it works for any non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V. The only difference in the general case is that when we're given a non-zero vector $v \in V$, we pass to the scaled vector $\widehat{v} = v/\sqrt{|\langle v, v\rangle|}$ to get a "unit vector" in the sense $\langle \widehat{v}, \widehat{v}\rangle = \pm 1$.

This carries over to vector bundles $E \to M$ equipped with a pseudo-Riemannian metric. We call a local trivialization of E over an open subset U *orthonormal* if it is fibrally orthonormal: the fibral vectors from the frame are pairwise orthogonal and have self-pairings that are locally equal to ± 1 (and so equal to such a sign on each connected component of U).

Theorem 3.2. *Let $E \to M$ be a C^p vector bundle endowed with a pseudo-Riemannian metric, with M a C^p manifold with corners, $0 \le p \le \infty$. Let $\{s_1, \dots, s_n\}$ be a trivializing frame over an open subset $U \subseteq M$. There exists a unique orthonormal frame $\{e_1, \dots, e_n\}$ for $E|_U$ such that $\{e_1, \dots, e_i\}$ spans the same subbundle as $\{s_1, \dots, s_i\}$ for $1 \le i \le n$ and such that the smooth function $\langle e_i, s_i\rangle_U$*

on U is positive. On fibers over $m \in U$, $\{e_i(m)\}$ is the orthonormal basis of $E(m)$ associated to the ordered basis $\{s_i(m)\}$ by the Gram-Schmidt algorithm.

Applying this to the tangent bundle of a Riemannian (smooth!) manifold with corners, the theorem says that locally orthonormal frames of smooth vector fields always exist on a Riemannian manifold with corners. But such frames will usually *not* be of the form $\{\partial_{x_i}\}$ for a local coordinate system $\{x_i\}$, so such frames do *not* relate to whether or not the Riemannian metric tensor is flat. As long as we are not too attached to working with the special local trivializing frames $\{\partial_{x_i}\}$ for the tangent bundle (which were the *only* ones considered in the 19th century), we can still make convenient use of orthonormal frames as we do in linear algebra. This is an example of E. Cartan's influential philosophy of "moving frames": one should always choose the trivializing frame adapted to the local problem at hand, and not just restrict attention to special trivializations arising from coordinate systems (which often have no geometric significance).

Proof. By the unique characterization of the output of the fibral Gram-Schmidt algorithm, such an orthonormal frame for the bundle over U must induce the Gram-Schmidt output on fibers. Hence, we get uniqueness. The problem is therefore one of existence. We just use the Gram-Schimidt process on the level of $E(U)$! For example, the function $\langle s_1, s_1\rangle$ is smooth and *non-vanishing*, so it is locally of a fixed sign (U might be disconnected). Hence, its absolute value $|\langle s_1, s_1\rangle|$ is a *smooth* positive function on U, whence its square root is a smooth positive function on U. Thus, $e_1 = s_1/\sqrt{|\langle s_1, s_1\rangle|}$ is a smooth section of E over U and

$$\langle e_1, e_1\rangle = \frac{\langle s_1, s_1\rangle}{|\langle s_1, s_1\rangle|}$$

is locally constant, equal to ± 1 on each connected component of U (depending on the local sign of the non-vanishing function $\langle s_1, s_1\rangle$).

Now we define $s_i' = s_i - \langle s_i, e_1\rangle e_1 \in E(U)$ for $2 \le i \le n$, recovering the next step of the Gram-Schmidt process on fibers over U. In particular, $\{e_1, s_2', \dots, s_n'\}$ is a fiberwise basis and so is a trivializing frame for $E|_U$. We repeat the game again (replacing s_2 with a suitably scaled e_2, and so on). This is inducing the Gram-Schmidt process on fibers, and so it goes through to the end, given an orthonormal frame for $E|_U$, with $\{e_1(u), \dots, e_i(u)\}$ and $\{s_1(u), \dots, s_i(u)\}$ having the same span in $E(u)$ for all $u \in U$. Hence, the fiberwise independent sections $\{e_1, \dots, e_i\}$ and $\{s_1, \dots, s_i\}$ span subbundles of $E|_U$ that coincide (inside each $E(u)$) on fibers over U and hence coincide as subbundles of $E|_U$. ■

Example 3.3. Let M be a Riemannian manifold. Taking $E = TM$ in the preceding discussions, we see in particular that $\Omega^n_M = \wedge^n(E^\vee)$ is endowed with a natural Riemannian structure for all n. In particular, the line bundle $\det(T^*M)$ of top-degree differential forms is endowed with a Riemannian metric.